On the Applications of Ricatti Differential Equations to Some Special Cases of the Two-Body Problem with Variable Masses

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Abstract In this paper, we obtain the solution of the two-body problem with variable masses by posing some assumptions on the classical equation and thereby reducing the solution of the problem to investigating the solutions of some Ricatti type differential equations. Along this process, we also give a simple proof of the well-known Mestscherskii Theorem and establish some related formal relations under these assumptions.

Keywords: two body problem with variable masses, ricatti equation, Mestscherskii’s therem


1. Introduction

The attempts to solve the two-body problem with variable masses go back to the middle of 19th century. The most meticulous and comprehensive of these endeavors appeared in the works of H. Glyden [1], J. Mestscherskii [2], G. Armellini [3], Sir James Jeans [4], William D. MacMillan [5], G. N. Doubouche [6,7], and K. Sawtchenko [8]. These were further expanded by the seven articles published by E. L. Martin in 1934 [9-15]. Most of these investigations relied upon the classical equations of motion in a gravitational field modified suitably to accommodate for the variability of the masses.

Then on, the problem kept on enjoying a distinguished existence at the confluence of physics, astrophysics, and applied mathematics, and was analyzed in many different ways and under many different assumptions.

Our goal in this paper is to investigate the solution of this problem under some reasonable restrictions. Throughout, we assume some basic familiarity with some physical and mathematical techniques, most of which can be found in Goldstein [16] and Betounes [17].

Since it is going to directly affect our system of equations, let us briefly talk about Kepler’s Second Law. As is well known, the law states that a line joining a planet and the Sun sweeps out equal areas during equal intervals of time, implying a planet travels faster when closer to the Sun, and slower when farther from the Sun.

We note that in a small time interval $dt$, the planet sweeps out a small triangle having base $r$ and height $rd\theta$ and consequently, an area of

$$dA = \frac{1}{2}r^2d\theta.$$ 

Thus, the rate of change of area would be

$$\frac{dA}{dt} = \frac{1}{2}r^2 \frac{d\theta}{dt}.$$ 

Note that the area enclosed by the elliptical orbit is $\pi ab$. Thus, the period $P$ satisfies the equation

$$\frac{1}{2}P r^2 \frac{d\theta}{dt} = \pi ab.$$ 

Now let us establish the system of differential equations that we will use to solve the problem.

Suppose the variable masses are given by two holomorphic functions $m_0(t)$ and $m_1(t)$. We will assume that

$$m_0(t) + m_1(t) = \mu(t)$$

where $\mu(t)$ is a positive analytic real function of $t$.

Let $G$ denote the constant of gravitation, which in SI units is approximately $6.674 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$. It can be shown that [18] the solution of the problem is equivalent to finding the solution of the system of differential equations
\[ r^2 \frac{d\theta}{dt} = C \]

\[ \frac{d^2 r}{dt^2} = \frac{C^2}{r^3} - \frac{G\mu(t)}{r^2}. \]

The first equation is, of course, Kepler’s Second Law with \( C = \frac{\pi ab}{P} \), where \( a, b \) are axes of the ellipse and \( P \) the period of revolution. Clearly, its integrability depends on that of the second one. Thus, from now on, we assume that the solution of our problem is reduced to the integration of the second equation.

### 2. Solution of the Differential Equation in the Astronomical Case

In the astronomical case, we assume that the total mass \( \mu(t) \) varies very slowly with respect to time and thus \( \mu'(t) \) and \( \mu''(t) \) are negligible compared to \( \mu(t) \).

Let \( p(t) = \frac{1}{\mu(t)} = \frac{1}{m_0(t) + m_1(t)} \)

Let \( \varepsilon \) denote the eccentricity and \( \phi \) denote the true anomaly, that is, the parameter that defines the position of a body moving along a Keplerian orbit.

Thus,

\[ r(t) = \frac{p(t)}{1 + \varepsilon \cos \phi} \]

Let us put

\[ \xi = 1 + \varepsilon \cos \phi \]

Hence,

\[ r = \frac{\xi}{\mu} \]

Let us now substitute this in the equation

\[ \frac{d^2 r}{dt^2} = \frac{C^2}{r^3} - \frac{G\mu(t)}{r^2} \]

We obtain

\[ \xi^* - \frac{\mu^A}{\xi^3} \left( C^2 - G\xi \right) = 0 \]

Taking the constants as unities, and putting

\[ \xi = 1 + W \]

the equation becomes

\[ W''(t) + W(t)\frac{\mu^A(t)}{(1 + W(t))^3} = 0 \]

We refer to this equation as the equation of the two-body problem in the astronomical case.

