American Journal of Applied Mathematics and Statistics, 2019, Vol. 7, No. 5, 171-177
Available online at http://pubs.sciepub.com/ajams/7/5/3
Published by Science and Education Publishing
DOI:10.12691/ajams-7-5-3

Optimal Design of Step Stress Partially Accelerated Life Test under Progressive Type-II Censored Data with Random Removal for Inverse Lomax Distribution

Yasser M. Amer[1,*], Rania M. Shalabi[2]

[1]Cairo Higher Institutes, Mokattam, Cairo, Egypt
[2]The Higher Institute of Managerial Science, Culture and Science City, 6th of October, Egypt
*Corresponding author: yasseramer4@yahoo.com

Received September 12, 2019; Revised October 24, 2019; Accepted November 08, 2019

Abstract Step Stress-Partially Accelerated Life Test SS-PALT under Type-II progressive censoring with Binomial or uniform removal assuming Inverse Lomax distribution has been presented. A comparison between both removals is shown. The Newton-Raphson method is applied to obtain maximum likelihood estimators MLE of the parameters and the optimal stress-change time which minimizes the generalized asymptotic variance. A simulation study is performed to illustrate the statistical properties of the parameters.

Keywords: *Partially accelerated life test, Binominal distribution, uniform distribution, Inverse Lomax distribution, optimal design, D-optimality, Monte Carlo Simulation*

Cite This Article: Yasser M. Amer, and Rania M. Shalabi, “Optimal Design of Step Stress Partially Accelerated Life Test under Progressive Type-II Censored Data with Random Removal for Inverse Lomax Distribution.” *American Journal of Applied Mathematics and Statistics*, vol. 7, no. 5 (2019): 171-177. doi: 10.12691/ajams-7-5-3.

1. Introduction

The engineers used the Partially accelerated life tests (PALTs) successfully to estimate the acceleration factor and thus extrapolating the accelerated data to normal conditions. In a PALT, items are tested at both normal and accelerated conditions. There are three Types of PALTs, these Types are step-stress, progressive-stress and constant-stress. Such testing conducted under stresses is called accelerated life test (ALT) or partially accelerated life test (PALT). In ALT, the units are put under stresses to get more failures in a short time. The main assumption in ALT is that the mathematical model relating the lifetime of the unit and the stress is known or can be assumed. In some cases, such model is neither known nor assumed. That is, ALT data cannot be extrapolated to normal use condition. So, in such cases, PALT is a more suitable test to be used to estimate the statistical model parameters.

In a SS-PALT, test unit starts at normal use condition for a specified time. If it does not fail at that time, it is putted under stress. Stress is repeatedly increased until the test unit fails or the test is terminated based on a certain censoring scheme. As indicated by [1], the step-stress method can reduce the testing time and save a lot of manpower, material sources and money. Specifically, SS-PALT should be used for reliability analysis to save time and money especially when the test units are with high reliability and the mathematical model indicated above is unknown or cannot be assumed Partially accelerated life tests (PALT) have been studied by several authors under step-stress scheme. For more details, see [2-16]. It is noted that no studies have been made on the step-stress PALT under progressive censoring. In this paper, we will combine progressive censoring and step-stress PALT to develop a step-stress PALT with Progressively Type-II Censored Data using the exponential distribution as a lifetime model.

When the experimenter does not observe the lifetimes of all test units the censored sampling arises in a life test. There are two censoring schemes, Type-I censoring and Type-II censoring. Both of these two censoring schemes do not allow for units to be removed from the test at the points other than the final termination point. This allowance may be desirable when a compromise between reduced the time of experimentation and the observation of at least some extreme lifetimes is sought. Progressively censored sampling allows to the experimenter to save time and cost. The most popular one is the progressive type-II censoring scheme and it can be briefly described as follows. Suppose n identical units are put on a life testing experiment. The integer $k < n$ is prefixed, and $r_1, \dots, r_k$ are k prefixed non-negative integers such that $\sum_{i=1}^{k} \boldsymbol{r_i} + \boldsymbol{k} = \boldsymbol{n}.\boldsymbol{r_1}$ of the surviving units are randomly selected and removed from the test. Similarly, at the time of the second failure, r_2 units are chosen randomly from the remaining $n-r_1-2$ units and they are removed, and so on. This experiment terminates when the m^{th} failure occurs at time

$t_{(m)}$, the remaining surviving units $r_m = n - m - r_1 - r_2 - \cdots - r_{m-1}$ are all removed from the test.

Extensive work has been done on various aspects of different progressive censoring schemes. [17] and [18] considered the progressive Type-II censoring scheme with fixed $r_1, r_2, \ldots, r_m$. But in some reliability experiments, the number of removals cannot be considered to be fixed. [19] and [20] considered progressive censoring with random (binomial) removals to estimate the unknown parameters of Weibull and Gompertz distribution using ordinary life testing. [21] used progressively Type-II censored data with binomial removals to estimate the parameters of exponential reliability model. [22] discuss step-stress partially-accelerated life test under progressive Type-II censoring with random removals.

