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On the Comparison of Classical and Bayesian Methods of Estimation of Reliability in Multicomponent Stress-Strength Model for a Proportional Hazard Rate Model

Taruna Kumari[1], Anupam Pathak[2,*]

[1]Department of Statistics, University of Delhi, Delhi-110007, India
[2]Department of Statistics, Ramjas College, University of Delhi, Delhi-110007, India
*Corresponding author: pathakanupam24@gmail.com

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Abstract In this article, we consider a multicomponent stress-strength model which has k independent and identical strength components $X_1, X_2, ..., X_k$ and each component is exposed to a common random stress Y. Both stress and strength are assumed to have proportional hazard rate model with different unknown power parameters. The system is regarded as operating only if at least s out of k $(1 \le s \le k)$ strength variables exceeds the random stress. Reliability of the system is estimated by using maximum likelihood, uniformly minimum variance unbiased and Bayesian methods of estimation. The asymptotic confidence interval is constructed for the reliability function. The performances of these estimators are studied on the basis of their mean squared error through Monte Carlo simulation technique.

Keywords: *proportional hazard rate model; maximum likelihood estimation, uniformly minimum variance unbiased estimation, Bayesian estimation; asymptotic confidence interval, multicomponent reliability.*

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1. Introduction

In the reliability context, the single-component stress-strength model is the probability $R = P(X > Y)$, which represents the reliability of an item or system of random strength X subject to random stress Y. In the single-component stress-strength model, a system fails if and only if at any given time, the applied stress exceeds the strength. Single-component stress-strength models have great utility in the fields of genetics, psychology, engineering and in so many others. A lot of work has been done in the literature for the classical and Bayesian estimation of single-component stress-strength reliability. For a brief review, one may refer to Basu [1], Kelley et al. [2], Awad and Gharraf [3], Tyagi and Bhattacharya [4], Chaturvedi and Kumar [5], Chaturvedi and Pathak [6,7], Chaturvedi et al. [8] and others. Inferences have been drawn for single-component stress-strength reliability for some families of lifetime distributions by Chaturvedi and Pathak [9], Chaturvedi and Kumari [10,11,12] and Kumari et al. [13].

The reliability in a multicomponent stress-strength model was developed by Bhattacharyya and Johnson [14]. Panday and Uddin [15] assumed the parameters not involved in reliability as known using Bayesian estimation. Rao and Kantam [16] studied the estimation of reliability in a multicomponent stress-strenght model for log-logistic distribution and Rao [17] developed an estimation procedure for reliability in multicomponent stress-strength based on generalized exponential distribution. Recently, Rao et al. [18] studied the estimation of reliability in a multicomponent stress-strength model for Burr-XII distribution and Kizilaslan and Nadar [19] developed an estimation procedure for reliability in multicomponent stress-strength based on Weibull distribution.

The multicomponent stress-strength system consists of k independent and identical strengths component and a common stress, functions when s $(1 \le s \le k)$ or more of the components simultaneously survive. This model corresponds to the s-out-of-k: G system. Multicomponent stress-strength models have great applications range from communication and industrial systems to logistic and military systems. For example, in suspension bridges, the deck is supported by a series of vertical cables hung from the towers. Suppose a suspension bridge consisting of k number of vertical cable pairs. The bridge will only survive if a minimum s number of vertical cable through the deck is not damaged when subjected to stresses due to wind loading, heavy traffic, corrosion etc. For extensive reviews of s-out-of-k and related systems one may refer to Kuo and Zuo [20].

Let the random samples $Y, X_1, X_2, ..., X_k$ be independent, $G(y)$ be the continuous distribution function of Y, and $F(x)$ be the common continuous distribution function of $X_1, X_2, ..., X_k$. The reliability in a multicomponent stress-strength model is given by [14].

$$R_{s,k} = P\left[\text{at least s of the } \left(X_1, X_2, ..., X_k\right) \text{exceed Y}\right] \quad (1.1)$$

$$= \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{\infty} [1-F(y)]^i [F(y)]^{k-i} \, dG(y), \qquad (1.2)$$

where $X_1, X_2, ..., X_k$ are identically independently distributed (*iid*) strength variables and subjected to common random stress, Y. The probability in (1.1) is termed as reliability in a multicomponent stress-strength model [14].

The random variable (*rv*) X follows the proportional hazard rate (PHR) model if the cumulative distribution function (*cdf*) of X have the following form

$$F(x;\theta) = 1 - \left[\bar{H}(x)\right]^{\theta}; x > 0, \qquad (1.3)$$

where $\bar{H}(x) = 1 - H(x)$ is the survival function of the baseline *rv* and $\theta > 0$ is power parameter. The probability density function (*pdf*) corresponding to the *cdf* (1.3) is given by

$$f(x;\theta) = \theta h(x)\left[\bar{H}(x)\right]^{\theta-1}; x > 0, \theta > 0, \qquad (1.4)$$

where $h(x)$ is the first derivative of $H(x)$ with respect to x. A *rv* X following PHR model with power parameter θ will be denoted by $X \sim$ PHR(θ).

The family represented by $f(\cdot)$ and $F(\cdot)$ is well known in lifetime experiments as the PHR model [21]. Ahmadi et al. [22], Wang and Shi [23] and Wang [24], mentioned some of it's particular cases such as exponential, Pareto, Lomax, Burr XII and others.

In this study, we consider the multicomponent stress-strength model which has k independent and identical strength components and a common stress. We assume that the strength variables and stress variable follow PHR model. The system functions if s $(1 \le s \le k)$ or more of the components simultaneously survive. The estimation of reliability for this system is obtained under the classical and Bayesian framework. The Lindley's approximation technique is carried out to obtain Bayesian estimates. Explicit expression for Bayes estimator of reliability is also obtained. Moreover, the asymptotic confidence interval (ACI) for reliability function is constructed.

The rest of the paper is organized as follows: In Section 2, the maximum likelihood (ML) estimator and ACI of $R_{s,k}$ are obtained. In Section 3, uniformly minimum variance unbiased (UMVU) estimator of $R_{s,k}$ is provided. In Section 4, Bayes estimator of $R_{s,k}$ is developed in both approximate and explicit forms under squared error loss function (SELF) [7]. In Section 5, simulation study is carried out to compare the estimates of $R_{s,k}$ by using Monte Carlo simulation technique and findings are illustrated by tables and plots. Finally, conclusions on the paper are provided in Section 6.