Assuming \( W(t) \ll 1 \), the equation further simplifies to

\[ W''(t) + W(t)\mu^A(t) = 0 \]

### A Special Case

If the total mass function \( \mu(t) \) is linear, say \( \mu(t) = \alpha t \), the equation

\[ W''(t) + \beta t^4 W(t) = 0 \]

with \( \beta = \alpha^4 \), can easily be solved by the Frobenius series method.

Putting

\[ W(t) = \sum_{k=0}^{\infty} a_k z^k \]

One easily obtains that for \( k \neq 6n, a_k = 0 \) and

\[ a_{6n} = (-1)^n \frac{\beta^{n/4}}{6^n n!(7 \cdot 13 \ldots (6n + 1))} \]

Giving us the solution

\[ W(t) = A t \left[ 1 - \sum_{n=1}^{\infty} (-1)^n \frac{\beta^{n/4} a_{6n}}{6^n n!(7 \cdot 13 \ldots (6n + 1))} \right] \]

where \( A \) is an arbitrary constant. It is easy to see that this series is convergent for all finite values of \( t \).

**Remark.**

In the equation

\[ W''(t) + W(t)\mu^A(t) = 0 \]

Let us now put

\[ W'(t) = W(t)U(t) \]

Thus, we can rewrite the equation as

\[ U'(t) + U^2(t) + \mu^A(t) = 0 \]

Recall that a differential equation of the form

\[ y''(x) = p_0(x) + p_1(x)y(x) + p_2(x)y^2(x) \]

with \( p_0(x) \neq 0, p_1(x) \neq 0, \) and \( p_2(x) \neq 0 \) is called a Ricatti equation, after the Italian mathematician Jacopo Ricatti (1676 – 1754). Thus, in this case, we can think of the solution of the two-body problem as a solution of a Ricatti equation. See Biernacki [19] and Milloux [20].

### 3. Solution of the Differential Equation in Case of Small Eccentricity

Recall we had

\[ r(t) = \frac{p(t)}{1 + \varepsilon \cos \phi} \]

with

\[ p(t) = \frac{1}{\mu(t)} \]

and
Let us now introduce a new variable $\rho$ by

$$\rho = \frac{\xi - 1}{\mu}$$

Thus,

$$r = \rho + \frac{1}{\mu}$$

Let us now substitute this in the equation

$$\frac{d^2r}{dt^2} = \frac{1}{r^3} - \frac{\mu(t)}{r^2}$$

to obtain

$$\frac{d^2\rho(t)}{dt^2} + \rho(t)\mu^A(t) + \frac{d^2(1/\mu(t))}{dt^2} = 0$$

Since

$$r = \rho + \frac{1}{\mu}$$

we have

$$\frac{d^2\rho(t)}{dt^2} = \frac{d^2r}{dt^2} - \frac{d^2(1/\mu(t))}{dt^2}$$

We thus have

$$\frac{d^2r(t)}{dt^2} - \frac{d^2(1/\mu(t))}{dt^2} + \left( r(t) - \frac{1}{\mu(t)} \right)\mu^A(t) + \frac{d^2(1/\mu(t))}{dt^2} = 0$$

which is, of course, the well-known Armellini equation

$$\frac{d^2r(t)}{dt^2} + r(t)\mu^A(t) = \mu^3(t)$$

[21]

Note that the homogenous form of this equation is a Ricatti equation, implying in these two cases the two-body problem with variable masses is reduced to finding the solution of a Ricatti equation, a theorem first proved by Armellini in 1935 [21].

Since in this case, $r \approx \frac{1}{\mu}$ the Armellini equation becomes

$$\frac{d^2(1/\mu(t))}{dt^2} = 0$$

that is,

$$2\left( \frac{d\mu(t)}{dt} \right)^2 = \mu(t)\frac{d^2\mu(t)}{dt^2}$$

We get,

$$A\mu^2 = \mu'$$

or

$$\frac{1}{\mu(t)} \cong r(t) = -At + B$$

which, of course, is an alternate way of obtaining Mestschekii’s theorem.

**Special Cases**

1. If $\frac{\xi}{\mu} = \alpha$, a constant, then

$$\mu(t) = \frac{1}{\alpha}, r = \alpha, \text{ and } \varphi = \frac{\pi}{2}$$

2. If $\frac{\xi}{\mu} = \alpha t$, $\alpha$ a constant, then

$$\mu(t) = \frac{1}{\alpha t}, r = \alpha t$$

Since $\xi = 1$,

$$\frac{1}{1 + \cos \varphi} = 1$$

and we still have $\varphi = \frac{\pi}{2}$.

