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# On the Estimation of the α-μ Channel Signal Fading Distribution Parameters

**Abdel Nasser S.A.A. Hassan, Ahmed M. Gad*, Wafaa M. Ibrahim**

Department of Statistics, Faculty of Economics and Political Science, Cairo University, Egypt
*Corresponding author: ahmed.gad@feps.edu.eg

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**Abstract** Radio channel signals are heavily used tool in telecommunications. A suitable probability distribution is needed to model signals. Many probability distributions have been introduced for this purpose. The α-μ probability distribution is a general channel signal fading model that encompasses many applied important distributions as a special case. This distribution is also known as generalized gamma, Stacy distribution. This distribution is used to describe the fading mobile radio signal under a general diffuse scattering. The main advantage of this probability distribution is that it is flexible and mathematically tractable. Also, many other distributions can be considered as a special case of α-μ probability distribution. In this article we discuss the model parameters' estimation. Two new maximum likelihood (ML) and Psi-inverse (PI) estimators for the α-μ channel signal fading distribution have been proposed. Simulation study is finally conducted to evaluate the performance of the proposed estimators. Simulation results show that the proposed methods perform well comparable to the existing estimators. This behavior is valid for limited sample size; n<1000 or large sample size; n≥1000.

***Keywords:*** *fading radio signals, α-μ distribution, Stacy distribution, gamma distribution, Erlang distribution, chi-squared, Nakagami distribution, size-biased distributions, ML estimators*

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## 1. Introduction

Radio channel signals are very important tool, in the field of telecommunications, to monitor the quality of the mobile signals within a specific range. The signal spread within a specific medium is interrupted by absorption, reflection, diffraction and scattering. The channel behavior has to be described using a suitable probability distribution model. This is enable us to set-up a good communication system. The short term fading that is resulted from a multiple path environments can be modeled using several distributions. The $\alpha-\mu$ distribution channel signal fading (CSF) model has been introduced by [14] as a general short-term fading distribution. This distribution is also known as generalized gamma, Stacy distribution. The channel signal envelope is modeled as a nonlinear function represented by the power parameter $\alpha > 0$, and the parameter $\mu > 0$ is related to the number of multipath components forming the signal. This means that the power parameter $\alpha$ represents the nonlinearity of the environment (propagation medium) and the parameter $\mu$ is related to the number of multipath signal clusters.

Reference [14] pointed out that the $\alpha-\mu$ distribution is flexible and mathematically tractable. Moreover, many other distributions can be considered as special types of the $\alpha-\mu$ distribution including Gamma distribution, Erlang distribution, Central Chi-squared distribution, Nakagami-m distribution, Exponential distribution, Weibull distribution, one-sided Gaussian distribution and Rayleigh distribution. Reference [15] discussed the relationship between the $\alpha-\mu$ distribution and other usable fading models. He also obtained the joint statistics of two for such variates among some other distributional characteristics. Reference [5] obtained an integral expression for the moment generating function of the $\alpha-\mu$ distribution and used it to evaluate the bit error rate of coherent modulation techniques. Reference [6] mentioned that the $\alpha-\mu$ model assumes that the channel radio signal is a composition of clusters of multipath waves propagating in a non-homogeneous environment. In this case the random phases of the scattered waves have similar delay times and the delay time spreads, of different clusters, are relatively large. Reference [3] presented a highly accurate closed form density and cumulative functions for the sum of independent identically distributed (i.i.d), $\alpha-\mu$ variates. They presented some numerical illustrative examples. Reference [1] obtained the maximum likelihood (ML) normal equations of the $\eta-\mu$ distribution parameters. They pointed out that many software packages can be used to solve these normal equations numerically. The asymptotic numerical estimators' variances were also obtained.

The probability density function and the cumulative function of sum of ratios of products and sum of products of independent $\alpha-\mu$ random variables are presented in [12]. Reference [2] introduced an estimator for the $\alpha-k-\mu$ distribution parameters and studied their true parameters' closeness. More applications can be found in [9], [4] and [10].

Now let $X_i$ and $Y_i, i=1,\ldots,\mu$ be a mutually independent Gaussian processes corresponding to the i[th] multipath component with a zero mean and equal variance $\sigma^2$. Define the random variable $R=\sqrt[\alpha]{\sum_{i=1}^{\mu}\left(X_i^2+Y_i^2\right)}$ as the envelope of the sum of the multipath components with the received channel signal. Then the fading signal envelop *R*, probability density function (PDF) has the form;

$$f_\alpha(r;\mu)=\frac{\alpha\,\mu^{\mu}\,r^{\alpha\mu-1}}{q^{\alpha\mu}\Gamma(\mu)}\exp\left(-\mu\frac{r^\alpha}{q^\alpha}\right) \qquad (1)$$

where $r\geq 0$ and $\alpha\geq 0$. The $q=\sqrt[\alpha]{E\left(R^\alpha\right)}=\sqrt[\alpha]{2\sigma^2\mu}$, $\Gamma(x)=\int_0^\infty u^{x-1}\exp(-u)\,du$ and $\mu\geq 0$ is the inverse of the normalized variance of $r^\alpha$. $\mu=\frac{E^2\left(R^\alpha\right)}{E\left(R^\alpha\right)^2-E^2\left(R^\alpha\right)}$, or simply $\mu=\frac{E^2\left(R^\alpha\right)}{V\left(R^\alpha\right)}$. Many probability distributions can be derived from the probability density function in Eq. (1). These include the Weibull distribution if $\mu=1$, the Gamma if $\alpha=1$, Nakagami-m if $\alpha=2$, Rayleigh when $\mu=1,\alpha=2$ and for $\alpha=2$ and $\mu=0.5$, the PDF will be that of the one-sided Gaussian. Reference [14] derived the k-moment for the distribution in Eq. (1) as

$$E\left(R^k\right)=\frac{q^k\Gamma\left(\mu+\frac{k}{\alpha}\right)}{\mu^{k/\alpha}\Gamma(\mu)},\ k=1, \qquad (2)$$

Reference [15] illustrated that the $\alpha-\mu$ channel signal fading distribution is another form of the Stacy (generalized Gamma) distribution. He also obtained the distribution level-crossing rate, average fade duration and some joint distribution characteristics. Reference [3] proposed a highly accurate closed-form approximations for the sum of i.i.d $\alpha-\mu$ random variables PDF and CDF. It is worth noting here that the model in (1), can be seen as the probability density of the random variable $v=R^\alpha$ not of *R*, as the model is presented totally in terms of $R^\alpha$.

The aim of this article is to propose two new estimators; the maximum likelihood (ML) and Psi-inverse (PI) estimators, for the $\alpha-\mu$ channel signal fading distribution. The performance of these two proposed estimators are discussed and compared with the existing ones through numerical simulations. The rest of the article is organized as follows. Section 2 is devoted to the estimation of the $\alpha-\mu$ channel signal fading distribution parameters. In section 2.1 we present two available estimation methods namely, the moment method (MM) and skewness logarithmic moment (SL) estimators. Sections 2.2 and 2.3 are devoted to the two new proposed estimators. The first are the ML estimators introduced in Section 2.2. Section 2.3 presented the second new set of estimators called the psi inverse (PI). In Section 3 a simulation study is presented to evaluate the proposed methods. Section 3.1, contains the small, moderate to large sample size performance and the very large sample size is discussed in Section 3.2.

## 2. Model Parameters' Estimation

Choosing the system behavioral model, up to and including its characteristic parameters, is the first step to design a controllable channel communication system. In this case the formula used to estimate the model parameters efficiently is the main challenge. This is can be done depending on data set. One of the oldest concepts in statistical science is the estimation techniques.

### 2.1. The Method of Moments (MM)

The sum of independent, possibly non-identical, lognormal random variables of $\alpha-\mu$ random variable in Eq. (1) are approximated by [12]. The sum of these lognormal random variables is used to evaluate an approximate MM estimators and a non-linear PDF least square estimators.

