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Reliability Estimation in Multicomponent Stress-strength Model based on Generalized Pareto Distribution

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Abstract The paper deals with the estimation of multicomponent system reliability where the system has k components with their strengths $X_1, X_2, \ldots X_k$ being independently and identically distributed random variables and each component is experiencing a random stress Y. The s-out-of-k system is said to function if atleast s out of k ($1 \leq s \leq k$) strength variables exceed the random stress. The reliability of such a system is derived when both strength and stress variables follow generalized Pareto distribution. The system reliability is estimated using maximum likelihood and Bayesian approaches. The maximum likelihood estimators are derived under both simple random sampling and ranked set sampling schemes. Lindley's approximation technique is used to get approximate Bayes estimators. The reliability estimators obtained from both the methods are compared by using mean squares error criteria and real data analysis is carried out to illustrate the procedure.

Keywords: *generalized Pareto distribution, stress-strength reliability, ranked set sampling, simple random sampling, maximum likelihood estimator, Bayes estimator*

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1. Introduction

The study of stress-strength model has received attention by the researchers because of its practical applications in the field of science and technology. The "stress-strength" reliability can be described as an assessment of reliability of a component in terms of random variable Y representing "stress" experienced by the component and X representing "strength" of a component available to overcome the possible stress, if the stress exceeds the strength, then the system will fail. The idea of stress-strength reliability $R = P(X > Y)$ was introduced by Birnbaum [1] and developed by Birnbaum and McCarty [2]. Estimation of reliability in stress-strength model are considered in the literature when the stress and strength variables follow distributions such as exponential, Weibull, normal, gamma etc. Raqab et. al., [3] estimated the reliability of the model for a 3-parameter generalized exponential distribution. Wong [4] obtained confidence intervals for $P(X > Y)$ when the underlying distribution is generalized Pareto. Angali et al. [5] considered Bayesian estimators of reliability for four parameter of bivariate exponential distribution under different loss functions.

The estimation of multicomponent stress-strength reliability for s-out-of-k and the other related systems have been extensively investigated by many authors in the literature. An s-out-of-k system functions when the system having k statistically independent and identically distributed components functions if s ($1 \leq s \leq k$) or more components with stand a common stress, which was first studied by Bhattacharyya and Johnson [6]. This type of systems can be seen in both industrial and military applications [7]. Estimation of multicomponent stress-strength reliability is considered for the log-logistic, generalized exponential, generalized inverted exponential, Rayleigh, Burr Type XII and generalized Rayleigh distributions respectively by Rao and Kantam [8], Rao [9,10,11,12] and Rao et al. [13]. Pandit and Kantu [14] considered estimation of multicomponent stress-strength reliability for parallel and series systems when strength and stress variables follow exponential distribution. Recently, Kizilaslan and Nadar [15] considered estimation of s-out-of-k stress-strength reliability using both classical and Bayesian approach when underlying distribution is Weibull.

In this paper, s-out-of-k system in stress-strength environment which has k independent and identical strength components and a common stress is studied. These kinds of situations may occur in real life. For example, in an electrical power station containing eight generating units which produce the electricity only if at least six units are working; the demand for the electricity of a district is fulfilled only if s-out-of-k wind rose are

operating at all times and in a communication system for a navy can be successful only if six transmitters out of ten are operational to cover a district [15]. This paper considers the above problem when the underlying distribution are to follow generalized Pareto for the strength and stress variables.

Rezaei et al. [16] proposed generalized Pareto distribution by assuming X and Y as independent generalized Pareto distribution with common scale parameter and different shape parameter.

The probability density function and cumulative function of generalized Pareto distribution are given by:

$$f(x) = \alpha\lambda(1+\lambda x)^{-\alpha+1}; x > 0, \alpha, \lambda > 0$$

and

$$F(x) = 1-(1+\lambda x)^{-\alpha}; x > 0, \alpha, \lambda > 0,$$

where α and λ are the shape and scale parameters, respectively.

Here, reliability in a multicomponent stress-strength based on X, Y are two independent random variables, which follow generalized Pareto distributions with shape parameters α and β and with common scale parameter λ.

