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A Modification of the Formula for the Average Velocity of a Planet

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Abstract In this paper, we will deal with the problem of calculating the average velocity of a celestial object revolving around another celestial object in an elliptical orbit. After proving our main theorem to this effect, we will give some alternate forms of the formula for the average velocity, and show that this average value is in fact, attained at certain points of the orbit. We will conclude the paper by providing an intuitively natural and straightforward amendment of this formula.

Keywords: *average velocity of a planet, first-order approximation, second-order approximation*

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1. Introduction

Human awareness of planetary motions dates back to prehistoric times. It is well-known that, scholars from the Mesopotamian, Greek and Egyptian civilizations, for reasons ranging from scientific to spiritual to paranormal, tried to observe the motions of celestial objects, their efforts culminating in the Ptolemaic model.

Eventually, medieval scientists, in particular, Nicolas Oresme (1320-1382) and Jean Buridan (1300-1361), paved the way to Johannes Kepler's (1571-1630) invention of a system that correctly described the major aspects of the motions of the planets around the sun. The crucially needed mathematical support of the theory was later furnished by Sir Isaac Newton (1643-1727), mainly through his law of gravitation. For a more detailed discussion of this fascinating historic development, see Thurston [1].

Our principal goal in this paper is to compute the average velocity of a planet rotating around the sun in the Keplerian model. For physical and astronomical background Dilgan [2], Gamow [3], and Curtis [4].

Suppose that a planet P of mass m is moving in an elliptical orbit about an object of mass M located at tone of the foci of the ellipse at distance c from the center. Let the semi-major and semi-minor axes of the orbit be a and b, respectively. Let v be the velocity of P when it is at a distance r from M. Let the average velocity be denoted as $\bar{v}$.

Setting the kinetic energy

$$KE = \frac{1}{2}mv^2$$

equal to the gravitational potential energy

$$U = -\frac{GMm}{r}$$

we get

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{GMm}{2a}$$

Solving this equation for the velocity v, we obtain

$$v = \sqrt{GM}\sqrt{\left(\frac{2}{r} - \frac{1}{a}\right)} = f(r)$$

the so-called **vis-viva equation**. Here $GM = \mu = \frac{4\pi^2 a^3}{T^2}$ is the **standard gravitational parameter**.

2. The Main Theorem

Before we state and prove our main theorem, let us prove a simple algebraic identity:

Lemma 1. For any two real numbers a and c

$$(a+c)\sqrt{\frac{2}{(a+c)} - \frac{1}{a}} = (a-c)\sqrt{\frac{2}{(a-c)} - \frac{1}{a}}$$

Proof. Clearly, the left hand-side can be written as

$$\sqrt{\frac{2a^2 - 2ac - a^2 + 2ac - c^2}{a}}$$

$$= \sqrt{2(a-c) - \frac{(a-c)^2}{a}} = (a-c)\sqrt{\frac{2}{(a-c)} - \frac{1}{a}}$$

Now for the main result we want to prove:

Theorem 1. *Suppose a planet P of mass m is moving in an elliptical orbit about an object of mass M located at one of the foci of the ellipse at distance c from the center. Let the semi-major and semi-minor axes of the orbit be a* and b, respectively. Then, the average velocity $\bar{v}$ is given as

$$\bar{v} = \frac{2\pi}{T}\frac{a^2}{b}$$

Proof.

Our goal is to compute the average value of the function $f(r)$, namely the integral

$$\bar{v} = \sqrt{GM}\,\frac{1}{2c}\int_{a-c}^{a+c}\sqrt{\left(\frac{2}{r}-\frac{1}{a}\right)}dr$$

Let us first compute the indefinite integral by making the change of variables

$$\left(\frac{2}{r}-\frac{1}{a}\right) = x^2$$

Then,

$$dr = \frac{-4a^2 x}{\left(ax^2+1\right)^2}dx$$

Thus, our integral becomes

$$-4a^2\int\frac{x^2}{\left(ax^2+1\right)^2}dx$$

Since

$$\frac{x^2}{\left(ax^2+1\right)^2} = \frac{1}{a}\left[\frac{1}{ax^2+1}-\frac{1}{\left(ax^2+1\right)^2}\right]$$

integrating we obtain

$$-4a^2\int\frac{x^2}{\left(ax^2+1\right)^2}dx$$

$$= -4a\left[\frac{1}{2\sqrt{a}}\arctan\left(\sqrt{a}x\right)-\frac{1}{2}\frac{x}{ax^2+1}\right]$$