Reference [2] suggested MLE for the $\alpha-K-\mu$ fading distribution using the so called Smith spectrum sampling generation and solving its normal equations numerically. They also discussed confidence interval for the single parameter of such distribution.

Reference [15] used the concept of the MM and numerically obtained its estimators of the two parameters $\alpha$ *and* $\mu$ of the distribution in Eq. (1). Reference [14] started with the measurable parameter $\vartheta_k$ $k=1$, which is defined as

$$\theta_k=\frac{E^2\left(R^k\right)}{E\left(R^{2k}\right)-E^2\left(R^k\right)} \qquad (3)$$

It can be easily seen that replacing $k$ with $\alpha$ in Eq. (1) gives $\theta_k=\mu$. Depending on Eq. (2) the expression of $\theta_k$ in Eq. (3) can be written as

$$\theta_k=\frac{\Gamma^2\left(\mu+\frac{k}{\alpha}\right)}{\Gamma(\mu)\Gamma\left(\mu+\frac{2k}{\alpha}\right)-\Gamma^2\left(\mu+\frac{k}{\alpha}\right)},\ k=1, \qquad (4)$$

The first two theoretical measurable parameters $\theta_{k1}$ *and* $\theta_{k2}$, according to the MM concept, are equated with the corresponding sample counterparts. So, we have

$$\theta_{k1} = \frac{\Gamma^2\left(\hat{\mu}_m + \frac{k_1}{\hat{\alpha}m}\right)}{\Gamma(\hat{\mu}m)\Gamma\left(\hat{\mu}m + \frac{2k_1}{\hat{\alpha}m}\right) - \Gamma^2\left(\hat{\mu}_m + \frac{k_1}{\hat{\alpha}m}\right)} = \frac{\left(\sum_{i=1}^{n} r_i^{k_1} \Big/ n\right)^2}{\left(\sum_{i=1}^{n} r_i^{2k_1} \Big/ n\right) - \left(\sum_{i=1}^{n} r_i^{k_1} \Big/ n\right)^2} \tag{5}$$

and

$$\theta_{k2} = \frac{\Gamma^2\left(\hat{\mu}_m + \frac{k_2}{\hat{\alpha}m}\right)}{\Gamma(\hat{\mu}m)\Gamma\left(\hat{\mu}m + \frac{2k_2}{\hat{\alpha}m}\right) - \Gamma^2\left(\hat{\mu}_m + \frac{k_2}{\hat{\alpha}m}\right)} = \frac{\left(\sum_{i=1}^{n} r_i^{k_2} \Big/ n\right)^2}{\left(\sum_{i=1}^{n} r_i^{2k_2} \Big/ n\right) - \left(\sum_{i=1}^{n} r_i^{k_2} \Big/ n\right)^2}, \tag{6}$$

where $\hat{\alpha}_m$ *and* $\hat{\mu}_m$ are the MM estimators for the parameters $\alpha$ *and* $\mu$, respectively. The equations (5) and (6) are solved numerically. The values of $k_1$ *and* $k_2$ for $\theta_k$ need to be chosen to conduct numerical solution.

Reference [8] introduced an MM estimators for the parameters $\alpha$ *and* $\mu$, based on the logarithmic $\alpha-\mu$ random variable, namely SL estimators. Assume that $Y = K\ln(R)$, where $R$ is the random variable in Eq. (1) and then based on the MM estimator $\hat{\alpha}m$ of the parameter $\alpha$, the statistic $\hat{\theta}_N \hat{\alpha}m$ can be estimated by

$$\hat{\theta}_N(\hat{\alpha}m) = \left(\sum_{i=1}^{n} r_i^{\hat{\alpha}m} \Big/ n\right)^{\frac{1}{\hat{\alpha}m}} \tag{7}$$

Also, define the estimator

$$\hat{\eta} = \frac{\hat{\mu}_2^{\frac{3}{2}}}{\hat{\mu}_3} = \frac{\left\{\frac{1}{n}\sum_{i=1}^{n}\left(y_i - \sum_{i=1}^{n} y_i \Big/ n\right)^2\right\}^{\frac{3}{2}}}{\frac{1}{n}\sum_{i=1}^{n}\left(y_i - \sum_{i=1}^{n} y_i \Big/ n\right)^3}, \tag{8}$$

where $y_1, y_2, \ldots, y_n$ is a simple random sample of size $n$ from the distribution of the random variable $Y = \ln(R)$, and the constant $K = 20/\ln(10)$.

Using the second and third theoretical moments of the random variable $K\ln(R)$ that are obtained in equations (11) and (12) of [8] and Eq. (8), we have SL as follows;

$$\hat{\eta} = \frac{\{\varphi'(\hat{\mu}_{SL})\}^{\frac{3}{2}}}{\varphi''(\hat{\mu}_{SL})}, \tag{9}$$

where $\varphi(x) = \frac{\partial \ln\Gamma(x)}{\partial x}$, $\varphi'(x) = \frac{\partial^2 \ln\Gamma(x)}{\partial x^2}$, and $\varphi''(x) = \frac{\partial^3 \ln\Gamma(x)}{\partial x^3}$, are the Psi-function (Digamma), Trigamma function and Tetragamma function respectively. Reference [8] solved Eq. (9) numerically for $\hat{\mu}_{SL}$ using the least squares method. They gave the following least squares approximation;

$$\hat{\mu}_{SL} = \begin{cases} \hat{\eta}^2 + 0.5 & \text{if } \hat{\eta} \le -2.85 \\ \begin{pmatrix} -0.0773\hat{\eta}^4 - 0.6046\hat{\eta}^3 \\ -0.7949\hat{\eta}^2 - 2.4675\hat{\eta} - 0.9208 \end{pmatrix} & \text{if } -2.85 < \hat{\eta} \le -0.6 \\ \begin{pmatrix} -132.8995\hat{\eta}^3 - 232.0659\hat{\eta}^2 \\ -137.6303\hat{\eta} - 27.3616 \end{pmatrix} & \text{if } -0.6 < \hat{\eta} < -0.5 \end{cases}$$

where, $\hat{\mu}_{SL} \downarrow 0$ *as* $\hat{\mu} \uparrow -0.5$.

Using the resulted estimate of Eq. (9), they recalculated a new estimate for $\alpha$;

$$\hat{\alpha}_{SL} = K\sqrt{\frac{\varphi'(\hat{\mu}_{SL})}{\hat{\mu}_2}} \tag{10}$$

where $\hat{\mu}_2$ is the estimator of the second central moment of the logarithmic $\alpha-\mu$ random variable of Eq. (1), given by

$$\hat{\mu}_2 = \frac{K^2}{n}\sum_{i=1}^{n}\left(y_i - \sum_{i=1}^{n} y_i \Big/ n\right)^2 \tag{11}$$

Reference [8] conducted a numerical comparison between the MM estimators suggested by [14] and the skewness logarithmic transformation estimators. They reported that both the MM and SL estimators are slightly biased, but the SL perform better. The comparison criteria was the normalized mean square error (NMSE) defined by;

$$NMSE(\hat{\zeta}) = \frac{1}{M}\sum_{i=1}^{M}\frac{(\hat{\zeta}_i - \zeta)^2}{\zeta^2}, \tag{12}$$

where $\hat{\zeta}_i, i = 1, 2, \ldots, M$ are the $i^{th}$ simulation trial estimate for the parameter $\zeta$ and $M$ are the simulation number of trails.