The removals from the test are assumed to have binomial distributions. The lifetimes of the test units are considered to be exponential distributed. Recently, [23] provided Step Stress Partially Accelerated Life Test under Progressive Type-II Censored Data with Random Removal for Gompertz Distribution and the removals from the test are assumed to have binomial and uniform distributions. Also, [24] introduced the same on Frechet Distribution.

In this paper, we will use SS-PAL under progressive Type-II censoring with random removals. The removals from the test are assumed to have binomial and uniform distributions. The lifetimes of the test units are considered to be Inverse Lomax distributed. Also, we will determine the optimal stress change time which minimizes the generalized asymptotic variance of the MLE of parameters. Section 2 presents the Inverse Lomax distribution and the assumptions of the partially accelerated model. Estimation of model parameters is given in Section 3; in Section 4 and 5 simulation study results and conclusion are given.

2. Inverse Lomax Distribution

The Lomax or Pareto II (the shifted Pareto) distribution was proposed in [25]. This distribution has found wide applications especially in analysis of the business failure life time data, income and wealth inequality, size of cities, actuarial science, medical and biological sciences, engineering, lifetime and reliability modeling. In lifetime, the Lomax model belongs to the family of decreasing failure rate in [26].

The Inverse Lomax distribution (ILD) belongs to inverted family of distributions and found to be very flexible to analyze the situation where the non-monotonicity of the failure rate has been realized in [27]. If a random variable Y has Lomax distribution, then $x = \frac{1}{y}$ has an Inverse Lomax distribution (ILD) [28]. (ILD) has an application in stochastic modeling of decreasing failure rate life components. Like other distributions belonging to the family of generalized Beta distribution, the (ILD) also has application in economics and actuarial sciences [29]. (ILD) was implemented on geophysical databases [30] on the sizes of land fires in the California state. [31], carried out research work regarding the statistical inference and Prediction on (ILD) through Bayesian inferences. [32] considered the (ILD) to possess the Lorenz ordering relationship between ordered statistics.

The probability density function (PDF) and the cumulative distribution function (CDF) for (ILD) respectively are as follows:

$$f(x) = \frac{\alpha\theta}{x^2}\left(1+\frac{\theta}{x}\right)^{-\alpha-1} \quad x > 0,\ \alpha, \theta > 0. \tag{1}$$

$$F(x) = \left(1+\frac{\theta}{x}\right)^{-\alpha} \quad x > 0,\ \alpha, \theta > 0. \tag{2}$$

3. Model and Assumptions

The following assumptions are used throughout the paper:

1. n identical and independent units are put on the life test.
2. The lifetime of each unit has an exponential distribution.
3. The test is terminated at the m^{th} failure, where m is prefixed $(m \le n)$.
4. Each of the n units is first run under normal use condition. If it does not fail or remove from the test by a pre-specified time τ. it is put under accelerated condition (stress).
5. At the i^{th} failure a random number of the surviving units, $R_i, i = 1,2,..m-1$, are randomly selected and removed from the test. Finally, at the m^{th} failure the remaining surviving units $R_m = n - m - \sum_{i=1}^{m-1} R_i$ are all removed from the test and the test is terminated.
6. The lifetime, say X, of a unit under SS-PALT can be rewritten as

$$X = \begin{cases} T & \text{if } T \le \tau \\ \tau + \dfrac{T-\tau}{\beta} & \text{if } T > \tau \end{cases}.$$

In this paper, we will have assumed that, probability density function (PDF) of X is given by

$$f(x) = \begin{cases} f_1(x) = \dfrac{\alpha\theta}{x^2}\left(1+\dfrac{\theta}{x}\right)^{-\alpha-1} & 0 < x \le \tau \\ f_2(x) = \dfrac{\alpha\theta\beta}{\left(\tau+\beta(x-\tau)\right)^2}\left(1+\dfrac{\theta}{\left(\tau+\beta(x-\tau)\right)}\right)^{-\alpha-1} & x > \tau \end{cases} \tag{3}$$

In addition, the survival functions (SF) under normal and accelerate use conditions respectively is given by

$$S_1(x) = 1-\left(1+\frac{\theta}{x}\right)^{-\alpha},\ 0 < x < \tau. \tag{4}$$

And

$$S_2(x) = 1-\left(1+\frac{\theta}{\left(\tau+\beta(x-\tau)\right)}\right)^{-\alpha},\ x > \tau. \tag{5}$$

4. Estimation of Parameters

4.1. Parameter Estimation with the Binomial Removals

The number of units removed from the test at each failure time follows a binomial distribution and any individual unit being removed is independent of others but with the same probability p. that is, $R_1 \sim bino(n-m,p)$ *and for* $i = 1,2,..3$, $R_i \sim bino\ (n-m-\sum_{j=1}^{m} r_j\ ,p)$ and $r_m = n-m-r_1-r_2-\cdots-r_{m-1}$.

