

On an Integer Sequence via Euler ϕ Function

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Received February 11, 2019; Revised March 20, 2019; Accepted May 08, 2019

Abstract Similar to Fibonacci sequence and Lucas sequences that are recursively defined we define an integer sequence using the Euler totient function ϕ and study some of its properties. We also verify that the sequence we have defined has some properties similar to Fibonacci sequence, but even then it is not a Lucas sequence.

Keywords: Fibonacci sequence, Lucas sequence, Euler ϕ function, integer sequences, perfect numbers

Cite This Article: Binu K P, Vinod S, and K Vishnu Namboothiri, "On an Integer Sequence via Euler ϕ Function." *American Journal of Applied Mathematics and Statistics*, vol. 7, no. 3 (2019): 112-114. doi: 10.12691/ajams-7-3-5.

1. Introduction

There have been various attempts in the recent literature to study properties related to integer sequences similar to Fibonacci sequence which is defined recursively by $F_0 = 1$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$. Fibonacci sequence has many interesting properties like even being consisting of very large terms after a few number of initial terms, the ratio of consecutive terms converges to a finite number. Another interesting property of Fibonacci sequence is that it has no perfect numbers in it (See [1]). A larger class of sequences is defined generalizing the Fibonacci sequence called Lucas sequences. They also share some of the important properties of Fibonacci sequences. In this article, we define an integer sequence via the arithmetical function. Euler ϕ function and try to obtain some properties of this sequence contrasting with the Fibonacci sequence

2. Notations and Basic Results

In what following, ϕ will denote the Euler totient function. Therefore, $\phi(n)$ is the number of positive integers $< n$ that are co prime to n , $n \geq 1$. By $\tau(n)$, we mean the number of divisors of n . $\sigma(n)$ will denote the sum of divisors of n including n itself. n is said to be perfect if $\sigma(n) = 2n$. Some examples for perfect numbers are 6 and 28.

Now we state a classical result on convergence of real numbers called as the Stolz-Cesaro theorem:

Theorem 2.1. [[3], Chapter 3, Theorem 1.2.2]

Let (a_n) be a sequence of real numbers and (b_n) a strictly monotone and divergent sequence. Then

$$\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = l \quad (l \in [-\infty, +\infty])$$

Implies

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = l.$$

The Fibonacci sequence, as we mentioned above in the introduction is defined recursively as $F_0 = 1$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For a fixed pair of integers (p, q) , Lucas sequence is defined as choosing l_0, l_1 arbitrary, and defining $l_n = pl_{n-1} - ql_{n-2}$. If we let $p = 1$ and $q = -1$, it becomes the definition of the Fibonacci sequence.

It is known that Fibonacci sequence or Lucas sequence do not contain any perfect numbers [1].

3. ϕ -Integer Sequences and Some of Its Properties

We now state a lemma which we are going to use frequently.

Lemma 3.1. The ϕ function satisfies the following bounds:

$$(1) \phi(n) \leq n-1 \quad \text{for } n \geq 2$$

$$(2) \phi(2n) \leq n \quad \text{for } n \geq 1$$

$$(3) \phi(n) \geq \frac{\sqrt{n}}{2} \quad \text{for } n \geq 1$$

Proof. The statement (1) is obvious. To see (2) note that for every $2n$, half of the number of numbers below or equal to that are even and so they are not co-prime to $2n$. Hence a maximum of n of them only can be co-prime to $2n$. Hence statement (2) follows. The claim in (3) has been established by J. Browkin while trying to see the weaker

result $\lim_{n \rightarrow \infty} \phi(n) = \infty$. This proof can be found, for example in [[4], Chapter 6, Theorem 4].

We now define a sequence (a_n) recursively as follows. Take $a_0 = 1$. Define $a_n = a_{n-1} + \phi(n), n \geq 1$. Some initial values of a_n are 1, 2, 3, 5, 7, 11, 13, 19, 23, 29, 33 . . . This sequence is clearly a monotone increasing sequence and goes to ∞ with n . Let us call it as the phi-sequence. This sequence is in fact called as the totient summatory function in the literature. We now show a_n can be restricted above by some polynomial in n . We summarize our claims in the next two propositions.

Proposition 3.2. $a_{2n} \leq 2 - \frac{1}{2}n + \frac{3}{2}n^2$ for every n .

Proof. By definition $a_{2n} = 1 + \phi(1) + \dots + \phi(2n)$. We now split this over even and odd indices. Thus

$$a_{2n} = 1 + \sum_{k \leq 2n, k \text{ even}} \phi(k) + \sum_{k \leq 2n, k \text{ odd}} \phi(k).$$

If k is even, then by lemma (3.1), $\phi(k) \leq \frac{k}{2}$ and if k odd (or in fact for any k), $\phi(k) \leq k - 1$. Then

$$\begin{aligned} a_{2n} &\leq 1 + \left(\frac{2}{2} + \frac{4}{2} + \dots + \frac{2n}{2} \right) \\ &\quad + (1 + (3-1) + (5-1) + \dots + (2n-1-1)) \\ &= 2 + \frac{n(n+1)}{2} + 2(1 + 2 + \dots + (n-1)) \\ &= 2 + \frac{n(n+1)}{2} + 2 \frac{n(n-1)}{2} \\ &= 2 + \frac{n}{2}(3n-1) \end{aligned}$$

which is what we claimed.