Reference [6] suggested an empirical procedure for estimating the parameters $\alpha$ *and* $\mu$ using the fact that $\theta_2 = 1/S_4^2$, where $\theta_k, k = 1, 2, \ldots$ is defined by Eq. (4). The amplitude index $S_4$, given by

$$S_4 = \sqrt{\frac{\langle I^2 \rangle - \langle I \rangle^2}{\langle I \rangle^2}} \quad (13)$$

where $I = |R|^2$, is the intensity, $R$ is the envelope random variable of Eq. (1) and $\langle . \rangle$ is the ensemble average. Note that the equation $\theta_2 = 1/S_4$ is in two unknown parameters $\alpha$ *and* $\mu$, they empirically searched for the estimates of $\alpha$ *and* $\mu$, say $\hat{\alpha}_E$ *and* $\hat{\mu}_E$ that produce the best PDF fit for the data. Reference [6] based on empirical study proposed a third degree polynomial approximation using the least square technique for the relation between the estimated $\alpha$ and the value of $0.3 \le S_4 \le 1$;

$$\hat{\alpha}_1 = -17.649 S_4^3 + 39.109 S_4^2 - 27.8218 S_4 + 7.498 \quad (14)$$

and $\hat{\alpha}_1 = 1/\log(10 S_4)$ for $S_4 > 1$. Using simple data set [6] studied the performance of the above approximations which produced a very close values compared to the real parameters.

## 2.2. The New Proposed Estimators

In this section two new estimators for the $\alpha - \mu$ distribution in Eq. (1) are discussed. These estimators are the ML estimator and the PI estimator. The estimators are derived via approximation of the resulted equations, since these equations cannot be presented in a closed form. The second proposed estimator; the PI estimator, is based on the cumulants of the random variable in hand, R. In Section 3, we show that our new method PI outperform all other existing methods in estimating the $\alpha - \mu$ distribution parameters.

### 2.2.1. The Maximum Likelihood Method (ML)

In the above the $\alpha - \mu$ distribution parameters' estimation problem is mainly handled using the MM. The maximum likelihood estimators (MLE's) are discussed below. Reference [1] discussed the ML estimators of the $\eta - \mu$ distribution which is another channel fading distribution introduced by [13]. They pointed out that the ML estimators can be obtained using the general maximization technique and these estimators will have the general MLE's asymptotic properties. In their simulation study they compared between two distribution formats based on the true parameter values. It can be seen from [1], that the derivation of the ML estimators of such distributions is difficult and only reachable through numerical methods. The ML estimators for the model in Eq. (1) are derived below. Also, an approximate form for these estimators are presented.

Let $r_1, r_2, \ldots, r_n$ be a simple random sample from the $\alpha - \mu$ distribution of Eq. (1), then the joint PDF of the sample is given by;

$$f(r_1, \ldots, r_n; \alpha, \mu) = \left( \frac{\alpha \mu^{\mu}}{q^{\alpha\mu} \Gamma(\mu)} \right)^n \left( \prod_{i=1}^{n} r_i \right)^{\alpha\mu - 1} \exp\left( -\frac{\mu}{q^{\alpha}} \sum_{i=1}^{n} r_i^{\alpha} \right),$$

Where $r_i > 0, i = 1, 2, \ldots, n$. Taking the natural log for the joint PDF above, we get

$$L(\alpha, \mu) = n \ln(\alpha) + n\mu \ln(\mu) - n\mu \ln q^{\alpha} - n \ln \Gamma(\mu) + (\alpha\mu - 1) \sum_{i=1}^{n} \ln(r_i) - \mu q^{-\alpha} \sum_{i=1}^{n} r_i^{\alpha}, \quad (15)$$

where $q^{\alpha} = E(R^{\alpha})$ as in Eq. (1). The bivariate function in Eq. (15) is very complicated with respect to differentiation especially for the parameter $\alpha$. Thus using rough approximations for the two terms $q^{\alpha} = E(R^{\alpha})$ and $\ln q^{\alpha}$, Eq. (15) may be written as

$$L(\alpha, \mu) \approx n \ln(\alpha) + n\mu \ln(\mu) - n\mu\alpha B - n \ln \Gamma(\mu) + (\alpha\mu - 1) \sum_{i=1}^{n} y_i - \mu \sum_{i=1}^{n} e^{(y_i - B)\alpha}, \quad (16)$$

Where $Y_i = \ln(R_i), i = 1, 2, \ldots, n$ and $E(Y) \approx B = \bar{Y}$. The above (16), may then be simplified as;

$$L(\alpha, \mu) \approx n \ln(\alpha) + n\mu \ln(\mu) - n \ln \Gamma(\mu) + \bar{Y} - \mu \sum_{i=1}^{n} e^{(y_i - B)\alpha}, \quad (17)$$

Differentiating Eq. (17) with respect to the two unknown parameters $\alpha$ *and* $\mu$; we have $\frac{\partial L(\alpha, \mu)}{\partial \alpha} = n\alpha^{-1} - \mu \sum_{i=1}^{n} e^{\alpha z_i} z_i$

and

$$\frac{\partial L(\alpha, \mu)}{\partial \alpha} = n + n \ln(\mu) - n\varphi(\mu) - \sum_{i=1}^{n} e^{\alpha z_i},$$

where $\varphi(x) = \frac{\partial \ln \Gamma(x)}{\partial(x)}$ is the psi (Digamma) function and $z_i = y_i - \bar{y}, i = 1, 2, \ldots, n.$ Equating the two above partial differentiations with zero, we get

$$\hat{\mu}_L \approx \frac{n}{\hat{\alpha}_L \sum_{i=1}^{n} \left( z_i e^{\hat{\alpha}_L z_i} \right)} \quad (18)$$

and

$$1 + \ln(\hat{\mu}_L) = \varphi(\hat{\mu}_L) - \frac{1}{n} \sum_{i=1}^{n} \left( z_i e^{\hat{\alpha}_L z_i} \right) = 0$$

$$\left\{ \begin{array}{l} 1 + \ln\left( n \Big/ \hat{\alpha}_L \sum_{i=1}^{n} \left( z_i e^{\hat{\alpha}_L z_i} \right) \right) \\ -\varphi\left( n \Big/ \hat{\alpha}_L \sum_{i=1}^{n} \left( z_i e^{\hat{\alpha}_L z_i} \right) \right) - \frac{1}{n} \sum_{i=1}^{n} \left( z_i e^{\hat{\alpha}_L z_i} \right) \end{array} \right\} = 0 \quad (19)$$

where $\hat{\alpha}_L$ *and* $\hat{\mu}_L$ are the MLE's of the parameters $\alpha$ *and* $\mu$ respectively.

Solving equation (19) iteratively for $\hat{\alpha}_L$ then we use the estimated value of $\hat{\alpha}_L$ of Eq. (19) in Eq. (18) we calculate $\hat{\mu}_L$.

### 2.2.2. The Psi-Inverse (PI) Method

Reference [7] used the cumulants of the beta random variable to estimate the beta distribution parameters. They called the method (estimators), Psi-inverse method (estimators). Here we use their method to estimate the parameters of the model in Eq. (1). Consider the random variable $Y = \ln(R)$ defined in the ML method above, then the moment generating function (MGF) of $Y$ is given by;

$$\begin{aligned} m_Y(t) &= E\left(e^{tY}\right) = E\left(e^{t\ln(R)}\right) \\ &= E\left(e^{\ln\left(R^t\right)}\right) = \frac{q^t\Gamma(\mu + t/\alpha)}{\Gamma(\mu)\mu^{t/\alpha}} \end{aligned} \tag{20}$$

It is known that the cumulant function (CF) of the random variable Y is defined as, $K_Y(t) = \ln\{m_Y(t)\}$, i.e.

$$\begin{aligned} K_Y(t) &= \frac{t}{\alpha}q^{\alpha} + \ln\{\Gamma(\mu + t/\alpha)\} \\ &- \frac{t}{\alpha}\ln(\mu) - \ln\{\Gamma(\mu)\} \end{aligned} \tag{21}$$

Using the first and second differentiations of $K_Y(t)$ with respect to t, then plug in $t = 0$, we have

$$\begin{aligned} K_Y'(t)\big|_{t=0} &= E(Y) \\ &= \left\{\ln\left(q^{\alpha}\right) + \varphi(\mu) - \ln(\mu)\right\}\Big/\alpha, \end{aligned} \tag{22}$$

and

$$K_Y''(t)\big|_{t=0} = \sigma_Y^2 = \varphi'(\mu)\big/\alpha^2. \tag{23}$$

Using a random sample $r_1, r_2, ..., r_n$ from the $\alpha - \mu$ distribution of (1), calculating the random values, $y_i, i = 1, 2, ..., n$ and the two equations (22) and (23) we get

$$\bar{Y} = \left\{\begin{matrix} \ln\left[E\left(R^{\alpha_\varphi}\right)\right] \\ +\varphi\left(\mu_\varphi\right) - \ln\left(\mu_\varphi\right) \end{matrix}\right\}\Bigg/\alpha_\varphi, \tag{24}$$

and

$$S_Y^2 = \varphi'\left(\mu_\varphi\right)\big/\alpha_\varphi^2, \tag{25}$$

where $\bar{Y}$ and $S_Y^2$ are the Y-sample mean and variance.