2. ML Estimation of $R_{s,k}$

This section deals with the ML estimation of $R_{s,k}$. Here, we assume that $X_1, X_2, ..., X_k, Y$ be independent; $X_1, X_2, ..., X_k \sim$ PHR(θ_1) and $Y \sim$ PHR(θ_2). Therefore from (1.1) and (1.3), $R_{s,k}$ is given by

$$\begin{aligned} R_{s,k} &= \sum_{i=s}^{k} \binom{k}{i} \theta_2 \int_0^{\infty} [1-H(y)]^{\theta_1 i+\theta_2-1} \left[1-(1-H(y))^{\theta_1}\right]^{k-i} h(y)dy \\ &= \sum_{i=s}^{k} \binom{k}{i} \theta_2 \int_0^{1} (1-z)^{\theta_1 i+\theta_2-1} \left(1-(1-z)^{\theta_1}\right)^{k-i} dz, \\ &\text{where } z = H(y) \\ &= \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (-1)^j \theta_2 \int_0^1 (1-z)^{\theta_1(i+j)+\theta_2-1} dz \\ &= \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (-1)^j \phi, \end{aligned} \qquad (2.1)$$

where $\phi = \phi(\theta_1, \theta_2) = \dfrac{\theta_2}{\left[\theta_1(i+j)+\theta_2\right]}$.

In order to obtain the estimators of $R_{s,k}$, suppose n systems are put on life-testing experiment from the strength population and m systems are from the stress population. In this case, we obtain the following observed data: $X_{i1}, X_{i2}, ..., X_{ik}$ and Y_l, $i = 1, 2, ..., n$ and $l = 1, 2, ..., m$. Then, the likelihood function of the observed sample is given by

$$\begin{aligned} L(\theta_1, \theta_2 \mid \underline{x}, \underline{y}) &= \prod_{i=1}^{n} \left(\prod_{j=1}^{k} f_X(x_{ij}; \theta_1) \right) \prod_{l=1}^{m} f_Y(y_l; \theta_2) \\ &= \theta_1^{nk} \theta_2^{n} \prod_{i=1}^{n} \prod_{j=1}^{k} h(x_{ij}) \prod_{i=1}^{n} \prod_{j=1}^{k} \left[1-H(x_{ij})\right]^{\theta_1 - 1} \\ &\quad \prod_{l=1}^{m} h(y_l) \prod_{l=1}^{m} [1-H(y_l)]^{\theta_2 - 1} \end{aligned} \qquad (2.2)$$

and the log-likelihood function is given by

$$\begin{aligned} l(\theta_1, \theta_2) &= \ln L(\theta_1, \theta_2 \mid \underline{x}, \underline{y}) \\ &= nk \ln \theta_1 + m \ln \theta_2 + \sum_{i=1}^{n} \sum_{j=1}^{k} \ln h(x_{ij}) \\ &\quad + (\theta_1 - 1) \sum_{i=1}^{n} \sum_{j=1}^{k} \ln\left[1 - H(x_{ij})\right] \\ &\quad + \sum_{l=1}^{m} \ln h(y_l) + (\theta_2 - 1) \sum_{l=1}^{m} \ln\left[1 - H(y_l)\right]. \end{aligned} \qquad (2.3)$$

From (2.3), the ML estimators of θ_1 and θ_2 are given by

$$\tilde{\theta}_1 = \frac{nk}{S^*} \text{ and } \tilde{\theta}_2 = \frac{m}{T^*}, \quad (2.4)$$

where $S^* = -\sum_{i=1}^{n} \sum_{j=1}^{k} \ln\left[1 - H(x_{ij})\right]$ and

$$T^* = -\sum_{l=1}^{m} \ln\left[1 - H(y_l)\right].$$

Hence, the ML estimator of $R_{s,k}$ is obtained from (2.1) and (2.4) by using the invariance property of ML estimators

$$\tilde{R}_{s,k} = \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (-1)^j \tilde{\phi}, \quad (2.5)$$

where $\tilde{\phi} = \frac{\tilde{\theta}_2}{\left[\tilde{\theta}_1 (i+j) + \tilde{\theta}_2\right]}$.

To obtain the ACI interval for $R_{s,k}$, we proceed as follows:

The Fisher information matrix of $\underline{\theta} = (\theta_1, \theta_2)$ is given as

$$I(\underline{\theta}) = -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial \theta_1^2}\right) & E\left(\frac{\partial^2 l}{\partial \theta_1 \partial \theta_2}\right) \\ E\left(\frac{\partial^2 l}{\partial \theta_2 \partial \theta_1}\right) & E\left(\frac{\partial^2 l}{\partial \theta_2^2}\right) \end{pmatrix} = \begin{pmatrix} \frac{nk}{\theta_1^2} & 0 \\ 0 & \frac{m}{\theta_2^2} \end{pmatrix}.$$

The ML estimator of $R_{s,k}$, $\tilde{R}_{s,k}$, is asymptotically normal with mean $R_{s,k}$ and variance

$$V(R_{s,k}) = \left[\begin{array}{l} \left(\frac{\partial R_{s,k}}{\partial \theta_1}\right)^2 V(\theta_1) \\ + \left(\frac{\partial R_{s,k}}{\partial \theta_2}\right)^2 V(\theta_2) \end{array} \right]_{|(\theta_1, \theta_2) = (\tilde{\theta}_1, \tilde{\theta}_2)},$$

where $V\left(\tilde{\theta}_1\right) = \frac{\theta_1^2}{nk}$, $V\left(\tilde{\theta}_2\right) = \frac{\theta_2^2}{m}$,

$$\frac{\partial R_{s,k}}{\partial \theta_1} = \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (-1)^{(j+1)} \frac{\theta_2 (i+j)}{\left[\theta_1 (i+j) + \theta_2\right]^2}$$

and

$$\frac{\partial R_{s,k}}{\partial \theta_2} = \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (-1)^j \frac{\theta_1 (i+j)}{\left[\theta_1 (i+j) + \theta_2\right]^2}.$$

Therefore, an asymptotic $100(1-\alpha)\%$ confidence interval of $R_{s,k}$ is given by

$$R_{s,k} \in \left(\tilde{R}_{s,k} - z_{\alpha/2}\sqrt{V\left(\tilde{R}_{s,k}\right)}, \tilde{R}_{s,k} - z_{\alpha/2}\sqrt{V\left(\tilde{R}_{s,k}\right)}\right), \quad (2.6)$$

where $z_{\alpha/2}$ is the upper $\alpha/2^{th}$ quantile of the standard normal distribution and $\sqrt{V\left(\tilde{R}_{s,k}\right)}$ is the value of $\sqrt{V\left(R_{s,k}\right)}$ at the ML estimate of the parameteres.