Let the random variables Y, X_1, X_2, … X_k be independent, $G(y)$ be the cummulative distribution function of Y and $F(x)$ be the common cummulative distribution function of X_1, X_2, … X_k. The reliability in a multicomponent stress-strength model developed by Bhattacharyya and Johnson [6] is

$$\begin{aligned} R_{s,k} &= P\left[at\ least\ s\ of\ the\ X_1, X_2, ..., X_k\ exceed\ Y\right] \\ &= \sum_{i=s}^{k}\binom{k}{i}\int_{-\infty}^{\infty}\left[1-F(y)\right]^i\left[F(y)\right]^{k-i}dG(y). \end{aligned} \quad (1)$$

The estimators of multicomponent system reliability are derived using maximum likelihood method under simple random sampling (SRS) and ranked set sampling (RSS) schemes. Also the Bayesian estimates are obtained using Lindley's approximation. The RSS method was introduced by McIntyre [17] and several authors are interested to study statistical inference related to reliability under RSS, as it has applications in different fields such as, reliability [18,19], environment [20,21,22].Estimation of reliability based on RSS is considered by Sengupta and Mukhati [23]. Muttlak et al. [24] estimated reliability when X and Y follow exponential distribution. Hussian [25] discussed estimation of stress-strength model for generalized inverted exponential distribution based on RSS and SRS, using maximum likelihood method to estimate R. Hassan et al. [26] studied the estimation of R when X and Y are independently distributed Burr XII random variables under different sampling schemes.

In section 2, reliability of the system is derived. Section 3 and 4, deals with Maximum likelihood estimation of $R_{s,k}$ based on Simple random sampling (SRS) and ranked set sampling(RSS) respectively. In section 5, Bayes estimators of $R_{s,k}$ is obtained using Lindley's approximation. Section 6 is devoted to simulation study in which the comparison of estimators of reliability are studied. In section 7, real data application is given and the conclusions are given in section 8.

The multicomponent stress-strength reliability

In this section multicomponent system reliability is considered when X and Y follow generalized Pareto distribution with parameters (α_1, λ) and (α_2, λ) respectively,

$$\begin{aligned} R_{s,k} &= \sum_{i=s}^{k}\binom{k}{i}\int_0^{\infty}\left\{\begin{matrix}\left[(1+\lambda y)^{-\alpha_1}\right]^i\left[1-(1+\lambda y)^{-\alpha_1}\right]^{k-i} \\ \alpha_2\lambda(1+\lambda y)^{-(\alpha_2+1)}\end{matrix}\right\}dy \\ &= \sum_{i=s}^{k}\binom{k}{i}\alpha_2\int_1^{\infty}[t^{-\alpha_1}]^i[1-t^{-\alpha_1}]^{k-i}t^{-(\alpha_2+1)}dt \end{aligned}$$

Where $1+\lambda y = t$

$$R_{s,k} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^j\frac{\alpha_2}{[\alpha_1(i+j)+\alpha_2]}. \quad (2)$$

2. MLE of $R_{s,k}$ under SRS

Let $(X_1, X_2, ..., X_n)$ and $(Y_1, Y_2, ..., Y_m)$ be two ordered random samples of size n, m respectively. Here, Strength and stress variables follow generalized Pareto distribution with shape parameters α_1, α_2 and scale parameter λ, then the likelihood function is given by

$$\begin{aligned} &L_s(\alpha_1, \alpha_2, \lambda) \\ &= \alpha_1^n\alpha_2^m\lambda^{n+m}\prod_{i=1}^{n}(1+\lambda x_i)^{-(\alpha_1+1)}\prod_{j=1}^{m}(1+\lambda y_j)^{-(\alpha_2+1)}. \end{aligned}$$

Thus, the log-likelihood function is

$$\begin{aligned} &\ln L_s(\alpha_1, \alpha_2, \lambda) \\ &= n\ln\alpha_1 + m\ln\alpha_2 + (n+m)\ln\lambda \\ &-(\alpha_1+1)\sum_{i=1}^{n}\ln(1+\lambda x_i) - (\alpha_2+1)\sum_{j=1}^{m}\ln(1+\lambda y_j). \end{aligned}$$

The likelihood equations for estimating α_1, α_2 and λ are

$$\frac{\partial \ln L_s}{\partial \alpha_1} = 0 \Rightarrow \frac{n}{\alpha_1} - \sum_{i=1}^{n}\ln(1+\lambda x_i) = 0 \quad (3)$$

$$\frac{\partial \ln L_s}{\partial \alpha_2} = 0 \Rightarrow \frac{m}{\alpha_2} - \sum_{j=1}^{m}\ln(1+\lambda y_j) = 0 \quad (4)$$

$$\begin{aligned} \frac{\partial \ln L_s}{\partial \lambda} = 0 \Rightarrow \frac{n+m}{\lambda} &- \sum_{i=1}^{n}\frac{x_i}{(1+\lambda x_i)}\left[\frac{n}{\ln(1+\lambda x_i)}+1\right] \\ &- \sum_{j=1}^{m}\frac{y_j}{(1+\lambda y_j)}\left[\frac{m}{\ln(1+\lambda y_j)}+1\right] = 0. \end{aligned} \quad (5)$$