Let $(x_i, r_i, u_{1i}, u_{2i})$, $i = 1,2,..,m$, denote the observation obtained form a progressively Type-II censored sample with random removals in a SS-PALT. Here $x_{(1)} \leq \ldots \leq x_{(m)}$. Given the pre-determined number of removals $R = (R_1 = r_1, .., R_{m-1} = r_{m-1})$, the conditional likelihood function of the observations $x = \{(y_i, r_i, u_{1i}, u_{2i}), i = 1,2,..,m\}$ takes the following form

$$L_1\left(x_i,\alpha,\beta,u_{1i},u_{2i}|R=r\right) = \prod_{i=1}^{m}\left(\left\{f_1\left(x_i\right)\left(S_1\left(x_i\right)\right)^{r_i}\right\}^{u_{1i}}\left\{f_1\left(x_i\right)\left(S_1\left(x_i\right)\right)^{r_i}\right\}^{u_{2i}}\right. \tag{6}$$

From (1), (2) and (3) is inserted in (6) and simplify, we get

$$\begin{aligned}
&L_1\left(y,\alpha,\beta,u_{1i},u_{2i}|R=r\right) \\
&= \prod_{i=1}^{m}\left[\frac{\alpha\theta}{x^2}\left(1+\frac{\theta}{x}\right)^{-\alpha-1}\left(1-\left(1+\frac{\theta}{x}\right)^{-\alpha}\right)^{r_i}\right]^{u_{1i}} \\
&\times\left[\frac{\alpha\ \theta\beta}{\left(\tau+\beta\left(x-\tau\right)\right)^2}\left(1+\frac{\theta}{\left(\tau+\beta\left(x-\tau\right)\right)}\right)^{-\alpha-1}\left(1-\left(1+\frac{\theta}{\left(\tau+\beta\left(x-\tau\right)\right)}\right)^{-\alpha}\right)^{r_i}\right]^{u_{2i}}
\end{aligned} \tag{7}$$

Where $\mathbf{u_{1i}} = \begin{cases}\mathbf{1} & \textbf{if } \mathbf{x_i} \leq \boldsymbol{\tau} \\ \mathbf{0} & \textbf{if } \mathbf{x_i} > \tau\end{cases}$, $\mathbf{u_{2i}} = \begin{cases}\mathbf{0} & \textbf{if } \mathbf{x_i} \leq \boldsymbol{\tau} \\ \mathbf{1} & \textbf{if } \mathbf{x_i} > \tau\end{cases}$.

And $\sum_{\mathbf{i=1}}^{\mathbf{m_1}} \mathbf{u_{1i}} + \sum_{\mathbf{i=1}}^{\mathbf{m_2}} \mathbf{u_{2i}} = \mathbf{m}$.

The number of units removed at each failure time follows a binomial distribution such that

$$P\left(R_1=r_1\right)=\binom{n-m}{r_1}p^{r_1}\left(1-p\right)^{n-m-r_1},$$

and for $\mathbf{i = 2,3,..,m-1}$

$$P\left(R_i=r_i|R_{i-1}=r_{i-1}..,R_1=r_1\right) = \binom{n-m-\sum_{j=1}^{i-1}r_j}{r_i}p^{r_i}\left(1-p\right)^{n-m-\sum_{j=1}^{i}r_j}. \tag{8}$$

At the parameter $\boldsymbol{k}$ is known, the likelihood of the sample of size n is given as

$$L\left(x_i,\alpha,\beta,p\right)=L_1\left(x_i,\alpha,\beta,u_{1i},u_{2i}|R=r\right)P\left(R=r\right) \tag{9}$$

Where,

$$\begin{aligned}
&\mathbf{P}\left(\mathbf{R}=\mathbf{r}\right)=\mathbf{P}\left(\mathbf{R}_1=\mathbf{r}_1,\mathbf{R}_2=\mathbf{r}_2,\ldots,\mathbf{R}_{\mathbf{m}-1}=\mathbf{r}_{\mathbf{m}-1}\right) \\
&=\mathbf{P}\left(\mathbf{R}_{\mathbf{m}-1}=\mathbf{r}_{\mathbf{m}-1}|\mathbf{R}_{\mathbf{m}-2}=\mathbf{r}_{\mathbf{m}-2}..,\mathbf{R}_1=\mathbf{r}_1\right) \\
&\times\mathbf{P}\left(\mathbf{R}_{\mathbf{m}-2}=\mathbf{r}_{\mathbf{m}-2}|\mathbf{R}_{\mathbf{m}-3}=\mathbf{r}_{\mathbf{m}-3}..,\mathbf{R}_1=\mathbf{r}_1\right) \\
&\times\ldots\mathbf{P}\left(\mathbf{R}_2=\mathbf{r}_2|\mathbf{R}_1=\mathbf{r}_1\right)\mathbf{P}\left(\mathbf{R}_1=\mathbf{r}_1\right).
\end{aligned}$$