Consequently,

$$\bar{v} = \sqrt{GM}\,\frac{1}{2c}\int_{a-c}^{a+c}\sqrt{\left(\frac{2}{r}-\frac{1}{a}\right)}dr$$

$$= \frac{\sqrt{GM}}{2c}\left[r\sqrt{\frac{2}{r}-\frac{1}{a}}-2\sqrt{a}\arctan\left(\sqrt{\frac{2a}{r}-1}\right)\right]_{a-c}^{a+c}$$

Invoking Lemma 1, we obtain

$$\bar{v} = \frac{\sqrt{GM}}{2c}\left[r\sqrt{\frac{2}{r}-\frac{1}{a}}-2\sqrt{a}\arctan\left(\sqrt{\frac{2a}{r}-1}\right)\right]_{a-c}^{a+c}$$

$$= \frac{\sqrt{GMa}}{c}\left(\arctan\sqrt{\frac{a+c}{a-c}}-\arctan\sqrt{\frac{a-c}{a+c}}\right)$$

and since

$$\arctan(\theta)+\arctan(\rho) = \arctan\left(\frac{\theta+\rho}{1-\theta\rho}\right)$$

we get,

$$\bar{v} = \frac{\sqrt{GMa}}{c}\arctan\left(\frac{c}{\sqrt{a^2-c^2}}\right)$$

Using the fact that the eccentricity of an ellipse is

$$\varepsilon = \frac{c}{a}$$

one can also write

$$\bar{v} = \frac{\sqrt{GMa}}{c}\arctan\left(\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\right)$$

Recalling that

$$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots$$

for $-1 \le x \le 1$, we have

$$\arctan\left(\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\right) \cong \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}$$

and subsequently,

$$\bar{v} = \frac{\sqrt{GMa}}{c}\cdot\frac{c}{b} = \frac{2\pi}{T}\frac{a^2}{b}$$

proving our theorem.

Lemma 2. (An Alternate Formulation) *Let R be the radius of curvature of the ellipse. Then*

$$\bar{v} = \frac{2\pi R}{T}$$

Proof. Suppose we parametrize the equation of an ellipse using

$$x(t) = b\cos t$$

and

$$y(t) = a\sin t$$

Then, using the formula

$$R = \frac{\left(x'^2+y'^2\right)^{3/2}}{x'y''-y'x''}$$

for the radius of curvature, we get at $t = 0$

$$R = \frac{a^2}{b}$$

and consequently,

$$\bar{v} = \frac{2\pi R}{T}$$

3. Some Additional Results

Here are some additional results of interest:

Lemma 2. *The average velocity $\bar{v}$ is the arithmetic mean of the maximum velocity attained at the pericenter and the minimum velocity attained at the apocenter.*

Proof. This follows immediately from the formulas of the maximum and minimum velocity [5]. Indeed,

$$\frac{1}{2}\left(v_{\min}+v_{\max}\right)=\frac{\pi a}{T}\left(\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}+\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\right)$$

$$=\frac{\pi a}{T}\frac{2}{\sqrt{1-\varepsilon^2}}=\frac{2\pi}{T}\frac{a^2}{b}=\overline{v}$$

Lemma 3. *Every planet attains its average velocity* $\bar{v}$.
Proof. Solving the equation

$$\overline{v}=\frac{2\pi}{T}\frac{a^2}{b}=v=\frac{2\pi}{T}a^{3/2}\sqrt{\left(\frac{2}{r}-\frac{1}{a}\right)}$$

we get that these velocities are equal whenever

$$r=\frac{2ab^2}{a^2+b^2}$$

4. A Modification of the Formula for the Average Velocity

The results of the previous sections were obtained using the first degree of approximation to the inverse tangent function. If we now take two terms in the series, and use the approximation

$$\arctan x \cong x-\frac{x^3}{3}$$

for $-1 \leq x \leq 1$, the equality

$$\overline{v}=\frac{\sqrt{GMa}}{c}\arctan\left(\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\right)$$

yields

$$\overline{v}\cong\frac{\sqrt{GMa}}{c}\left[\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}-\left(\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\right)^3\right]$$

and subsequently,

$$\overline{v}=\frac{\sqrt{GMa}}{c}\cdot\left[\frac{c}{b}-\left(\frac{c}{b}\right)^3\right]=\frac{\sqrt{GMa}}{b}\left[1-\left(\frac{c}{b}\right)^2\right]$$

Implying the formula

$$\overline{v}=\frac{2\pi}{T}\frac{a^2}{b}\left[1-\left(\frac{c}{b}\right)^2\right]$$

would yield a more accurate approximation to the average velocity without introducing any additional computational complexities. Of course, if $c \ll b$, then this second-degree approximation will be close to the first-degree approximation obtained in Section 2.

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