3. UMVU Estimation of $R_{s,k}$

This section deals with the UMVU estimation of $R_{s,k}$. To find the UMVU estimator of $R_{s,k}$, $\hat{R}_{s,k}$ say, it is enough to find UMVU estimator of ϕ by using the linearity property of UMVU estimators. From (2.2), it is seen that (S^*, T^*) is a complete sufficient statistics for (θ_1, θ_2). Moreover, $S^* \sim gamma(nk, \theta_1)$ and $T^* \sim gamma(m, \theta_2)$. Let

$$\psi(S,T) = \begin{cases} 1; S > (i+j)T, \\ 0; \text{otherwise}, \end{cases}$$

where $S = -\ln\left[1 - H(x_{11})\right]$ and $T = -\ln\left[1 - H(y_1)\right]$. Obviously, (S,T) have exponential distribution with means $1/\theta_1$ and $1/\theta_2$, respectively. Then, $\psi(s,T)$ is an unbiased estimator of ϕ.

The UMVU estimator of ϕ, $\hat{\phi}$ say, can be obtained by using Lehmann-Scheffe Theorem and is given by

$$\begin{aligned} \hat{\phi} &= E\left(\psi(S,T) \mid S^* = s^*, T^* = t^*\right) \\ &= P\left(S > (i+j)T \mid S^* = s^*, T^* = t^*\right) \\ &= \iint_{\mathbb{C}} f_{S|S^*=s^*}\left(S \mid S^* = s^*\right) f_{T|T^*=t^*}\left(T \mid T^* = t^*\right) dsdt, \end{aligned}$$

where $\mathbb{C} = \left\{(s,t) : 0 < s < s^*, 0 < t < t^*, s > (i+j)t\right\}$. Notice that, $S \mid S^* = s^* \sim beta_1(1, nk-1)$ and

$$T \mid T^* = t^* \sim beta_1(1, m-1).$$

Thus,

$$\hat{\phi} = \iint_{\mathbb{C}} \frac{(m-1)(nk-1)}{s^* t^*} \left(1 - \frac{s}{s^*}\right)^{nk-2} \left(1 - \frac{t}{t^*}\right)^{m-2} dsdt. \quad (3.1)$$

The integral in (3.1) is considered in two cases, i.e., $(i+j)t^* > s^*$ and $(i+j)t^* < s^*$.

When $(i+j)t^* > s^*$, the integral (3.1) can be expressed as

$$\begin{aligned} \hat{\phi} &= \int_{t=0}^{t^*} \int_{s=t(i+j)}^{s^*} \frac{(m-1)(nk-1)}{s^* t^*} \left(1 - \frac{s}{s^*}\right)^{nk-2} \left(1 - \frac{t}{t^*}\right)^{m-2} dsdt \\ &= \int_{t=0}^{t^*} \frac{(m-1)(nk-1)}{t^*} \left(\int_{z=\frac{t}{s^*}(i+j)}^{s^*} (1-z)^{nk-2} dz \right) \left(1 - \frac{t}{t^*}\right)^{m-2} dt, \end{aligned}$$

where $z = \frac{s}{s^*}$

$$= \int_{t=0}^{t^*} \frac{(m-1)}{t^*}\left(1-\frac{(i+j)t}{s^*}\right)^{nk-1}\left(1-\frac{t}{t^*}\right)^{m-2} dt$$

$$= \int_{w=0}^{1} (m-1)\left(1-\frac{(i+j)t^*}{s^*}w\right)^{nk-1}(1-w)^{m-2}\, dw,$$

where $w = \frac{t}{t^*}$

$$= \int_{w=0}^{1} (m-1)(1-Cw)^{nk-1}(1-w)^{m-2}\, dw,$$

where $C = \frac{t^*(i+j)}{s^*}$

$$= \sum_{l=0}^{nk-1}(-1)^l (m-1)\binom{nk-1}{l} C^l \int_0^1 w^l (1-w)^{m-2} dw$$

$$= \sum_{l=0}^{nk-1}(-1)^l (m-1)\binom{nk-1}{l} C^l B(l+1, m-1). \quad (3.2)$$

Similarly, when $(i+j)t^* < s^*$, the double integral in (3.1) can be expressed as

$$\hat{\phi} = \int_{s=0}^{s^*}\int_{t=0}^{s/t(i+j)} \frac{(m-1)(nk-1)}{s^* t^*}\left(1-\frac{s}{s^*}\right)^{nk-2}\left(1-\frac{t}{t^*}\right)^{m-2} dsdt.$$

Proceeding on the similar lines as earlier, we get

$$\hat{\phi} = 1 - \sum_{l=0}^{m-1}(-1)^l (nk-1) C^{-l}\binom{m-1}{l} B(l+1, nk-1). \quad (3.3)$$

Therefore, $\tilde{R}_{s,k}$ can be obtained by using (2.1), (3.2) and (3.3) as

$$\hat{R}_{s,k} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^j \hat{\phi}. \quad (3.4)$$

4. Bayesian Estimation of $R_{s,k}$

This section deals with the Bayesian estimation of $R_{s,k}$. To obtain the Bayes estimator of $R_{s,k}$, we have considered two independent non-informative priors, $\pi(\theta_1)$ and $\pi(\theta_2)$ say, where

$$\pi(\theta_1) = \frac{1}{\theta_1};\ \theta_1 > 0 \text{ and } \pi(\theta_2) = \frac{1}{\theta_2};\ \theta_2 > 0. \quad (4.1)$$

Looking at (2.2) and (4.1), the joint posterior density of (θ_1, θ_2) comes out to be

$$\pi(\theta_1, \theta_2 \mid \underline{x}, \underline{y}) = \frac{(S^*)^{nk}(T^*)^m}{\Gamma(nk)\Gamma(m)}\theta_1^{nk-1}\theta_2^{m-1} \prod_{i=1}^{n}\prod_{j=1}^{k}\left[1-H(x_{ij})\right]^{\theta_1}\prod_{l=1}^{m}\left[1-H(y_l)\right]^{\theta_2}. \quad (4.2)$$

Therefore, the Bayes estimator of $R_{s,k}$ under SELF is given by

$$\breve{R}_{s,k} = \int_0^\infty\int_0^\infty R_{s,k}\pi(\theta_1, \theta_2 \mid \underline{x}, \underline{y})d\theta_1 d\theta_2. \quad (4.3)$$

4.1. Lindley's Approximation

In this section, we consider the Lindley's approximation technique for the estimation of $R_{s,k}$. To find the Bayes estimator of $R_{s,k}$, $\breve{R}_{s,k}^L$ say, using the Lindley's approximation technique, consider the posterior expectation $I(\underline{x})$ is expressible in the form of ratio of integral as given below

$$I(\underline{x}) = E(\phi \mid \underline{x}) = \frac{\int_{(\theta_1,\theta_2)} \phi e^{l(\theta_1,\theta_2)+\rho(\theta_1,\theta_2)} d(\theta_1,\theta_2)}{\int_{(\theta_1,\theta_2)} e^{l(\theta_1,\theta_2)+\rho(\theta_1,\theta_2)} d(\theta_1,\theta_2)}, \quad (4.4)$$

where $\rho(\theta_1, \theta_2)$ is the log of joint prior of θ_1 and θ_2, given by

$$\rho(\theta_1,\theta_2) = -\ln(\theta_1\theta_2) = -\ln(\theta_1) - \ln(\theta_2). \quad (4.5)$$