From (3), (4) and (5), the MLE of α_1, α_2 and λ is

$$\hat{\alpha}_1 = \frac{n}{\sum_{i=1}^{n}\ln(1+\hat{\lambda}x_i)} \quad (6)$$

$$\hat{\alpha}_2 = \frac{m}{\sum_{j=1}^{m}\ln(1+\hat{\lambda}y_j)} \quad (7)$$

where $\hat{\lambda}$ can be obtained as the solution of non-linear equation of the form,

$$H(\lambda)=\lambda$$

$$H(\lambda)=(n+m)\left[\begin{array}{l}\sum_{i=1}^{n}\frac{x_i}{(1+\lambda x_i)}\left[\frac{n}{\ln(1+\lambda x_i)}+1\right]\\ -\sum_{j=1}^{m}\frac{y_j}{(1+\lambda y_j)}\left[\frac{m}{\ln(1+\lambda y_j)}+1\right]\end{array}\right]^{-1} \quad (8)$$

here, $\hat{\lambda}$ can be obtained by using any iterative scheme

$$\lambda(a+1)=H(\lambda_{(a)})$$

where, $\lambda_{(a)}$ is the a^{th} iterate of $\hat{\lambda}$. The iteration procedure should be stopped when $\left|\lambda_{(a)}-\lambda_{(a+1)}\right|$ is sufficiently small. After obtaining $\hat{\lambda}$, the MLEs of α_1 and α_2 are obtained from (6) and (7). Hence, the MLE of $\hat{R}_{s,k}$ is obtained by using the invariance property of MLEs, that is,

$$\hat{R}_{s,k}^{Msrs}=\sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^j\frac{\hat{\alpha}_2}{[\hat{\alpha}_1(i+j)+\hat{\alpha}_2]}. \quad (9)$$

3. MLE of $R_{s,k}$ under RSS

Let $X_{(i:m_x)j}, i=1,...,m_x, j=1,...,r_x$ be a ranked set samples with sample size $n_x=m_x r_x$, where m_x is the set size and r_x is the number of cycles drawn from generalized Pareto distribution with parameters α_1 and λ and $Y_{(k:m_y)l}, k=1,...,m_y, l=1,...,r_y$ be a ranked set samples with sample size $n_y=m_y r_y$, where m_y is the set size and r_y is the number of cycles drawn from generalized Pareto distribution with parameters α_2 and λ respectively. For convenience, denote $X_{(i:m_x)j}$ and $Y_{(k:m_y)l}$ as X_{ij} and Y_{kl} respectively.

The pdf of the random variables X_{ij} and Y_{kl} are given by

$$g_1(x_{ij})=\frac{m_x!}{(i-1)!(m_x-i)!}\alpha_1\lambda\left[1-(1+\lambda x_{ij})^{-\alpha_1}\right]^{i-1}\times(1+\lambda x_{ij})^{-\alpha_1(m_x-i+1)+1}; \quad (10)$$
$$x_{ij}>0,\alpha_1,\lambda>0$$

$$g_2(y_{kl})=\frac{m_y!}{(k-1)!(m_y-k)!}\alpha_1\lambda\left[1-(1+\lambda y_{kl})^{-\alpha_2}\right]^{k-1}\times(1+\lambda y_{kl})^{-\alpha_2(m_y-k+1)+1}; \quad (11)$$
$$y_{kl}>0,\alpha_2,\lambda>0$$

The likelihood function of α_1, α_2 and λ is given by

$$L_r(\alpha_1,\alpha_2,\lambda)=\xi^*\prod_{i=1}^{m_x}\prod_{j=1}^{r_x}\alpha_1\lambda\left[1-(1+\lambda x_{ij})^{-\alpha_1}\right]^{i-1}(1+\lambda x_{ij})^{-\alpha_1(m_x-i+1)+1}*\prod_{k=1}^{m_y}\alpha_1\lambda\left[1-(1+\lambda y_{kl})^{-\alpha_2}\right]^{k-1}(1+\lambda y_{kl})^{-\alpha_2(m_y-k+1)+1}.$$

Thus, the log-likelihood function of α_1, α_2 and λ is

$$\begin{aligned}\ln L_r(\alpha_1,\alpha_2,\lambda)&=\xi^*+m_x r_x\ln\alpha_1+m_y r_y\ln\alpha_2+(m_x r_x+m_y r_y)\ln\lambda\\&+(i-1)\sum_{i=1}^{m_x}\sum_{j=1}^{r_x}\ln\left[1-(1+\lambda x_{ij})^{-\alpha_1}\right]\\&+(k-1)\sum_{k=1}^{m_y}\sum_{l=1}^{r_y}\ln\left[1-(1+\lambda y_{kl})^{-\alpha_2}\right]\\&-[\alpha_1(m_x-i+1)+1]\sum_{i=1}^{m_x}\sum_{j=1}^{r_x}\ln(1+\lambda x_{ij})\\&-[\alpha_2(m_y-k+1)+1]\sum_{k=1}^{m_y}\sum_{l=1}^{r_y}\ln(1+\lambda y_{kl})\end{aligned}$$