Proposition 3.3. $a_{2n+1} \leq 2 - \frac{3}{2}n + \frac{3}{2}n^2$ for every n .

Proof. Note that $a_{2n+1} = a_{2n} + \phi(2n+1) \leq a_{2n} + 2n$. Now use the bound obtained in the above proposition to conclude.

Fibonacci sequence is monotonically increasing to ∞ with n . One of the key properties of Fibonacci sequence is that the ratio of consecutive terms converges to the number $\phi = \frac{1 + \sqrt{5}}{2} = 1.6180339887 \dots$. Though the terms

are increasing by large leaps, the ratio of terms $\frac{F_{n+1}}{F_n}$ surprisingly approaches a finite number. In the next result, we show that the sequence we defined above has the same property. That is, the limit of ratio of terms $\frac{a_{n+1}}{a_n}$ converges to a finite number.

Theorem 3.4. The sequence of ratios $\left(\frac{a_{n+1}}{a_n} \right)$ converges to 1 as $n \rightarrow \infty$.

Proof. Since $a_n = 1 + \phi(1) + \dots + \phi(n)$ we have

$$\frac{a_{n+1}}{a_n} = 1 + \frac{\phi(n+1)}{1 + \phi(1) + \dots + \phi(n)}.$$

By (1) and (3) in lemma (3.1),

$$\begin{aligned} \frac{a_{n+1}}{a_n} &\leq 1 + \frac{n}{1 + \frac{\sqrt{1}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \dots + \frac{\sqrt{n}}{2}} \\ &= 1 + \frac{1}{\frac{1}{n} + \frac{1}{2n} (\sqrt{1} + \sqrt{2} + \dots + \sqrt{n})} \end{aligned}$$

Write $x_n = \sqrt{1} + \sqrt{2} + \dots + \sqrt{n}$ and $y_n = n$. Then

$$\frac{x_{n+1} - x_n}{y_{n+1} - y_n} = \frac{\sqrt{n+1}}{1} \rightarrow \infty$$

by theorem (2.1), we have $\frac{x_n}{y_n} \rightarrow \infty$. Thus $\frac{a_{n+1}}{a_n}$ is bounded above by a sequence which converges to 1. Also, since $\frac{a_{n+1}}{a_n} \geq 1$ by definition, it is bounded below by the constant sequence (1). Hence by squeeze theorem for real number sequences, $\frac{a_{n+1}}{a_n}$ also converges to 1.

We now prove that the sequence we have defined is not a Lucas sequence.

Theorem 3.5. There do not exist integers r, s such that $a_n = ra_{n-1} - sa_{n-2}$ which works for every n . In other words no tail of the ϕ sequence is Lucas.

Proof. On the contrary, suppose such r, s exists. Then since $a_n = a_{n-1} + \phi(n)$ we get

$$1 = r \frac{a_{n-1}}{a_n} - s \frac{a_{n-2}}{a_n} = r \frac{a_{n-1}}{a_n} - s \frac{a_{n-1}}{a_n} + s \frac{\phi(n-1)}{a_n}.$$

Now the $\frac{a_{n-1}}{a_n} \rightarrow 1$. Since $\frac{\phi(n-1)}{a_n} < \frac{n}{a_n} \rightarrow 0$ as can be seen in the proof of theorem (3.4). So as $n \rightarrow \infty$ we get $r - s \rightarrow 1$. Since r, s fixed integers, $r - s = 1$. Thus

$$\begin{aligned} a_n &= (1 + s)a_{n-1} - sa_{n-2} \\ &= (1 + s)[a_{n-2} + \phi(n-1)] - sa_{n-2} \\ &= a_{n-1} + s\phi(n-1). \end{aligned}$$

Therefore $s\phi(n-1) = \phi(n)$. Note that s is positive by this equation. If we let $n - 1$ to be a prime, then $\phi(n-1) = n - 2$. Since n is even, $\phi(n) \leq \frac{n}{2}$. If we take $s\phi(n-1) = \phi(n) = c$, this means that $n - 2 \leq c \leq \frac{n}{2}$, which is not true for $n > 4$. So such an s cannot exist. The infinitude of primes proves our claim.

We have already seen that Lucas sequences do not contain perfect numbers. Though we are unable to state such a result for our phi-sequence, we have experimental evidence to believe that such a statement holds for our sequence as well. We have tested it for all terms up to 10^6

(with the support of SageMath software) and could not locate a single perfect number. So we state it as a conjecture and leave it for future studies:

Conjecture 3.6. The ϕ sequence does not contain any perfect numbers.

There are many other questions one may find interesting with these type of sequences. For example, the

set $\left\{ \frac{\phi(F_{n+1})}{\phi(F_n)} : n \geq 1 \right\}$ is dense in the set of all positive

numbers (see [2]). Such a similar result can be discussed for our phi-sequence also. Further, we believe that similar properties may hold if we modify our sequence replacing

the phi-function with τ or σ . We plan to discuss these sequences in a sequel to this article.

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