If n and m are sufficiently large, according to Lindley [25], $I(\underline{x})$ can be approximately evaluated as

$$I(\underline{x}) = \tilde{\phi} + \frac{1}{2}\Big[\left(\tilde{\phi}_{\theta_1\theta_1} + 2\tilde{\phi}_{\theta_1}\tilde{\rho}_{\theta_1}\right)\tilde{\sigma}_{\theta_1\theta_1} + \left(\tilde{\phi}_{\theta_2\theta_1} + 2\tilde{\phi}_{\theta_2}\tilde{\rho}_{\theta_1}\right)\tilde{\sigma}_{\theta_2\theta_1} + \left(\tilde{\phi}_{\theta_1\theta_2} + 2\tilde{\phi}_{\theta_1}\tilde{\rho}_{\theta_2}\right)\tilde{\sigma}_{\theta_1\theta_2} + \left(\tilde{\phi}_{\theta_2\theta_2} + 2\tilde{\phi}_{\theta_2}\tilde{\rho}_{\theta_2}\right)\tilde{\sigma}_{\theta_2\theta_2}\Big]$$
$$+ \frac{1}{2}\Big[\left(\tilde{\phi}_{\theta_1}\tilde{\sigma}_{\theta_1\theta_1} + \tilde{\phi}_{\theta_2}\tilde{\sigma}_{\theta_1\theta_2}\right)\begin{pmatrix}\tilde{L}_{\theta_1\theta_1\theta_1}\tilde{\sigma}_{\theta_1\theta_1} + \tilde{L}_{\theta_1\theta_2\theta_1}\tilde{\sigma}_{\theta_1\theta_2} \\ +\tilde{L}_{\theta_2\theta_1\theta_1}\tilde{\sigma}_{\theta_2\theta_1} + \tilde{L}_{\theta_2\theta_2\theta_1}\tilde{\sigma}_{\theta_2\theta_2}\end{pmatrix} + \left(\tilde{\phi}_{\theta_1}\tilde{\sigma}_{\theta_2\theta_1} + \tilde{\phi}_{\theta_2}\tilde{\sigma}_{\theta_2\theta_2}\right)\begin{pmatrix}\tilde{L}_{\theta_1\theta_1\theta_2}\tilde{\sigma}_{\theta_1\theta_1} + \tilde{L}_{\theta_1\theta_2\theta_2}\tilde{\sigma}_{\theta_1\theta_2} \\ +\tilde{L}_{\theta_2\theta_1\theta_2}\tilde{\sigma}_{\theta_2\theta_1} + \tilde{L}_{\theta_2\theta_2\theta_2}\tilde{\sigma}_{\theta_2\theta_2}\end{pmatrix}\Big].$$

Thus

$$I(\underline{x}) = \tilde{\phi} + \frac{1}{2}\begin{bmatrix}\left(\tilde{\phi}_{\theta_1\theta_1} + 2\tilde{\phi}_{\theta_1}\tilde{\rho}_{\theta_1}\right)\tilde{\sigma}_{\theta_1\theta_1} \\ +\left(\tilde{\phi}_{\theta_2\theta_2} + 2\tilde{\phi}_{\theta_2}\tilde{\rho}_{\theta_2}\right)\tilde{\sigma}_{\theta_2\theta_2}\end{bmatrix} + \frac{1}{2}\left[\tilde{\phi}_{\theta_1}\tilde{\sigma}^2_{\theta_1\theta_1}\tilde{l}_{\theta_1\theta_1\theta_1} + \tilde{\phi}_{\theta_2}\tilde{\sigma}^2_{\theta_2\theta_2}\tilde{l}_{\theta_2\theta_2\theta_2}\right], \quad (4.6)$$

where,

$$\tilde{\phi} = \frac{\tilde{\theta}_2}{\left[\tilde{\theta}_1(i+j) + \tilde{\theta}_2\right]},\quad \tilde{\phi}_{\theta_1} = -\frac{\tilde{\theta}_2(i+j)}{\left[\tilde{\theta}_1(i+j) + \tilde{\theta}_2\right]^2},$$

$$\tilde{\phi}_{\theta_2} = \frac{\tilde{\theta}_1(i+j)}{\left[\tilde{\theta}_1(i+j) + \tilde{\theta}_2\right]^2},\quad \tilde{\phi}_{\theta_1\theta_1} = \frac{2\tilde{\theta}_2(i+j)^2}{\left[\tilde{\theta}_1(i+j) + \tilde{\theta}_2\right]^3},$$

$$\tilde{\phi}_{\theta_2\theta_2} = -\frac{2\tilde{\theta}_1(i+j)}{\left[\tilde{\theta}_1(i+j)+\tilde{\theta}_2\right]^3}, \tilde{l}_{\theta_1\theta_1\theta_1} = \frac{2nk}{\tilde{\theta}_1^3}, \tilde{l}_{\theta_2\theta_2\theta_2} = \frac{2m}{\tilde{\theta}_2^3},$$

$$\tilde{\rho}_{\theta_1} = -\frac{1}{\tilde{\theta}_1}, \tilde{\rho}_{\theta_2} = -\frac{1}{\tilde{\theta}_2}, \tilde{\sigma}_{\theta_1\theta_1} = -\frac{\tilde{\theta}_1^2}{nk}$$

and $\tilde{\sigma}_{\theta_2\theta_2} = -\frac{\tilde{\theta}_2^2}{m}$.

Therefore,

$$I(\underline{x}) = \frac{\tilde{\theta}_2}{\left[\tilde{\theta}_1(i+j)+\tilde{\theta}_2\right]} + \frac{1}{nkm}\frac{\tilde{\theta}_1\tilde{\theta}_2(i+j)\left[m\tilde{\theta}_1(i+j)-nk\tilde{\theta}_2\right]}{\left[\tilde{\theta}_1(i+j)+\tilde{\theta}_2\right]^3}.$$

Hence,

$$\breve{R}_{s,k}^{L} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^j \frac{\tilde{\theta}_2}{\left[\tilde{\theta}_1(i+j)+\tilde{\theta}_2\right]} \times\left\{1+\frac{\tilde{\theta}_1(i+j)\left[m\tilde{\theta}_1(i+j)-nk\tilde{\theta}_2\right]}{nkm\left[\tilde{\theta}_1(i+j)+\tilde{\theta}_2\right]^2}\right\}. \quad (4.7)$$