That is,

$$\begin{aligned}
\mathbf{P}\left(\mathbf{R}=\mathbf{r}\right)&=\frac{\left(\mathbf{n}-\mathbf{m}\right)!}{\left(\mathbf{n}-\mathbf{m}-\sum_{\mathbf{i}=1}^{\mathbf{m}-1}\mathbf{r_i}\right)!\prod_{\mathbf{i}=1}^{\mathbf{m}-1}\mathbf{r_i}}\mathbf{p}^{\sum_{\mathbf{i}=1}^{\mathbf{m}-1}\mathbf{r_i}} \\
&\times\left(1-\mathbf{p}\right)^{\left(\mathbf{m}-1\right)\left(\mathbf{n}-\mathbf{m}\right)-\sum_{\mathbf{i}=1}^{\mathbf{m}-1}\left(\mathbf{m}-\mathbf{i}\right)\left(\mathbf{r_i}\right)}.
\end{aligned} \tag{10}$$

The log-likelihood function $\boldsymbol{lnL(x_i,\alpha,\beta,p)} = \boldsymbol{l(x_i,\alpha,\beta,p)}$ can be written as follows:

$$\mathbf{l}\left(\mathbf{x_i},\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{p}\right)=\ln\mathbf{L}_1\left(\mathbf{x_i},\boldsymbol{\alpha},\boldsymbol{\beta}\right)+\mathbf{lnP}\left(\mathbf{R}=\mathbf{r}\right). \tag{11}$$

First partial derivatives are derived to obtain the estimate of the parameters $\boldsymbol{\alpha},\boldsymbol{\theta}$ and acceleration factor $\boldsymbol{\beta}$. The log likelihood which is associated with (10) is given

$$\begin{aligned}
&\mathbf{l}\left(\mathbf{x_i},\boldsymbol{\alpha},\boldsymbol{\theta},\boldsymbol{\beta},\mathbf{p}\right)=\mathbf{m}\ln\boldsymbol{\alpha}+\mathbf{m}\ln\boldsymbol{\theta}+\mathbf{m}_2\ln\boldsymbol{\beta} \\
&-2\sum_{\mathbf{i}=1}^{\mathbf{m}_1}\ln\mathbf{x_i}-\left(\boldsymbol{\alpha}+1\right)\sum_{\mathbf{i}=1}^{\mathbf{m}_1}\ln\left(1+\frac{\boldsymbol{\theta}}{\mathbf{x_i}}\right) \\
&+\sum_{\mathbf{i}=1}^{\mathbf{m}_1}\mathbf{r_i}\ln\left(1-\left(1+\frac{\boldsymbol{\theta}}{\mathbf{x_i}}\right)^{-\boldsymbol{\alpha}}\right)-2\sum_{\mathbf{i}=1}^{\mathbf{m}_2}\ln\left(\boldsymbol{\tau}+\boldsymbol{\beta}\left(\mathbf{x_i}-\boldsymbol{\tau}\right)\right) \\
&-\left(\boldsymbol{\alpha}+1\right)\sum_{\mathbf{i}=1}^{\mathbf{m}_2}\ln\left(1+\frac{\boldsymbol{\theta}}{\boldsymbol{\tau}+\boldsymbol{\beta}\left(\mathbf{x_i}-\boldsymbol{\tau}\right)}\right) \\
&+\sum_{\mathbf{i}=1}^{\mathbf{m}_2}\mathbf{r_i}\ln\left(1-\left(1+\frac{\boldsymbol{\theta}}{\left(\boldsymbol{\tau}+\boldsymbol{\beta}\left(\mathbf{x_i}-\boldsymbol{\tau}\right)\right)}\right)^{-\boldsymbol{\alpha}}\right)+\mathbf{lnp}\sum_{\mathbf{i}=1}^{\mathbf{m}-1}\mathbf{r_i} \\
&+\mathbf{lnn}\left(1-\mathbf{p}\right)\left[\left(\mathbf{m}-1\right)\left(\mathbf{n}-\mathbf{m}\right)-\sum_{\mathbf{i}=1}^{\mathbf{m}-1}\left(\mathbf{m}-\mathbf{i}\right)\left(\mathbf{r_i}\right)\right]+\mathbf{c}
\end{aligned}$$

Where, $\mathbf{c}=\left(\frac{\left(\mathbf{n}-\mathbf{m}\right)!}{\left(\mathbf{n}-\mathbf{m}-\sum_{\mathbf{i}=1}^{\mathbf{m}-1}\mathbf{r_i}\right)!\prod_{\mathbf{i}=1}^{\mathbf{m}-1}\mathbf{r_i}}\right)$.