Equation (25) gives $\mu_\varphi = \varphi'^{-1}\left(\alpha_\varphi^2 S_Y^2\right)$ and substituting this in equation (24);

$$\begin{aligned} \bar{Y} &= \left\{\begin{matrix} \ln\left[E\left(R^{\alpha_\varphi}\right)\right] + \varphi\left[\varphi'^{-1}\left(\alpha_\varphi^2 S_Y^2\right)\right] \\ -\ln\varphi'^{-1}\left(\alpha_\varphi^2 S_Y^2\right) \end{matrix}\right\}\Bigg/\alpha_\varphi \\ &\approx \left\{\begin{matrix} \ln\left(\dfrac{\sum_{i=1}^{n} e^{Y_i\alpha_\varphi}}{n}\right) + \varphi\left[\varphi'^{-1}\left(\alpha_\varphi^2 S_Y^2\right)\right] \\ -\ln\varphi'^{-1}\left(\alpha_\varphi^2 S_Y^2\right) \end{matrix}\right\}\Bigg/\alpha_\varphi \end{aligned} \tag{26}$$

The $\varphi^{-1}(.)$ estimates $\alpha_\varphi$ *and* $\mu_\varphi$ of Eq. (25) and Eq. (26) can be obtained by using any simple computer program package available for $\varphi'^{-1}(.)$, as the one given by MTLAB or R-package.

Alternatively, the following simple approximation of $\varphi'^{-1}(.)$, derived by [11] can be used,

$$\varphi'^{-1}(\theta) = \begin{cases} \left(g(\theta) - 1\right)^{-1} & 0 < \theta < \dfrac{\pi^2}{6} \\ \exp\left(\begin{matrix} 0.321 - 0.673\ln(\theta) \\ +0.025\ln^2(\theta) \end{matrix}\right) & \dfrac{\pi^2}{6} \le \theta < 40 \\ \left(\theta - \dfrac{\pi^2}{6}\right)^{-1} & \theta \ge 40, \end{cases} \tag{27}$$

where

$$\begin{aligned} g(\theta) &= \left(2 + 3\theta + \sqrt{(2+3\theta)^2 + 1}\right)^{\frac{1}{3}} \\ &+ \left(2 + 3\theta - \sqrt{(2+3\theta)^2 + 1}\right)^{\frac{1}{3}} \end{aligned}$$

## 3. Simulation Studies

Two simulation studies have been conducted. The aim of the first simulation study is to evaluate the performance and applicability of the two new proposed estimators, for relatively small sample sizes that are common in statistical applications. The second simulation study compares our work with [8] using large sample sizes that are common in telecommunications. The codes for simulation are written using R-package.

### 3.1. The First Simulation Study

The aim of this simulation study is to assess the performance and applicability of the two new proposed estimators; the PI and the ML estimators. These two new estimators are compared to the existing estimators; the MM of [14], and the SL of [8]. Limited number of observations is very common in statistical applications;

sample size below 500 observations. So, in this first simulation study the focus is on these limited samples; $1 < n < 500$. The sample sizes are chosen as 20, 50, 100 and 500 to cover small, moderate and large samples. The normalized mean square errors (NMSE's) as in Eq. (12) is used as a comparison criteria. They are obtained for each estimator. The smaller the Normalized mean square error (NMSE) the better the estimator.

**Simulation Setting**

Samples of size $n$ are generated from the $\alpha-\mu$ distribution in Eq. (1). The generation process is conducted using two different methods. The first method is to generate two mutually independent Gaussian random variables with mean zero and variance one; $X_i$ and $Y_i, i = 1, \ldots \mu$. Then, define a random draw from the $\alpha-\mu$ distribution, $R$, as $R = \sqrt[\alpha]{\sum_{i=1}^{\mu}\left(X_i^2 + Y_i^2\right)}$. The process is repeated $m$ times to obtain the required sample size. The second method is to simulate a random sample of size $n$ from Gamma distribution $G_i = Gamma(\mu,1)$, $i = 1,2,...,n$ and then $R_i = \sqrt[\alpha]{G_i}, i = 1,2,...,n$ as in [8]. This means that the comparison study has been conducted twice depending on the generation method.

The parameters $\mu$ and $\alpha$ are chosen as $\mu = 0.7,1, 1.3,1.8,1.9,4.3,5,10,15,30,50$ and $\alpha = 0.8,1,1.1,1.5,1.6, 2,2.2,2.5,3,$ 3.9. Different combinations of these parameters $\alpha$ and $\mu$ are used to generate Normal and Gamma variables. The sample sizes are chosen as $n = 20,50,100$ and 500 to cover small, and moderate sample size. The samples are simulated depending on the different combinations of the two parameters $\mu$ and $\alpha$. The number of replications is fixed at 100000 replications. For each of the 100000 replications, the two parameters $\mu$ and $\alpha$ are estimated using all four methods of estimation; the MM, the SL, the ML and the PI. More combinations of the parameters have been tried; $\mu = 5,10$ and $\alpha = 1.5$, but the results are not reported because they are similar to the reported results.

**Simulation results**

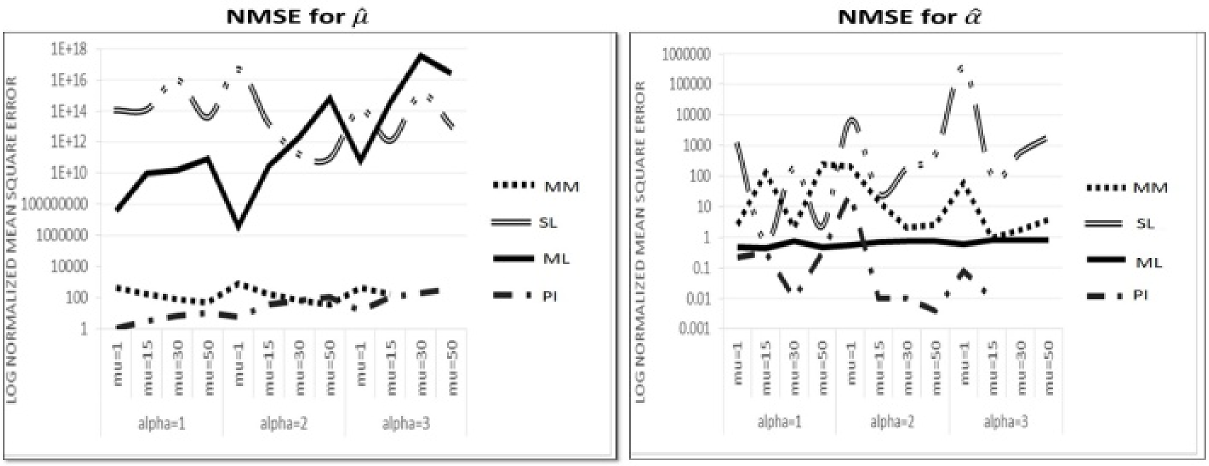


**Figure 1.** Log Normalized mean square error (NMSE) of $\hat{\mu}$ and $\hat{\alpha}$ at n=20, and 100000 replications when simulating from Gaussian distribution.