4.2. Explicit Expression for Bayes Estimator

Following Kotz et al. [26] and Ventura and Racugno [27], we obtain the posterior *pdf* of $R_{s,k}$ by mean of a one-to-one transformation of the type $U:(\theta_1,\theta_2)\to\left(R_{s,k},\lambda\right)$. So, putting $\phi = \frac{\theta_2}{\left[\theta_1(i+j)+\theta_2\right]}$; $\lambda = \theta_1(i+j)+\theta_2$, taking into account that the Jacobian of the transformation is $\frac{\lambda}{(i+j)}$, by (4.2) the joint *pdf* of (ϕ,λ) is

$$\pi(\phi,\lambda\,|\,\underline{x},\underline{y}) = \frac{(S^*)^{nk}(T^*)^m\phi^{m-1}(1-\phi)^{nk-1}}{\Gamma(nk)\Gamma(m)(i+j)^{nk}} \times\lambda^{nk+m-1}e^{-\left\{\frac{(1-\phi)}{(i+j)}S^*+\phi T^*\right\}\lambda}. \quad (4.8)$$

Consequently, we can obtain the posterior *pdf* of ϕ marginalizing (4.8) with respect to λ, i.e.,

$$\pi(\phi,\lambda\,|\,\underline{x},\underline{y}) = \frac{(i+j)^m}{B(nk,m)}\left(\frac{T^*}{S^*}\right)^m \phi^{m-1}(1-\phi)^{nk-1}(1+b\phi)^{-(nk+m)}, \quad (4.9)$$

where $b = \frac{(i+j)T^*}{S^*}-1$.

The Bayes estimator of ϕ, $\breve{\phi}^E$ say, without using Lindley's approximation technique, under SELF can be easily obtained by using a result of Gradshteyn and Ryzhik ([28], p.286, section 3.197(3)), as

$$\breve{\phi}^E = \begin{cases} \frac{m}{(nk+m)}\left(\frac{(i+j)T^*}{S^*}\right)^m \\ \times{}_2F_1\left(nk+m, n+1, nk+m+1; 1-\frac{(i+j)T^*}{S^*}\right); \\ \qquad (i+j)T^* > 2S^*, \\ \frac{m}{(nk+m)}\left(\frac{S^*}{(i+j)T^*}\right)^m \\ \times{}_2F_1\left(nk+m, nk, nk+m+1; 1-\frac{S^*}{(i+j)T^*}\right); \\ \qquad 2(i+j)T^* > S^*, \end{cases} \quad (4.10)$$

where ${}_2F_1(\cdot,\cdot,\cdot,\cdot)$ denotes the well-known hypergeometric function [see, for example, Gradshteyn and Ryzhik ([29], p.1005, Eq. 9.111)].

Therefore, $\breve{R}_{s,k}^{E}$ can be obtained by using (2.1) and (4.10) as

$$\breve{R}_{s,k}^{E} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^j\breve{\phi}^E. \quad (4.11)$$

4.3. MCMC Method

It is seen that the marginal densities of θ_1 and θ_2 are gamma distribution with parameters (nk,S^*) and (m,T^*), respectively. To obtain the Bayes estimate of $R_{s,k}^{MCMC}$ under SELF the following algorithm is used:

(1) Set i=1

(2) Generate $\theta_1^{(i)}$ from $gamma(nk,S^*)$

(3) Generate $\theta_2^{(i)}$ from $gamma(m,T^*)$

(4) Compute $R_{s,k}^{(i)}$ at $\theta_1^{(i)},\theta_2^{(i)}$

(5) Set i=i+1

(6) Repeat steps 2-5, N times and get the posterior sample $R_{s,k}^{(i)}$, $i=1,2,...,N$.

Then the Bayes estimate of $R_{s,k}^{MCMC}$ under SELF is given by

$$\breve{R}_{s,k}^{MCMC} = \frac{1}{N-M}\sum_{i=M+1}^{N-M}R_{s,k}^{(i)}. \quad (4.12)$$

5. Simulation Study

This section deals with some experimental results to examine the behavior of the proposed methods for different parametric values and sample sizes. Simulation study is carried out by using Monte Carlo simulation technique and comparisons are made on the basis of mean squared errors (MSEs) of different estimates. Throughout the simulation, we have considered exponential distribution by taking $H(x)=1-e^{-x}, x>0$. All the computations are done on *statistical software-R*.

In order to obtain the $R_{s,k}^{MCMC}$, we ran a MCMC chain. We generate 30000 iteration and to diminish the effect of the starting distribution, we discard first 5000 observations and focus on the remaining.

We have generated 3000 random samples each of size n from strength and of size m from the stress populations for different vales of θ_1 and θ_2. For θ_1 =2 and θ_2 =1(1)4, we have computed $R_{s,k}$, average values of $\tilde{R}_{s,k}$, $\hat{R}_{s,k}$, $\breve{R}_{s,k}^{L}$, $\breve{R}_{s,k}^{E}$, $\breve{R}_{s,k}^{MCMC}$ and their corresponding MSEs. We have also computed the ACI and length of the ACI. For different values of n and (s,k) =(1,3), (2,3), these results are reported in Table 1. Under the same set-up for θ_1=3, θ_2 =1(1)4 and different values of n, m and (s,k) =(1,4), (2,4), the results are presented in Table 2.

In order to compare the performances of different estimators of, $R_{s,k}$ graphically, for different values of n, m, we have conducted the simulation experiment based on the above mentioned procedure. For θ_1 =2 and θ_2 =1(1)4, we have computed the MSEs and the biases corresponding to the different estimators of $R_{s,k}$. For different values of n, m and (s,k) =(1,3), obtained MSEs and biases have scaled by multiplying 10^2, thereafter these results are plotted in Figure 1-3, respectively.