The following likelihood equations are obtained by equating the partial derivatives of $\boldsymbol{l(x_i,\alpha,\theta,\beta,p)}$ with respect to $\boldsymbol{\alpha},\boldsymbol{\theta},\boldsymbol{\beta}$ and $\boldsymbol{p}$ to zero:

$$\begin{aligned}
&\frac{\partial l}{\partial\alpha}=\frac{m}{\alpha}-\sum_{i=1}^{m_1}\ln\left(1+\frac{\theta}{x_i}\right)-\sum_{i=1}^{m_2}\ln\left(1+\frac{\theta}{\tau+\beta\left(x_i-\tau\right)}\right) \\
&+\sum_{i=1}^{m_1}\left[\frac{r_i\ln\left(1+\frac{\theta}{x_i}\right)}{\left(1+\frac{\theta}{x_i}\right)^{\alpha}-1}\right]+\sum_{i=1}^{m_2}\left[\frac{r_i\ln\left(1+\frac{\theta}{\tau+\beta\left(x_i-\tau\right)}\right)}{\left(1+\frac{\theta}{\tau+\beta\left(x_i-\tau\right)}\right)^{\alpha}-1}\right]=0
\end{aligned} \tag{13}$$

$$\frac{\partial l}{\partial \theta} = \frac{m}{\theta} - (\alpha+1)\sum_{i=1}^{m_1}(x_i+\theta)^{-1}$$
$$+\alpha\sum_{i=1}^{m_1}\left[\frac{r_i}{(x_i+\theta)\left\{\left(1+\frac{\theta}{x_i}\right)^{\alpha}-1\right\}}\right]$$
$$-(\alpha+1)\sum_{i=1}^{m_2}(\tau+\beta(x_i-\tau)+\theta)^{-1} \tag{14}$$
$$+\alpha\sum_{i=1}^{m_2}\left[\frac{r_i}{(\tau+\beta(x_i-\tau)+\theta)\left\{\left(1+\frac{\theta}{\tau+\beta(x_i-\tau)}\right)^{\alpha}-1\right\}}\right]$$
$$=0$$

$$\frac{\partial l}{\partial \beta} = \frac{m_2}{\beta} - 2\sum_{i=1}^{m_2}\left[\frac{x_i-\tau}{\tau+\beta(x_i-\tau)}\right]$$
$$+(\alpha+1)\times\theta\sum_{i=1}^{m_2}\left[\frac{(x_i-\tau)}{(\tau+\beta(x_i-\tau))(\tau+\beta(x_i-\tau)+\theta)}\right]$$
$$-\alpha\theta\sum_{i=1}^{m_2}\left[\frac{r_i(x_i-\tau)(\tau+\beta(x_i-\tau))^{-2}}{\left(1+\frac{\theta}{\tau+\beta(x_i-\tau)}\right)\left\{\left(1+\frac{\theta}{\tau+\beta(x_i-\tau)}\right)^{\alpha}-1\right\}}\right] \tag{15}$$
$$=0$$

$$\frac{\partial l}{\partial p} = \frac{\sum_{i=1}^{m-1} r_i}{p} - \frac{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)(r_i)}{1-p} = 0. \tag{16}$$

From Equation (16), $\boldsymbol{p}$ is estimated as follow

$$\hat{p} = \frac{\sum_{i=1}^{m-1} r_i}{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i-1)(r_i)}. \tag{17}$$

There is no closed-from solution to this system of equations (13-15), so we will solve for $\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\beta}}$ iteratively, using the Newton-Raphson method, a tangent method for root finding. In our case we will estimate $\boldsymbol{\Omega} = (\boldsymbol{\alpha}, \boldsymbol{\theta}, \boldsymbol{\beta})$ iteratively

$$\hat{\boldsymbol{\Omega}}_{j+1} = \hat{\boldsymbol{\Omega}}_j - G^{-1}\mathbf{g} \tag{18}$$

where g is the vector of normal equations for which we want $g = [g_1\ g_2\ g_3]$ with $g_1 = \frac{\partial l}{\partial \alpha}$, $g_2 = \frac{\partial l}{\partial \theta}$, and $g_3 = \frac{\partial l}{\partial \beta}$, and G is the matrix of second derivatives

$$G = \begin{bmatrix} \frac{\partial g_1}{\partial \alpha} & \frac{\partial g_1}{\partial \theta} & \frac{\partial g_1}{\partial \beta} \\ \frac{\partial g_2}{\partial \alpha} & \frac{\partial g_2}{\partial \theta} & \frac{\partial g_2}{\partial \beta} \\ \frac{\partial g_3}{\partial \alpha} & \frac{\partial g_3}{\partial \theta} & \frac{\partial g_3}{\partial \beta} \end{bmatrix}. \tag{19}$$

The Newton-Raphson algorithm converges, as our estimates of $\boldsymbol{\alpha}, \boldsymbol{\theta}$ and $\boldsymbol{\beta}$ change by less than a tolerated amount with each successive iteration, to $\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\beta}}$.