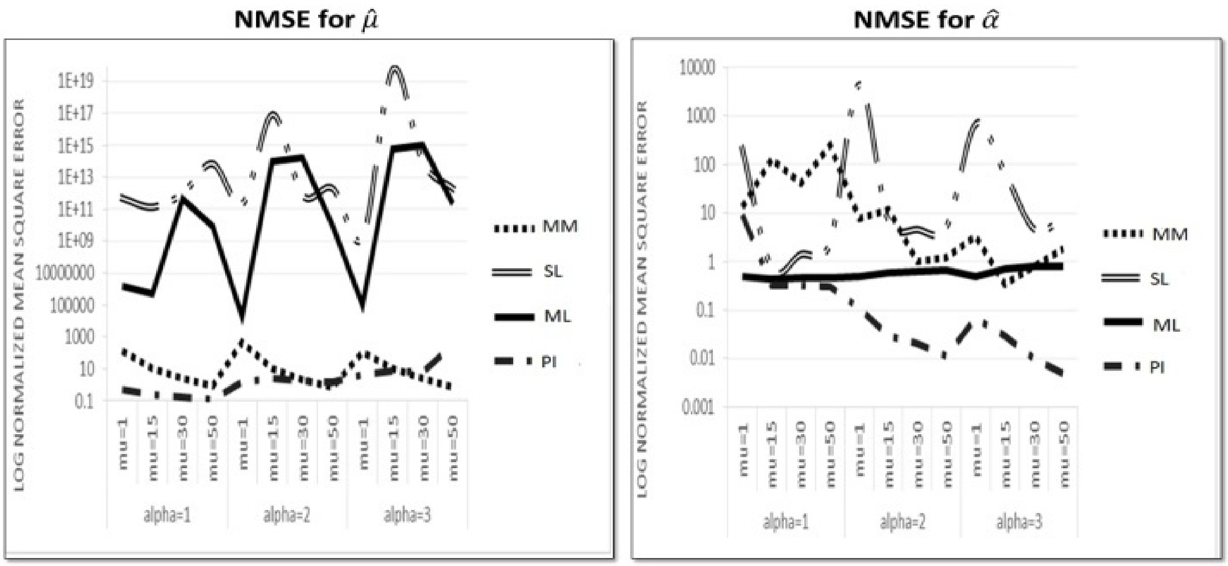


**Figure 2.** Log Normalized mean square error (NMSE) of $\hat{\mu}$ and $\hat{\alpha}$ at n=50, and 100000 replications when simulating from Gaussian distribution.

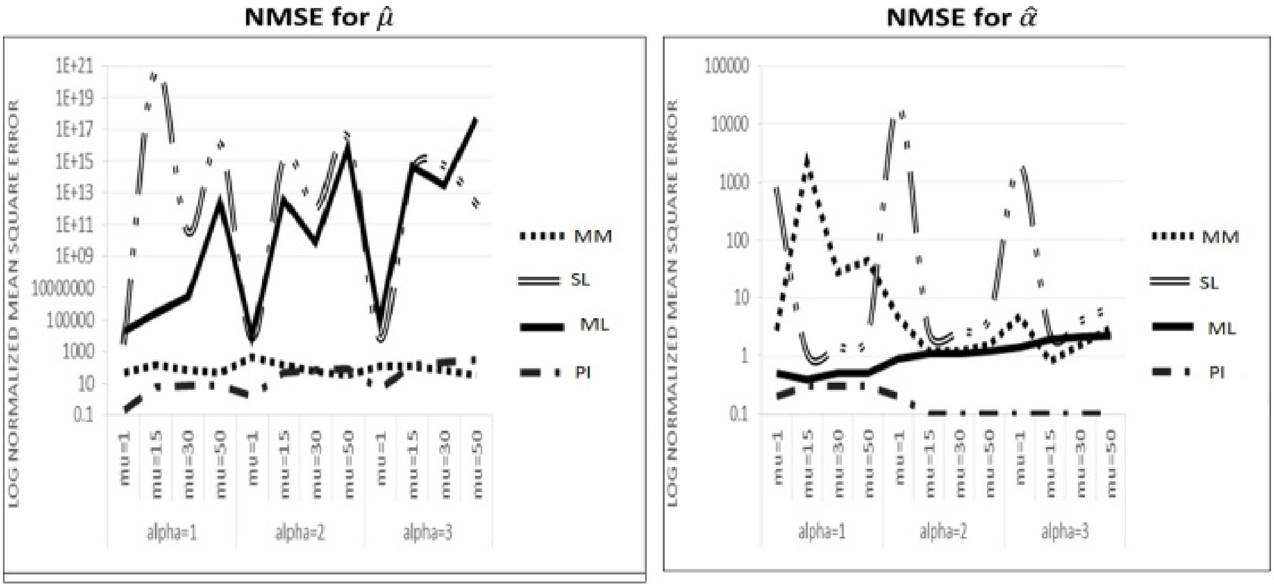


**Figure 3.** Log Normalized mean square error (NMSE) of $\hat{\mu}$ and $\hat{\alpha}$ at n=100, and 100000 replications when simulating from Gaussian distribution.

The simulation results are shown in Figure 1 - 3 and and Table 1 - 8. Depending on the Normalized mean square error (NMSE) different estimators have been compared. The smaller the Normalized mean square error (NMSE) the better the estimator. The smallest values of the Normalized mean square error (NMSE) in each case are in **bold**. From the results it is obvious that the new proposed PI method is superior, by a significant factor for the two parameters, to the other three methods. The PI method is performing much better than the other three methods, followed by the MM then by ML for estimating $\mu$ and by ML then the MM for estimating $\alpha$. However, the SL estimator of [8] performs poorly due to the sample size limitation, $n < 500$. It is anticipated that this estimator will improve as the sample size increases, for $n \geq 1000$. The generation method of the variables does not affect performance of the results. That is the seed variable generation, Gaussian or Gamma, does not affect the estimators' performance. The simulation study also indicates that the general NMSE values decreased with the increase of the sample size. This means that as the sample size increases the performance of estimators gets better.

**Table 1. Normalized mean square error (NMSE) of $\hat{\mu}$ and $\hat{\alpha}$ at n=500, and 100000 replications simulating from Gaussian distribution**

| | $\mu$ | 1 | | 15 | | 30 | | 50 | |
|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | Method | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ |
| 1 | MM | 8.36471 | 0.362551 | 3.60697 | 0.21049 | 1.3302 | 0.31081 | 0.49361 | 0.4663 |
| | SL | 267.785 | 128.3681 | 9.8E+10 | 0.18301 | 7.E+15 | 0.31073 | 5.9E+13 | 0.4633 |
| | ML | 36.6750 | 0.452851 | 5564.89 | 0.39354 | 6529.7 | 0.40199 | 5635.27 | 0.4042 |
| | PI | 0.02836 | 0.031491 | 0.31966 | 0.15764 | 0.4372 | 0.24130 | 0.42180 | 0.2820 |
| 2 | MM | 124.125 | 0.832607 | 3.33824 | 0.16815 | 1.2469 | 0.29557 | 0.3423 | 0.3458 |
| | SL | 274.586 | 217.0968 | 1.1E+09 | 0.18316 | 8E+13 | 0.30902 | 6.9E+13 | 0.4616 |
| | ML | 36.5967 | 0.452596 | 5558.07 | 0.39430 | 2E+08 | 0.40703 | 4.5E+08 | 0.4271 |
| | PI | 0.58879 | 0.030525 | 2.88114 | 0.04674 | 3.4109 | 0.05341 | 3.32836 | 0.0522 |
| 3 | MM | 91.7316 | 1.585671 | 3.12517 | 0.16611 | 0.2206 | 0.19973 | 0.22272 | 0.3428 |
| | SL | 281.605 | 336.3533 | 1.1E+09 | 0.18316 | 8E+13 | 0.30902 | 6.9E+13 | 0.4616 |
| | ML | 36.6030 | 0.452834 | 4.7E+12 | 0.44140 | 1E+13 | 0.51672 | 2.4E+13 | 0.5708 |
| | PI | 2.12475 | 0.010433 | 8.50622 | 0.04237 | 9.9583 | 0.05033 | 9.84479 | 0.0497 |