Table 1. Estimates of $R_{s,k}$

$\theta_1=2$ $R_{1,3}\downarrow$	$\theta_2\downarrow$	$n\downarrow$	$m\downarrow$	$\tilde{R}_{s,k}\downarrow$	$\hat{R}_{s,k}\downarrow$	$\breve{R}_{s,k}^{L}\downarrow$	$\breve{R}_{s,k}^{E}\downarrow$	$\breve{R}_{s,k}^{MCMC}\downarrow$	ACI ↓	Length of ACI ↓
0.5428571	1	10	10	0.5554(0.0127)	0.5445(0.0138)	0.546(0.0115)	0.5463(0.0116)	0.5463(0.0116)	(0.3435,0.7672)	0.4237
		20	20	0.5452(0.0059)	0.5395(0.0062)	0.5407(0.0056)	0.5408(0.0056)	0.5408(0.0056)	(0.3920,0.6985)	0.3065
		20	30	0.545(0.0045)	0.5419(0.0046)	0.5429(0.0043)	0.5429(0.0043)	0.5429(0.0043)	(0.4116,0.6784)	0.2668
		30	30	0.5458(0.0042)	0.542(0.0044)	0.5428(0.0041)	0.5429(0.0041)	0.5429(0.0041)	(0.4199,0.6717)	0.2518
0.75	2	10	10	0.7524(0.0089)	0.7523(0.0099)	0.736(0.0087)	0.7364(0.0087)	0.7364(0.0087)	(0.5666,0.9381)	0.3715
		20	20	0.7497(0.0046)	0.7496(0.0049)	0.7413(0.0045)	0.7414(0.0045)	0.7414(0.0045)	(0.6153,0.8842)	0.2688
		20	30	0.7487(0.0036)	0.7497(0.0037)	0.7434(0.0036)	0.7435(0.0036)	0.7435(0.0036)	(0.6316,0.8658)	0.2342
		30	30	0.7487(0.0032)	0.7485(0.0034)	0.743(0.0032)	0.743(0.0032)	0.743(0.0032)	(0.6381,0.8593)	0.2212
0.847619	3	10	10	0.8419(0.0055)	0.8464(0.0059)	0.8248(0.0061)	0.825(0.006)	0.825(0.006)	(0.6925,0.9912)	0.2987
		20	20	0.8449(0.0028)	0.8472(0.0029)	0.836(0.003)	0.8361(0.003)	0.836(0.003)	(0.7396,0.9502)	0.2106
		20	30	0.8462(0.002)	0.8488(0.0021)	0.8403(0.0021)	0.8404(0.0021)	0.8404(0.0021)	(0.7553,0.9372)	0.1819
		30	30	0.8457(0.0018)	0.8473(0.0019)	0.8397(0.0019)	0.8397(0.0019)	0.8397(0.0019)	(0.7597,0.9317)	0.1720
0.9	4	10	10	0.8954(0.0034)	0.9015(0.0035)	0.88(0.004)	0.8798(0.004)	0.8798(0.004)	(0.7787,1.0122)	0.2335
		20	20	0.8968(0.0016)	0.9(0.0016)	0.8887(0.0018)	0.8887(0.0018)	0.8887(0.0018)	(0.8153,0.9784)	0.1631
		20	30	0.8976(0.0013)	0.9006(0.0013)	0.8922(0.0014)	0.8922(0.0014)	0.8922(0.0014)	(0.8274,0.9678)	0.1404
		30	30	0.8983(0.0011)	0.9004(0.0011)	0.8928(0.0012)	0.8928(0.0012)	0.8928(0.0012)	(0.8324,0.9642)	0.1318
$R_{2,3}\downarrow$										
0.3142857	1	10	10	0.3291(0.0083)	0.3139(0.0082)	0.3279(0.0078)	0.3279(0.0078)	0.327990.0078)	(0.1592,0.4990)	0.3398
		20	20	0.3219(0.0039)	0.3142(0.0038)	0.3215(0.0038)	0.3215(0.0038)	0.3215(0.0038)	(0.2020,0.4417)	0.2397
		20	30	0.3192(0.0029)	0.3145(0.0029)	0.32(0.0028)	0.32(0.0028)	0.32(0.0028)	(0.2158,0.4227)	0.2070
		30	30	0.3198(0.0025)	0.3147(0.0025)	0.3196(0.0025)	0.3196(0.0025)	0.3196(0.0025)	(0.2220,0.4175)	0.1955
0.5	2	10	10	0.5154(0.0106)	0.5041(0.0113)	0.5073(0.0096)	0.5076(0.0096)	0.5076(0.0096)	(0.3133,0.7175)	0.4042
		20	20	0.5072(0.0057)	0.5014(0.0059)	0.5034(0.0054)	0.5034(0.0054)	0.5034(0.0054)	(0.3622,0.6523)	0.2901
		20	30	0.5035(0.004)	0.5003(0.0041)	0.5019(0.0039)	0.5019(0.0039)	0.5019(0.0039)	(0.3773,0.6297)	0.2524
		30	30	0.5063(0.0039)	0.5024(0.004)	0.5038(0.0038)	0.5038(0.0038)	0.5038(0.0038)	(0.3872,0.6255)	0.2383
0.6190476	3	10	10	0.6285(0.0107)	0.6224(0.0117)	0.6162(0.0099)	0.6166(0.0099)	0.6166(0.0099)	(0.4304,0.8266)	0.3962
		20	20	0.6221(0.0053)	0.6189(0.0056)	0.6159(0.0051)	0.616(0.0051)	0.616(0.0051)	(0.4785,0.7657)	0.2872
		20	30	0.6181(0.0039)	0.6168(0.0041)	0.6147(0.0038)	0.6147(0.0038)	0.6147(0.0038)	(0.4928,0.7434)	0.2506
		30	30	0.6202(0.0038)	0.618(0.0039)	0.616(0.0037)	0.6161(0.0037)	0.6161(0.0037)	(0.5021,0.7382)	0.2361
0.7	4	10	10	0.7012(0.0091)	0.6988(0.01)	0.6867(0.0088)	0.6871(0.0088)	0.687(0.0088)	(0.5152,0.8871)	0.3720
		20	20	0.7019(0.0046)	0.7006(0.0049)	0.6944(0.0045)	0.6945(0.0045)	0.6945(0.0045)	(0.5681,0.8356)	0.2675
		20	30	0.6995(0.0035)	0.6997(0.0037)	0.695(0.0035)	0.6951(0.0035)	0.6951(0.0035)	(0.5829,0.8162)	0.2334
		30	30	0.7012(0.0031)	0.7004(0.0032)	0.6962(0.0031)	0.6963(0.0031)	0.6963(0.0031)	(0.5913,0.8112)	0.2199