3. If $\frac{\xi}{r} = \mu = \alpha$, $\alpha$ a constant, then we get the equation

$$\xi'(t) = \alpha^4 \frac{(1 - \xi)}{\xi^3} = \Psi(\xi)$$

which can be solved easily. To this end, we write

$$\xi'(t) = v(\xi)$$

This implies

$$\xi'(t) = \frac{dv}{d\xi} = v'(\xi) v(\xi)$$

and the equation becomes

$$v'(\xi) = \frac{\Psi(\xi)}{v(\xi)} = Y(\xi)$$

4. The case $\frac{\xi}{r} = \mu = (\alpha + \beta t)^2$ was also analyzed by Armellini. In this case, if we put

$$\alpha + \beta t = \sigma(t)$$

in the equation

$$\xi'(t) - (1 - \xi(t))\frac{d\sigma(t)}{dt} = 0$$

since

$$\frac{d\xi}{dt} = \frac{d\xi}{d\sigma} \frac{d\sigma}{dt} = \mu \frac{d\xi}{d\sigma}$$

and consequently,

$$\frac{d^2\xi}{dt^2} = \beta^2 \frac{d^2\xi}{d\sigma^2}$$
Thus, we now get a Sturm-Liouville type equation
\[
\frac{d^2 \xi}{d\sigma^2} + \frac{\xi}{\beta^2} - \frac{\sigma^{4\nu}}{\beta^2} = 0
\]

We will now show that this equation can be transformed into a Bessel equation. To this end, let us put
\[
4\nu = \frac{1}{n} - 2
\]

\[
\frac{2n}{\beta} \sigma^{1/2n} = x
\]

\[
\xi \sigma^{-1/2} = y
\]

Since
\[
\frac{d\xi}{d\sigma} = \frac{d\xi}{dx} \frac{dx}{d\sigma} + \frac{d\xi}{dy} \frac{dy}{d\sigma}
\]

we get
\[
\frac{d^2\xi}{d\sigma^2} = \sigma^{-1/2} \left[ \left( \frac{1}{\beta} \sigma^{(1/2n)-1} \right) \frac{d\xi}{dx} + \left( -\frac{1}{2} \xi \sigma^{-1} \right) \frac{d\xi}{dy} \right]
\]

implying
\[
\frac{d\xi}{d\sigma} = \left( \frac{1}{\beta} \sigma^{(1/2n)-1} \right) \frac{dy}{dx} + \left( -\frac{1}{2} \xi \sigma^{-1} \right)
\]

Differentiating both sides with respect to \(\sigma\), one obtains
\[
\frac{d^2\xi}{d\sigma^2} = \left( \frac{1}{\beta^2} \sigma^{1/n} \frac{d^2y}{dx^2} + \frac{1}{2n\beta} \sigma^{1/2n} \frac{dy}{dx} - \frac{1}{4} \right) \sigma^{-3/2}
\]

Substituting this in the equation
\[
\frac{d^2\xi}{d\sigma^2} + \frac{\xi}{\beta^2} - \frac{\sigma^{4\nu}}{\beta^2} = 0
\]

we get
\[
\left( \frac{1}{\beta^2} \sigma^{1/n} \frac{d^2y}{dx^2} + \frac{1}{2n\beta} \sigma^{1/2n} \frac{dy}{dx} - \frac{1}{4} \right) \sigma^{-3/2} + \left( \frac{\sigma^{1/n-3/2}}{\beta^2} \right) y = 0
\]

Since
\[
\sigma^{-1/n} = \left( \frac{\beta x}{2n} \right)^2
\]

equation can be rewritten as
\[
\frac{x^2}{4n^2} \frac{d^2y}{dx^2} + \frac{x}{4n^2} \frac{dy}{dx} - \frac{1}{4} y + \frac{x^2}{4n^2} y = 0
\]

which is of course the Bessel equation
\[
\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left( 1 - \frac{n^2}{x^2} \right) y = 0
\]

We can again apply the Frobenius series solution method to this equation and get
\[
y(x) = C_1 J_n(x) + C_2 J_{-n}(x)
\]

where
\[
J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{n+2k}}{k! \Gamma(n+k+1)}
\]

Here, \(\Gamma(s)\) is the gamma function defined as
\[
\Gamma(s) = \int_0^{\infty} u^{s-1} e^{-u} du
\]

over all complex \(s\) with \(Re(s) > 0\).

References