The bias and the root of mean squared error (RMSE) of an estimator $\hat{\boldsymbol{\Omega}}$ of the parameter $\boldsymbol{\Omega}$, easily obtained by

$$Bias(\hat{\boldsymbol{\Omega}}) = \frac{1}{M}\sum_{i=1}^{M}\left[\hat{\boldsymbol{\Omega}}-\boldsymbol{\Omega}\right]$$
$$\text{and } RMSE(\hat{\boldsymbol{\Omega}}) = \sqrt{\frac{1}{M}\sum_{i=1}^{M}\left[\hat{\boldsymbol{\Omega}}-\boldsymbol{\Omega}\right]^2}. \tag{20}$$

4.2. Parameter Estimation with the Uniform Removals

The number of units removed from the test at each failure time follows a uniform distribution and any individual unit being removed is independent of others but with the same probability p. that is, $\boldsymbol{R_1 \sim unif(0, n-m)}$ and for i=1,2,..3. $\boldsymbol{R_i \sim unif\ (0, n-m-\sum_{j=1}^{i-1} r_j)}$ and $\boldsymbol{r_m = n-m-r_1-r_2-\cdots-r_{m-1}}$.

The number of units removed at each failure time follows a uniform distribution such that

$$P(R_1 = r_1) = \frac{1}{n-m+1}. \tag{21}$$

And for $\mathbf{i = 2, 3, .., m-1.}$

$$P(R_i = r_i | R_{i-1} = r_{i-1}.., R_1 = r_1)$$
$$= \frac{1}{n-m-\sum_{j=1}^{i-1} r_j + 1}$$

where, the joint probability distribution of $\boldsymbol{R} = (\boldsymbol{R_1 = r_1, .., R_{m-1} = r_{m-1}})$ and is given by

$$P(R = r) = \frac{1}{n-m-\sum_{i=1}^{m-1} r_i + 1} \tag{22}$$

where $\boldsymbol{0 \le r_i \le\ n-m-\sum_{j=1}^{i-1} r_j\ , i = 0, 1, \dots, m-1.}$

It is clear that $\boldsymbol{P(R = r)}$ does not depend on the parameters $\boldsymbol{\alpha}, \boldsymbol{\theta}$ and $\boldsymbol{\beta}$, hence the maximum likelihood estimators can be derived directly by maximizing the equations (9) and then solving the equations (13-16).

4.3. Fisher Information Matrix

The asymptotic Fisher information matrix can be formulated as follows;

$$F = -\begin{pmatrix} \frac{\partial g_1}{\partial \alpha} & \frac{\partial g_1}{\partial \theta} & \frac{\partial g_1}{\partial \beta} \\ \frac{\partial g_2}{\partial \alpha} & \frac{\partial g_2}{\partial \theta} & \frac{\partial g_2}{\partial \beta} \\ \frac{\partial g_3}{\partial \alpha} & \frac{\partial g_3}{\partial \theta} & \frac{\partial g_3}{\partial \beta} \end{pmatrix}_{\boldsymbol{\Omega}=\hat{\boldsymbol{\Omega}}}. \tag{23}$$

In relation to the asymptotic variance-covariance matrix of the ML estimators of the parameters, it can be approximated by numerically inverting the above Fisher's information matrix $\boldsymbol{F}$.

4.4. D-optimality

In this section we explore the choice of $\boldsymbol{\tau}$ in a SS-PALT with Type-II progressive censoring. We propose one selection criterion which enable one to choose the optimal value of $\boldsymbol{\tau}$. The proposed criterion is based on the determinant of the Fisher's information matrix. Maximizing that determinant is equivalent to minimizing the generalized asymptotic variance (GAV) of the MLE of the model parameters. The GAV is the reciprocal of the determinant of the Fisher's information matrix F, for example, see [33]. That is,

$$GAF\left(\hat{\alpha},\hat{\theta},\hat{\beta}\right)=\frac{1}{|F|}. \tag{24}$$

So, the optimal value of $\boldsymbol{\tau}$ is chosen such that the determinant of the Fisher's information matrix $\boldsymbol{F}$ is maximized and then the GAV is minimized. This is called the D-optimality criterion.