**Table 2. Normalized mean square error (NMSE) of $\hat{\mu}$ and $\hat{\alpha}$ at n=20, and 100000 replications simulating from Gamma distribution**

| | $\mu$ | 1 | | 15 | | 30 | | 50 | |
|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | Method | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ |
| 1 | MM | 398.02 | 1316.86 | 11.51746 | 127 | 2.7232 | 40.726 | 0.92094 | 246.661 |
| | SL | 1.E+14 | 292.2162 | 6.8E+12 | 0.73 | 3E+14 | 1.378 | 6.16E+11 | 2.28 |
| | ML | 375360 | 0.557534 | 6620000 | 0.44 | 416666 | 0.452 | 1.62E+09 | 0.469 |
| | PI | 1.0110 | 598.0757 | 0.216666 | 0.33 | 0.2058 | 0.32 | 0.212665 | 0.31 |
| 2 | MM | 499.63 | 212.378 | 11.36293 | 204.51 | 2.3748 | 2.19 | 0.737176 | 2.665 |
| | SL | 2E+11 | 541.038 | 7.73E+11 | 145.36 | 566666 | 9596.35 | 1.75E+09 | 300.380 |
| | ML | 20083 | 0.559 | 1806666 | 0.68 | 7E+10 | 0.73 | 1.22E+13 | 0.756 |
| | PI | 1.5175 | 150.873 | 2.249553 | 0.01 | 2.1222 | 0.01 | 2.133398 | 0.004 |
| 3 | MM | 120.08 | 57.355 | 11.60886 | 1.06 | 2.5130 | 1.80 | 0.753845 | 3.698 |
| | SL | 1E+09 | 520.710 | 7.73E+10 | 8972.23 | 4E+13 | 31018.27 | 2.22E+11 | 44.655 |
| | ML | 110317 | 0.575 | 2.43E+13 | 0.80 | 1E+16 | 0.82 | 5.06E+14 | 0.828 |
| | PI | 4.5145 | 0.113 | 7.3428 | 0.01 | 6.9736 | 0.00 | 299.822 | 0.000 |

**Table 3. Normalized mean square error (NMSE) of $\hat{\mu}$ and $\hat{\alpha}$ at n=20, and 100000 replications simulating from Gamma distribution using non-integer values for α and μ.**

| Method | α=1.1, μ=1.3 | | α=1.6, μ=1.3 | | α=2.2, μ=1.8 | |
|---|---|---|---|---|---|---|
| | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ |
| MM | 276.5705 | 1047.634 | 441.3752 | 240.5957 | 125.6602 | 45.29593 |
| SL | 8818525 | 1.438052 | 1332810 | 1.296705 | 5E+08 | 5.507658 |
| ML | 39198392 | 0.550133 | 27118245 | 0.540466 | 37823757 | 0.541258 |
| PI | 1.542043 | 0.93611 | 2.829862 | 0.236036 | 5.50375 | 0.059498 |
| Method | α=0.8, μ=4.3 | | α=3.9, μ=0.7 | | α=2.5, μ=1.9 | |
| | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ |
| MM | 498.2039 | 17.14023 | 605.8959 | 83.4929 | 95.28845 | 14.24005 |
| SL | 7592426 | 4.408869 | 10383.55 | 0.99685 | 85395188 | 4.438051 |
| ML | 1.14E+09 | 0.65136 | 4.43E+08 | 0.603757 | 32712704 | 0.555202 |
| PI | 25.50683 | 0.076529 | 22.24618 | 0.09391 | 3.664399 | 0.069466 |

**Table 4. Normalized mean square error (NMSE) of $\hat{\mu}$ and $\hat{\alpha}$ at n=50, and 100000 replications simulating from Gamma distribution**

| | $\mu$ | 1 | | 15 | | 30 | | 50 | |
|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | Method | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ |
| 1 | MM | 121.17 | 18.47027 | 10.88361 | 127 | 2.7424853 | 40.726 | 0.918987 | 246.661 |
| | SL | 1.E+09 | 145.532 | 2.36E+17 | 0.73 | 2.216E+11 | 1.378 | 1.37E+12 | 2.28 |
| | ML | 155579 | 0.502578 | 131793.7 | 0.44 | 133666666 | 0.452 | 1.63E+10 | 0.469 |
| | PI | 0.5221 | 23.13945 | 0.261261 | 0.33 | 0.1577809 | 0.32 | 0.12425 | 0.31 |
| 2 | MM | 499.63 | 7.601 | 10.48987 | 95.41 | 2.234092 | 0.95 | 0.693044 | 1.186 |
| | SL | 2E+11 | 29249.055 | 1.17E+13 | 4.77 | 2.97E+13 | 4.99 | 3.84E+15 | 5.159 |
| | ML | 20083 | 0.503 | 1.78E+16 | 0.57 | 4.666E+10 | 0.64 | 1.12E+15 | 0.680 |
| | PI | 1.5175 | 0.109 | 2.355422 | 0.03 | 1.717 | 0.02 | 1.459747 | 0.011 |
| 3 | MM | 120.08 | 3.157 | 10.09518 | 0.35 | 2.353828 | 0.78 | 0.708074 | 1.903 |
| | SL | 1E+09 | 425.247 | 3.43E+12 | 2.34 | 8.06E+10 | 4.62 | 5.84E+17 | 5.859 |
| | ML | 110317 | 0.504 | 4.79E+13 | 0.72 | 1.09E+11 | 0.77 | 5.44E+12 | 0.783 |
| | PI | 4.5145 | 0.069 | 7.44474 | 0.03 | 5.9748066 | 0.01 | 299.822 | 0.005 |

**Table 5. Normalized mean square error (NMSE) of $\hat{\mu}$ and $\hat{\alpha}$ at n=50, and 100000 replications simulating from Gamma distribution using non-integer values for α and μ.**

| Method | α=1.1, μ=1.3 | | α=1.6, μ=1.3 | | **α=2.2, μ=1.8** | |
|---|---|---|---|---|---|---|
| | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ |
| MM | 363.3865 | 930.2021 | 360.9082 | 250.7539 | 81.98651 | 1.940368 |
| SL | 3095688 | 3.736328 | 30200278 | 3.131442 | 76147.12 | 8.821595 |
| ML | 29743080 | 0.524846 | 34234187 | 0.559461 | 2489811 | 0.478868 |
| PI | 1.097822 | 18.32337 | 3.137915 | 0.245365 | **4.707902** | **0.066347** |
| Method | α=0.8, μ=4.3 | | α=3.9, μ=0.7 | | **α=2.5, μ=1.9** | |
| | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ |
| MM | 30.44225 | 6.126682 | 319.6473 | 2.298922 | 76.93467 | 0.600999 |
| SL | 4.91E+08 | 4.484801 | 3205.069 | 16.83392 | 342200.6 | 62.84481 |
| ML | 1647324 | 0.448978 | 62068635 | 0.540921 | 990993.5 | 0.473275 |
| PI | 0.321248 | 0.481059 | 10.13117 | 0.059549 | **5.990333** | **0.063457** |