Table 2. Estimates of $R_{s,k}$

$\theta_1=3$ $R_{1,4}\downarrow$	$\theta_2\downarrow$	$n\downarrow$	$m\downarrow$	$\tilde{R}_{s,k}\downarrow$	$\hat{R}_{s,k}\downarrow$	$\breve{R}^{L}_{s,k}\downarrow$	$\breve{R}^{E}_{s,k}\downarrow$	$\breve{R}^{MCMC}_{s,k}\downarrow$	ACI↓	Length of ACI↓
0.5428571	1	10	10	0.4832(0.0117)	0.4676(0.0122)	0.4756(0.0106)	0.4759(0.0107)	0.4759(0.0107)	(0.2771,0.6893)	0.4122
		20	20	0.4748(0.0059)	0.4668(0.006)	0.4712(0.0056)	0.4713(0.0056)	0.4713(0.0056)	(0.3272,0.6223)	0.2950
		20	30	0.4718(0.0042)	0.467(0.0043)	0.4703(0.0041)	0.4703(0.0041)	0.4703(0.0041)	(0.3451,0.5985)	0.2534
		30	30	0.4722(0.0039)	0.4656(0.0045)	0.4699(0.0037)	0.4699(0.0038)	0.4699(0.0037)	(0.3512,0.5931)	0.2419
0.75	2	10	10	0.6898(0.0114)	0.684(0.0128)	0.674(0.0108)	0.6745(0.0108)	0.6745(0.0108)	(0.4877,0.8920)	0.4043
		20	20	0.6863(0.0055)	0.6831(0.0058)	0.6782(0.0053)	0.6783(0.0053)	0.6783(0.0053)	(0.5393,0.8333)	0.2940
		20	30	0.6836(0.0042)	0.6822(0.0043)	0.6786(0.0041)	0.6787(0.0041)	0.6787(0.0041)	(0.5567,0.8105)	0.2538
		30	30	0.6864(0.0038)	0.6836(0.0035)	0.6809(0.0037)	0.681(0.0037)	0.681(0.0037)	(0.5655,0.8073)	0.2418
0.847619	3	10	10	0.7987(0.0071)	0.7996(0.008)	0.7802(0.0074)	0.7805(0.0073)	0.7805(0.0073)	(0.6258,0.9715)	0.3456
		20	20	0.7992(0.0039)	0.7996(0.0041)	0.7896(0.004)	0.7897(0.004)	0.7897(0.004)	(0.6754,0.9230)	0.2476
		20	30	0.7986(0.0029)	0.7997(0.003)	0.7923(0.0029)	0.7924(0.0029)	0.7924(0.0029)	(0.6919,0.9052)	0.2133
		30	30	0.8007(0.0027)	0.8004(0.0031)	0.7943(0.0027)	0.7943(0.0027)	0.7943(0.0027)	(0.6995,0.9020)	0.2026
0.9	4	10	10	0.8609 (0.005)	0.8651(0.0054)	0.8428(0.0057)	0.8428(0.0056)	0.8428(0.0056)	(0.7180,1.0038)	0.2858
		20	20	0.8639(0.0025)	0.8661(0.0026)	0.8545(0.0027)	0.8545(0.0027)	0.8545(0.0027)	(0.7634,0.9645)	0.2011
		20	30	0.8652(0.0018)	0.8675(0.0019)	0.8589(0.0019)	0.8589(0.0019)	0.8589(0.0019)	(0.7794,0.9510)	0.1716
		30	30	0.8664(0.0016)	0.8683(0.0017)	0.86(0.0017)	0.86(0.0017)	0.86(0.0017)	(0.7850,0.9478)	0.1628
$R_{1,4}\downarrow$										
0.3142857	1	10	10	0.3074(0.0078)	0.2906(0.0073)	0.3059(0.0073)	0.306(0.0073)	0.306(0.0073)	(0.1441,0.4707)	0.3265
		20	20	0.296 (0.0035)	0.2877(0.0034)	0.2956(0.0034)	0.2956(0.0034)	0.2955(0.0034)	(0.1826,0.4094)	0.2267
		20	30	0.2926(0.0024)	0.2873(0.0023)	0.2931(0.0023)	0.2932(0.0023)	0.2931(0.0023)	(0.1962,0.3891)	0.1929
		30	30	0.2926(0.0022)	0.2862(0.002)	0.2924(0.0021)	0.2925(0.0022)	0.2924(0.0021)	(0.2007,0.3846)	0.1840
0.5	2	10	10	0.4907(0.012)	0.4759(0.0126)	0.4826(0.0109)	0.4829(0.0109)	0.4829(0.0109)	(0.2869,0.6944)	0.4075
		20	20	0.4812(0.0058)	0.4736(0.0059)	0.4774(0.0055)	0.4775(0.0055)	0.4775(0.0055)	(0.3350,0.6273)	0.2923
		20	30	0.4778(0.0041)	0.4732(0.0041)	0.4761(0.0039)	0.4761(0.0039)	0.4761(0.0039)	(0.3522,0.6034)	0.2511
		30	30	0.4816(0.0038)	0.4805(0.0055)	0.4791(0.0036)	0.4791(0.0036)	0.4791(0.0036)	(0.3614,0.6017)	0.2403
0.6190476	3	10	10	0.6098(0.0116)	0.6(0.0128)	0.5969(0.0107)	0.5974(0.0107)	0.5974(0.0107)	(0.4027,0.8170)	0.4142
		20	20	0.6056(0.006)	0.6005(0.0063)	0.5991(0.0058)	0.5992(0.0058)	0.5992(0.0058)	(0.4558,0.7554)	0.2996
		20	30	0.6049(0.0047)	0.6022(0.0048)	0.6012(0.0045)	0.6013(0.0045)	0.6012(0.0045)	(0.4759,0.7339)	0.2580
		30	30	0.6019(0.0038)	0.5985(0.0037)	0.5976(0.0037)	0.5977(0.0037)	0.5977(0.0037)	(0.4784,0.7254)	0.2470
0.7	4	10	10	0.6966(0.0103)	0.6917(0.0115)	0.6807(0.0097)	0.6812(0.0097)	0.6812(0.0097)	(0.5003,0.8930)	0.3927
		20	20	0.6893(0.0057)	0.6865(0.006)	0.6812(0.0056)	0.6813(0.0056)	0.6813(0.0056)	(0.5467,0.8318)	0.2851
		20	30	0.6892(0.0039)	0.6881(0.0041)	0.6842(0.0039)	0.6842(0.0039)	0.6843(0.0039)	(0.5662,0.8122)	0.2460
		30	30	0.6885(0.0036)	0.6856(0.0039)	0.6831(0.0036)	0.6831(0.0036)	0.6831(0.0036)	(0.571,0.806)	0.235