5. Simulation Study

A simulation study is performed to obtain MLEs of β, θ and p. Also, to study the properties of these estimates through the root of the mean squared errors (RMSEs),) and the confidence intervals for different sample sizes. Moreover, we will determine the optimal stress change time which minimizes the generalized asymptotic variance of the MLE of parameters. To perform the simulation study, we followed the same steps has been introduced in [24];

a) First specify the value of n and m.
b) The value of the parameters are chosen to be $\alpha = 2\,, \theta = 0.5\,, \beta = 2.3\,, \tau = 2, p = 0.4$.
c) Generate a random sample with size n and censoring size m with random removals, $r_i, i = 1,2,\ldots,m-1$ from the random variable X given by (3).
d) Generate a group value $R_i \sim bino\,(n-m-\sum_{j=1}^{m} r_j\,, p)$ and also $R_i \sim unif\;(0, n-m-\sum_{j=1}^{i-1} r_j)$ where $0 \le r_i \le n-m-\sum_{j=1}^{i-1} r_j\,, i = 0,1,\ldots,m-1$ and $r_m = n-m-r_1-r_2-\cdots-r_{m-1}$.
e) For different sample sizes n= 20, 50, 80 and 100, compute the ML estimates.

The bias and the root of mean squared error (RMSE) are obtained associated with the MLE of the parameters, optimal value of τ and also the Optimal GAV of the MLEs of the model parameters are obtained numerically for each sample size.

Table 1. Simulation study results with Binomial Removals for $\boldsymbol{\alpha = 2\,, \theta = 0.5\,, \beta = 2.3\,, \tau = 2, p = 0.4}$

n	m	Estimates			The root of mean squared error			The bias			$\boldsymbol{\tau}$	$\boldsymbol{\lvert F^{-1}\rvert}$
		$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$RMSE_{\hat{\theta}}$	$RMSE_{\hat{\alpha}}$	$RMSE_{\hat{\beta}}$	$Bias_{\hat{\theta}}$	$Bias_{\hat{\alpha}}$	$Bias_{\hat{\beta}}$		
20	9	0.541	2.015	2.274	0.199	0.159	0.179	0.041	0.015	-0.026	2.103	1.110
	19	0.537	2.040	2.303	0.198	0.159	0.173	0.037	0.04	0.003	2.148	1.225
50	9	0.552	2.035	2.283	0.197	0.155	0.172	0.052	0.035	-0.017	2.146	1.107
	19	0.547	2.021	2.274	0.193	0.153	0.172	0.047	0.021	-0.026	2.188	1.243
	29	0.544	2.013	2.269	0.194	0.152	0.173	0.044	0.013	-0.031	2.206	1.233
	39	0.554	2.017	2.263	0.190	0.149	0.169	0.054	0.017	-0.037	2.144	1.311
	49	0.481	1.955	2.274	0.175	0.149	0.167	-0.019	-0.045	-0.026	2.178	1.365
80	9	0.489	1.953	2.264	0.172	0.146	0.165	-0.011	-0.047	-0.036	2.176	1.111
	19	0.483	1.943	2.260	0.169	0.144	0.164	-0.017	-0.057	-0.04	2.123	1.228
	29	0.480	1.940	2.260	0.169	0.145	0.165	-0.02	-0.06	-0.04	2.137	1.479
	39	0.487	1.940	2.253	0.169	0.143	0.164	-0.013	-0.06	-0.047	2.247	1.572
	49	0.489	1.945	2.256	0.168	0.142	0.163	-0.011	-0.055	-0.044	2.130	1.535
	59	0.512	1.947	2.235	0.168	0.138	0.162	0.012	-0.053	-0.065	2.102	1.609
	69	0.530	1.953	2.223	0.168	0.135	0.161	0.03	-0.047	-0.077	2.133	1.650
	79	0.530	1.980	2.250	0.167	0.134	0.155	0.03	-0.02	-0.05	2.151	1.763
100	9	0.533	2.024	2.291	0.167	0.134	0.147	0.033	0.024	-0.009	2.142	1.100
	19	0.525	2.028	2.303	0.165	0.133	0.145	0.025	0.028	0.003	2.113	1.202
	29	0.537	2.046	2.309	0.164	0.131	0.141	0.037	0.046	0.009	2.114	1.253
	39	0.570	2.072	2.302	0.163	0.125	0.135	0.07	0.072	0.002	2.196	1.338
	49	0.562	2.067	2.305	0.161	0.124	0.134	0.062	0.067	0.005	2.130	1.429
	59	0.542	2.055	2.313	0.157	0.123	0.132	0.042	0.055	0.013	2.219	1.588
	69	0.542	2.052	2.310	0.156	0.123	0.133	0.042	0.052	0.01	2.239	1.636
	69	0.524	2.058	2.334	0.150	0.121	0.126	0.024	0.058	0.034	2.135	1.709
	79	0.530	2.061	2.331	0.153	0.120	0.127	0.03	0.061	0.031	2.116	1.774
	89	0.536	2.056	2.320	0.149	0.118	0.125	0.036	0.056	0.02	2.216	1.810
	99	0.548	2.060	2.312	0.149	0.116	0.124	0.048	0.06	0.012	2.195	1.825