**Table 6. Normalized mean square error (NMSE) of $\hat{\mu}$ and $\hat{\alpha}$ at n=100, and 100000 replications simulating from Gamma distribution.**

| | $\mu$ | 1 | | 15 | | 30 | | 50 | |
|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | Method | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ |
| 1 | MM | 50.02322 | 2.786005 | 9.19013 | 127 | 2.52678 | 40.7 | 0.8764 | 246.6 |
| | SL | 4552.653 | 707.5228 | 3.4E+19 | 0.73 | 1066666 | 1.37 | 2.E+14 | 2.28 |
| | ML | 20287.81 | 0.476596 | 18656.8 | 0.44 | 83955.2 | 0.45 | 4E+10 | 0.469 |
| | PI | 0.208247 | 0.216498 | 0.38769 | 0.33 | 0.23576 | 0.32 | 0.159 | 0.31 |
| 2 | MM | 499.6382 | 2.301 | 9.53033 | 9.49 | 2.10215 | 0.61 | 0.6482 | 0.793 |
| | SL | 2.23E+11 | 10312.51 | 8.9E+13 | 0.75 | 3.3E+10 | 1.32 | 5E+14 | 2.251 |
| | ML | 20083.27 | 0.476 | 2.4E+11 | 0.49 | 2563333 | 0.56 | 1E+14 | 0.601 |
| | PI | 1.517595 | 0.084 | 3.0782 | 0.05 | 2.15528 | 0.03 | 1.7159 | 0.021 |
| 3 | MM | 120.0871 | 1.632 | 8.5054 | 0.26 | 2.16951 | 0.53 | 0.6631 | 1.019 |
| | SL | 1.66E+09 | 653.520 | 1.4E+13 | 0.77 | 1.6E+13 | 1.34 | 3E+10 | 2.274 |
| | ML | 110317.1 | 0.476 | 3.16E13 | 0.64 | 1E+12 | 0.71 | 1E+16 | 0.736 |
| | PI | 4.514539 | 0.045 | 9.16833 | 0.04 | 7.08238 | 0.03 | 299.82 | 0.017 |

**Table 7. Normalized mean square error (NMSE) of $\hat{\mu}$ and $\hat{\alpha}$ at n=100, and 100000 replications simulating from Gamma distribution using non-integer values for α and μ.**

| Method | α=1.1, μ=1.3 | | α=1.6, μ=1.3 | | α=2.2, μ=1.8 | |
|---|---|---|---|---|---|---|
| | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ |
| MM | 52.40374 | 1.923051 | 204.4898 | 2.249282 | 35.70785 | 1.620354 |
| SL | 367.0863 | 0.776423 | 202.8599 | 0.780174 | 5.4E+08 | 0.680482 |
| ML | 4429.429 | 0.450467 | 888.9694 | 0.463341 | 226739.8 | 0.438629 |
| PI | 0.325413 | 0.181788 | 0.957086 | 0.126601 | 2.140547 | 0.077104 |
| Method | α=0.8, μ=4.3 | | α=3.9, μ=0.7 | | α=2.5, μ=1.9 | |
| | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ |
| MM | 18.76111 | 1.792048 | 214.4723 | 0.998233 | 20.63483 | 0.159805 |
| SL | 4613161 | 0.283859 | 2233.297 | 14.70996 | 164.3369 | 0.293215 |
| ML | 329803 | 0.443901 | 157.6046 | 0.527266 | 237811.5 | 0.445787 |
| PI | 0.289312 | 0.235699 | 6.012686 | 0.040923 | 4.417651 | 0.051187 |

**Table 8. Normalized mean square error (NMSE) of $\hat{\mu}$ and $\hat{\alpha}$ at n=500, and 100000 replications simulating from Gamma distribution.**

| | $\mu$ | 1 | | 15 | | 30 | | 50 | |
|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | Method | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ | $\hat{\mu}$ | $\hat{\alpha}$ |
| 1 | MM | 0.49361 | 0.466323 | 3.67333 | 0.21173 | 1.3371 | 0.31179 | 0.49270 | 0.4600 |
| | SL | 5.9E+13 | 0.463397 | 2.6E+20 | 0.18535 | 5E+11 | 0.31143 | 4.4E+12 | 0.4584 |
| | ML | 5635.27 | 0.404207 | 6049.75 | 0.39471 | 6896.6 | 0.40311 | 5677.62 | 0.4039 |
| | PI | 0.42180 | 0.282001 | 0.33614 | 0.15944 | 0.461 | 0.24192 | 0.42241 | 0.2852 |
| 2 | MM | 118.310 | 0.788064 | 3.35258 | 0.17064 | 1.2521 | 0.29634 | 0.34274 | 0.3441 |
| | SL | 280.332 | 2966.876 | 4.8E+12 | 0.18587 | 8E+12 | 0.31240 | 9.1E+11 | 0.4604 |
| | ML | 36.7754 | 0.452925 | 155583 | 0.39423 | 6E+10 | 0.40685 | 7.1E+11 | 0.4259 |
| | PI | 0.59172 | 0.030668 | 2.91004 | 0.04691 | 3.3895 | 0.05320 | 3.33242 | 0.0524 |
| 3 | MM | 88.9055 | 1.608563 | 3.13893 | 0.16801 | 0.2224 | 0.20014 | 0.22246 | 0.3432 |
| | SL | 267.884 | 30.19459 | 2.1E+11 | 0.18557 | 1E+13 | 0.31198 | 7.2E+11 | 0.4628 |
| | ML | 36.6010 | 0.452639 | 1.0E+12 | 0.44358 | 1E+14 | 0.51901 | 6.4E+19 | 0.5705 |
| | PI | 2.12635 | 0.010452 | 8.65200 | 0.04327 | 10.223 | 0.05195 | 9.84479 | 0.0501 |

To sum up, the tables and graphs show that the new Psi-inverse estimator outperforms the other three estimators. This suggest that it is statistically, in case of limited sample sizes, reasonable to use the new estimation method, the Psi-inverse method, over the other three methods. This is due to its good performance. Moreover, the method superiority is not affected by the sample size. Hence, the new estimators provide an attractive and reliable alternative parameters' estimators to the available traditional ones.

## 3.2. The Second Simulation Study

The aim of this simulation study is to compare the proposed estimators with existing estimators, in [8], in the case of very large samples; $n \geq 1000$. Similar to the first simulation study the comparison criteria is the normalized mean square error (NMSE).

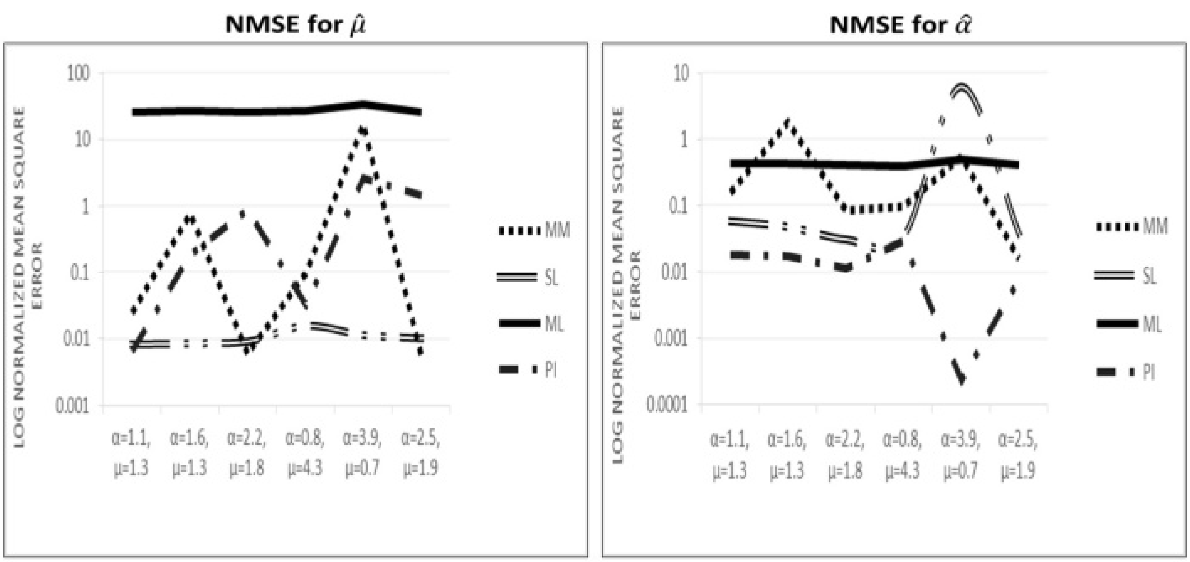


**Figure 4.** Log Normalized mean square error (NMSE) of $\hat{\mu}$ and $\hat{\alpha}$ at n=1000, and 500 replications when simulating from Gamma distribution.