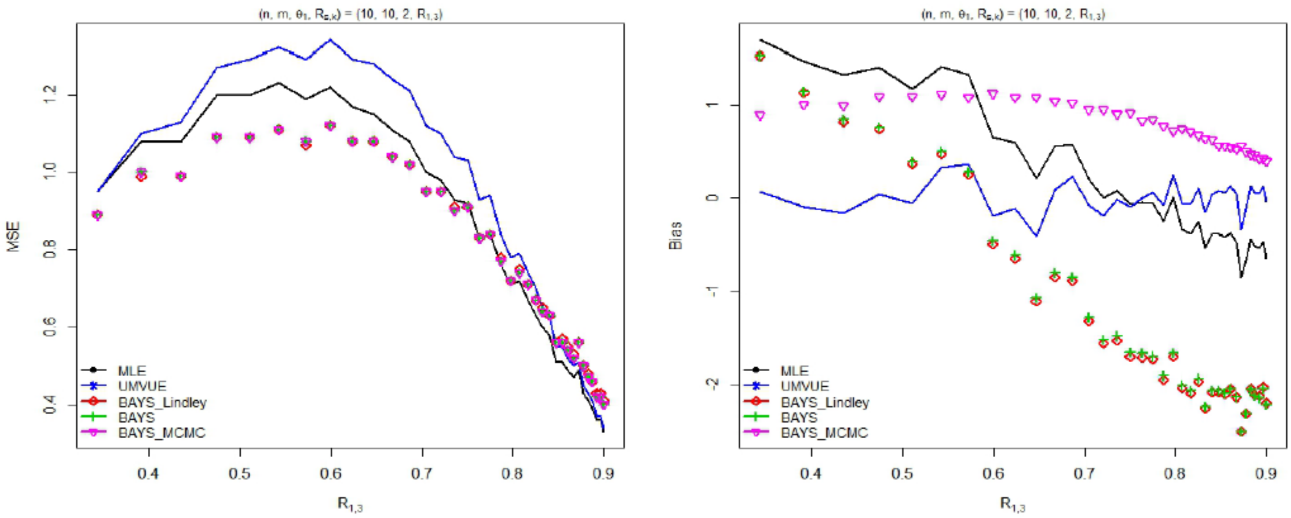

Figure 1. MSEs and Biases of $R_{1,3}$, for n = m=10

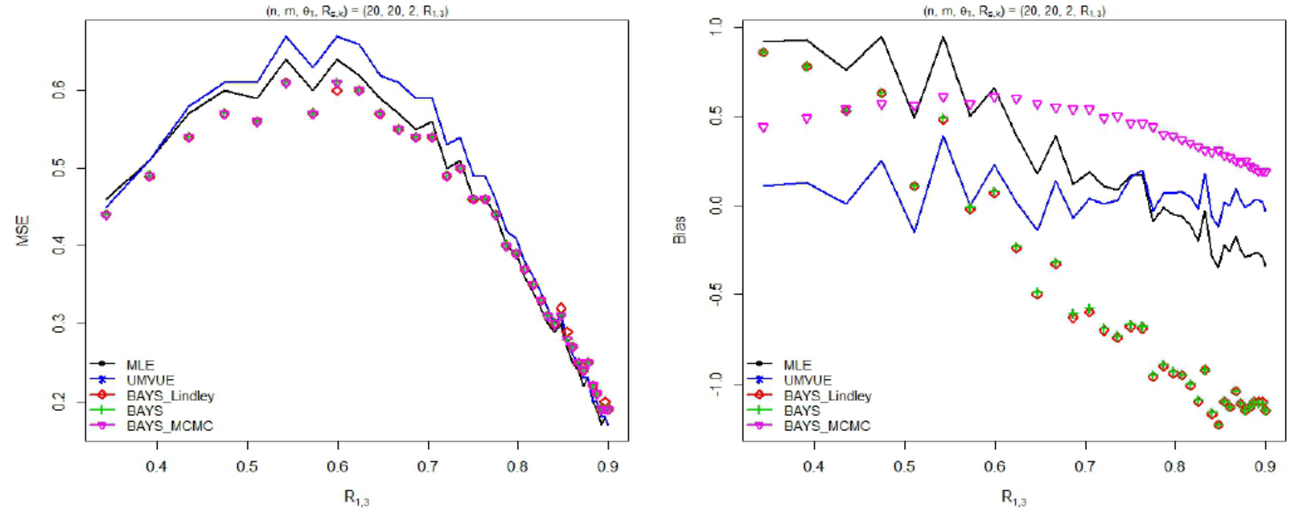

Figure 2. MSEs and Biases of $R_{1,3}$, for n = m=20

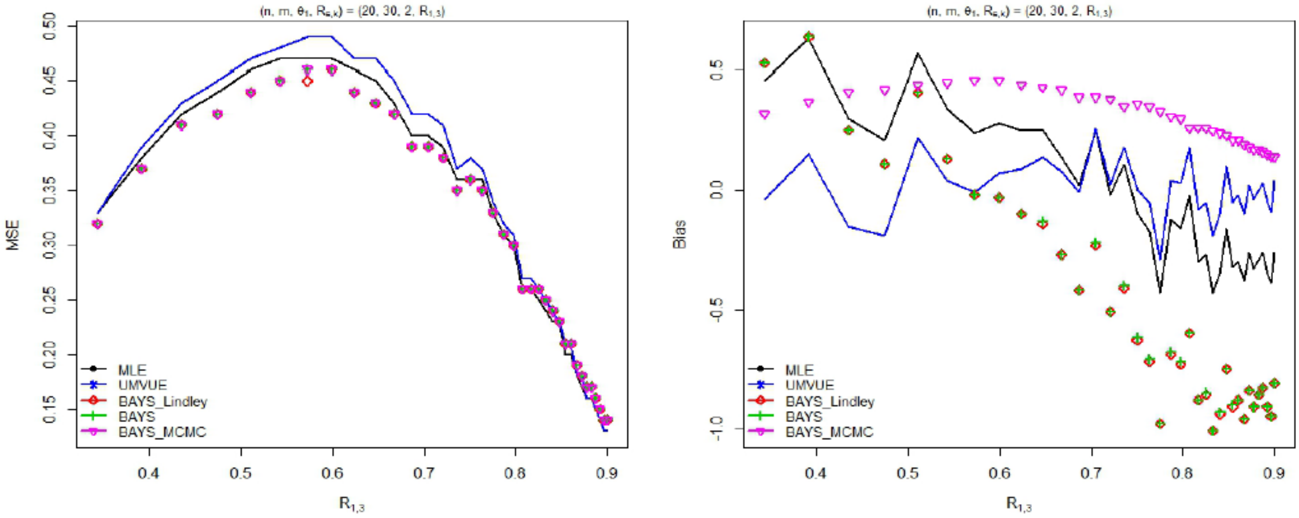

Figure 3. MSEs and Biases of $R_{1,3}$, for n = m=30

6. Conclusions

The Table 1 - Table 2 and Figure 1 - Figure 3, illustrate the following:

(1) Minimum MSE is depicted by the Bayes estimators of $R_{s,k}$,

(2) both the Bayes estimators of $R_{s,k}$ depicting the same behavior of MSEs and biases,

(3) the MSEs and biases of the estimates decreases when the sample size increases,

(4) all the estimates come close to each other when the sample size increases,

(5) when $R_{s,k}$ is around 0.5 the corresponding MSEs are maximum,

(6) when $R_{s,k}$ is small or large the corresponding MSEs are minimum for all estimates, and

(7) length of the ACI decreases as sample size increase.

Here, we have studied the multicomponent system, which has k independent and identical strength components and each component is exposed to a common random stress, where the underlying distribution of stress and strength variables is assumed to be PHR model. Using Monte Carlo simulation technique UMVU, ML and Bayes estimates are obtained and compared on the basis of their corresponding MSEs. Further this paper give the clear picture of comparison of classical and Bayesian methods of estimation for the members of PHR family of distributions.

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Conflicts of Interest

Both authors declare no competing interest.

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