Table 2. Simulation study results with uniform Removals for $\alpha = 0.5, \theta = 1.5, \beta = 1.3, \tau = 3$

n	m	Estimates			The root of mean squared error			The bias			τ	$\lvert F^{-1}\rvert$
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$RMSE_{\hat{\alpha}}$	$RMSE_{\hat{\theta}}$	$RMSE_{\hat{\beta}}$	$\text{Bias}_{\hat{\alpha}}$	$\text{Bias}_{\hat{\theta}}$	$\text{Bias}_{\hat{\beta}}$		
20	9	1.567	0.580	1.313	0.147	0.111	0.119	0.067	0.08	0.013	3.107	1.113
	19	1.569	0.586	1.317	0.146	0.111	0.117	0.069	0.086	0.017	3.137	1.114
50	9	1.568	0.577	1.309	0.144	0.109	0.117	0.068	0.077	0.009	3.138	1.196
	19	1.569	0.566	1.297	0.142	0.107	0.117	0.069	0.066	-0.003	3.118	1.230
	29	1.571	0.573	1.302	0.140	0.105	0.114	0.071	0.073	0.002	3.112	1.319
	39	1.568	0.592	1.324	0.139	0.105	0.110	0.068	0.092	0.024	3.114	1.339
	49	1.568	0.591	1.323	0.139	0.104	0.109	0.068	0.091	0.023	3.146	1.235
80	9	1.562	0.41	1.328	0.136	0.103	0.108	0.062	-0.09	0.028	3.185	1.116
	19	1.567	0.586	1.319	0.135	0.102	0.107	0.067	0.086	0.019	3.149	1.216
	29	1.552	0.572	1.320	0.132	0.101	0.107	0.052	0.072	0.02	3.110	1.295
	39	1.506	0.532	1.274	0.122	0.103	0.105	0.006	0.032	-0.026	3.125	1.349
	49	1.504	0.525	1.321	0.120	0.100	0.105	0.004	0.025	0.021	3.107	1.453
	59	1.515	0.526	1.311	0.122	0.098	0.104	0.015	0.026	0.011	3.143	1.467
	69	1.521	0.529	1.308	0.120	0.096	0.109	0.021	0.029	0.008	3.133	1.510
	79	1.512	0.546	1.334	0.118	0.096	0.099	0.012	0.046	0.034	3.111	1.594
100	9	1.512	0.551	1.339	0.117	0.096	0.097	0.012	0.051	0.039	3.165	1.269
	19	1.507	0.543	1.336	0.119	0.094	0.096	0.007	0.043	0.036	3.111	1.302
	29	1.517	0.546	1.329	0.114	0.093	0.098	0.017	0.046	0.029	3.128	1.469
	39	1.495	0.528	1.328	0.111	0.092	0.095	-0.005	0.028	0.028	3.079	1.547
	49	1.502	0.609	1.407	0.110	0.091	0.080	0.002	0.109	0.107	3.172	1.610
	59	1.498	0.607	1.409	0.109	0.093	0.079	-0.002	0.107	0.109	3.135	1.721
	69	1.512	0.640	1.428	0.108	0.088	0.073	0.012	0.14	0.128	3.209	1.786
	69	1.523	0.633	1.410	0.106	0.085	0.073	0.023	0.133	0.11	3.150	1.815
	79	1.527	0.634	1.407	0.106	0.084	0.072	0.027	0.134	0.107	3.163	1.286
	89	1.523	0.626	1.403	0.104	0.083	0.072	0.023	0.126	0.103	3.109	1.250
	99	1.523	0.625	1.402	0.103	0.082	0.071	0.023	0.125	0.102	3.102	1.284

6. Conclusion

This paper presented the SS-PALT under Type-II progressive censoring with Binomial or uniform removals assuming (ILD). Comparison between both removals is shown. The Newton-Raphson method is applied to obtain MLE estimators of the parameters and the optimal stress-change time which minimizes the GAV.

The numerical study for obtaining the optimum plan for binomial removal is tabulated in Table 1 for different sample size and Table 2 describes uniform removal for possible values of the parameters. We derive the MLE of the parameters. Also, we compute the RMSE associated with the MLE. The above results it is easy to find that for the fixed values of the parameters, the error and optimal time decrease with increasing sample size n. Performance of testing plans and model assumptions are usually evaluated by the properties of the maximum likelihood estimates of model parameters. From the numerical results, we can conclude that both the average value of τ and the average value of GAV for Type-II progressive censoring are getting close to those of complete sample with the bigger m and close faster for bigger n. Hence from the numerical result, we can conclude that estimates of binomial or uniform are stable with relatively small RMSE with increasing sample size. Therefore, the test design obtained here is a robust design and work well for binomial or uniform removal.

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