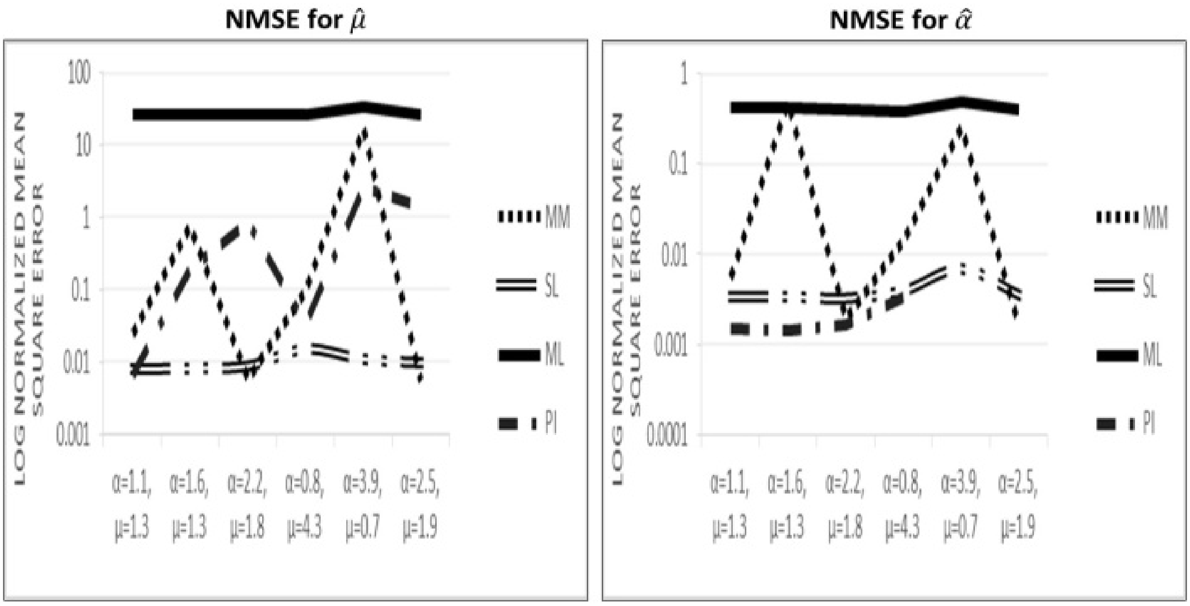


**Figure 5.** Log Normalized mean square error (NMSE) of $\hat{\mu}$ and $\hat{\alpha}$ at n=10000, and 500 replications when simulating from Gamma distribution.

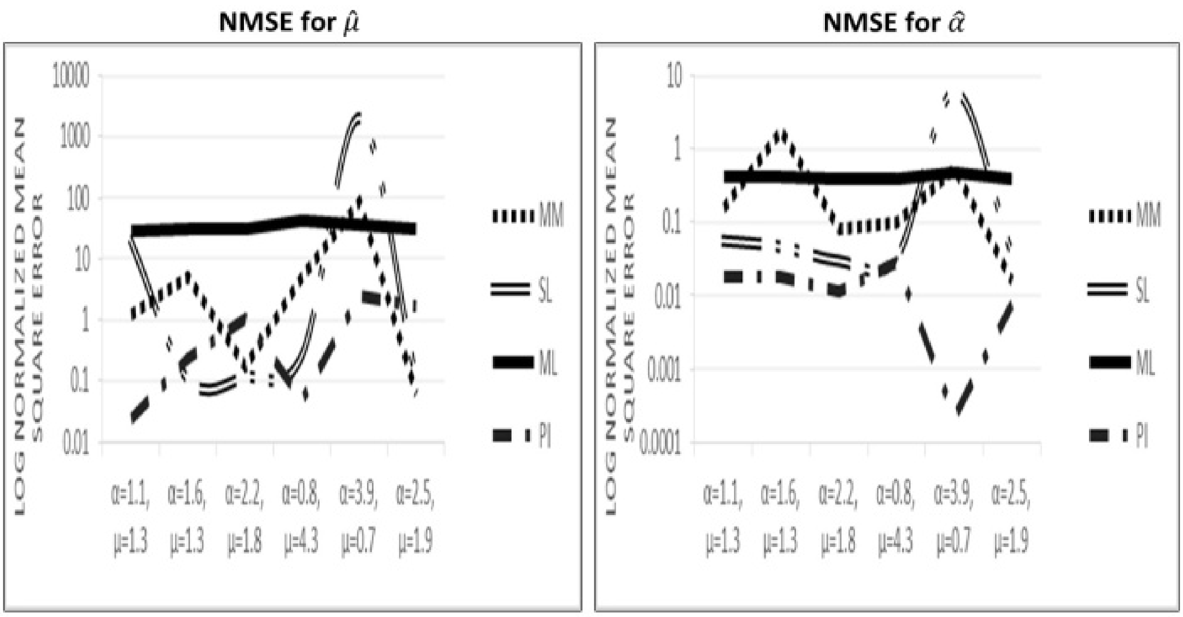


**Figure 6.** Log Normalized mean square error (NMSE) of $\hat{\mu}$ and $\hat{\alpha}$ at n=100000, and 500 replications when simulating from Gamma distribution.

**Simulation setting**

Samples are generated from the $\alpha-\mu$ distribution in Eq. (1). The samples are generated in the special case of gamma distribution using the setup of [8]. The objective of generating from gamma distribution is to mimic the work [8] and to utilize non-integer values for the parameter $\mu$. The same sample sizes are fixed at 1000, 10000 and 100000; the same as in [8]. The set of parameters used in [8] are $\alpha = 0.8, 1.1, 1.2, 1.6, 2.2, 2.5, 3.9$ and $\mu = 0.7, 1.3, 1.8, 1.9, 4.3$. Note that the Gaussian seed generation for the variable in Eq. (1), requires integer values for $\mu$ which is not the case for Gamma distribution. The number of replications is fixed at 500 replications.

**Simulation results**

Simulation results are displayed in Figure 4-6. The results are reported the new estimators and the existing ones; namely the MM estimator and the SL estimator. It can be seen from the results that the PI method still outperform the other three methods except in few cases where the SL method produces better estimates for parameter $\mu$ but not $\alpha$. It is also clear that whenever the difference between $\mu$ and $\alpha$ gets larger the SL estimator deteriorates.

## 4. Discussions

In telecommunications field the radio channel signals are very important tool. As any phenomena a model is needed to accommodate the behavior of radio channel signals. In literature the $\alpha-\mu$ probability distribution has been introduced for this purpose. It is a general channel signal fading model that encompasses many applied important distributions as a special case. This distribution is also known as generalized gamma, Stacy distribution. Many other distributions can be considered as special cases of the $\alpha-\mu$ distribution including Gamma distribution, Erlang distribution, Central Chi-squared distribution, Nakagami-m distribution, Exponential distribution, Weibull distribution, one-sided Gaussian distribution and Rayleigh distribution. In this article we propose two methods to estimate the unknown parameters for the $\alpha-\mu$ probability distribution. They are the maximum likelihood (ML) and Psi-inverse (PI) estimators. The proposed estimators are compared with the MM estimator of [14] and the SL estimator of [8]. Depending on simulation studies the proposed methods perform well comparable to the existing estimators; the MM estimator and the SL estimator. This behavior is valid apart from the sample size.

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