where ξ^* is a constant. The likelihood equations are

$$\frac{\partial\ln L_r}{\partial\alpha_1}=0\Rightarrow\frac{m_x r_x}{\alpha_1}-(i-1)\sum_{i=1}^{m_x}\sum_{j=1}^{r_x}\frac{(1+\lambda x_{ij})^{-\alpha_1}\ln(1+\lambda x_{ij})}{[1-(1+\lambda x_{ij})^{-\alpha_1}]}-(m_x-i+1)\sum_{i=1}^{m_x}\sum_{j=1}^{r_x}\ln(1+\lambda x_{ij})=0 \quad (12)$$

$$\frac{\partial\ln L_r}{\partial\alpha_2}=0\Rightarrow\frac{m_y r_y}{\alpha_2}-(k-1)\sum_{k=1}^{m_y}\sum_{l=1}^{r_y}\frac{(1+\lambda y_{kl})^{-\alpha_2}\ln(1+\lambda y_{kl})}{[1-(1+\lambda y_{kl})^{-\alpha_2}]}-(m_y-k+1)\sum_{k=1}^{m_y}\sum_{l=1}^{r_y}\ln(1+\lambda y_{kl})=0 \quad (13)$$

$$\frac{\partial\ln L_r}{\partial\lambda}=0\Rightarrow\frac{m_x r_x+m_y r_y}{\lambda}+(i-1)\sum_{i=1}^{m_x}\sum_{j=1}^{r_x}\frac{\alpha_1 x_{ij}(1+\lambda x_{ij})^{-(\alpha_1+1)}}{[1-(1+\lambda x_{ij})^{-\alpha_1}]}+(k-1)\sum_{k=1}^{m_y}\sum_{l=1}^{r_y}\frac{\alpha_2 y_{kl}(1+\lambda y_{kl})^{-\alpha_2}}{[1-(1+\lambda y_{kl})^{-\alpha_2}]}-[\alpha_1(m_x-i+1)+1]\sum_{i=1}^{m_x}\sum_{j=1}^{r_x}\frac{x_{ij}}{(1+\lambda x_{ij})}-[\alpha_2(m_y-k+1)+1]\sum_{k=1}^{m_y}\sum_{l=1}^{r_y}\frac{y_{kl}}{(1+\lambda y_{kl})}=0 \quad (14)$$

A closed form expression for equations (12)-(14) is difficult to obtain analytically. Hence, one can use any iterative technique to solve these equations. The MLE of

$\hat{R}_{s,k}$ is obtained by using the invariance property of MLEs, that is,

$$\hat{R}^{Mrss} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^j \frac{\hat{\alpha}_2}{[\hat{\alpha}_1(i+j)+\hat{\alpha}_2]}. \quad (15)$$

4. Bayes Estimation of $R_{s,k}$

Here we assume that all the parameters α_1, α_2 and λ are unknown and independent random variables with gamma priors $(a_i, b_i), i = 1,2,3.$

The pdf of gamma random variables X with parameters (a_i, b_i) is

$$f(x) = \frac{b_i^{a_i}}{\Gamma(a_i)} x^{a_i-1}e^{-xb_i}\ ;\ x>0, a_i, b_i>0,$$

where $a_i, b_i > 0,\ i = 1,2,3$.

Thus the joint prior α_1, α_2 and λ is

$$g(\alpha_1,\alpha_2,\lambda) = \frac{b_1^{a_1}}{\Gamma(a_1)}\frac{b_2^{a_2}}{\Gamma(a_2)}\frac{b_3^{a_3}}{\Gamma(a_3)} \times \alpha_1^{a_1-1}\alpha_2^{a_2-1}\lambda^{a_3-1}e^{-\alpha_1 b_1}e^{-\alpha_2 b_2}e^{-\lambda b_3}\ ;$$
$$\alpha_1,\alpha_2,\lambda>0, a_i, b_i>0$$

substituting $L_s(\alpha_1,\alpha_2,\lambda)$ and $g(\alpha_1,\alpha_2,\lambda)$, the corresponding joint posterior distribution is given by

$$\pi(\alpha_1,\alpha_2,\lambda \mid X,Y) = A\alpha_1^{n+a_1-1}\alpha_2^{m+a_2-1}\lambda^{n+m+a_3-1} e^{-\alpha_1(b_1+\sum_{i=1}^{n}\ln(1+\lambda x_i))-\alpha_2(b_2+\sum_{j=1}^{m}\ln(1+\lambda y_j))} e^{-\lambda b_3-\sum_{i=1}^{n}\ln(1+\lambda x_i)-\sum_{j=1}^{m}\ln(1+\lambda y_j)},$$

where

$$A^{-1} = \Gamma(n+a_1)\Gamma(m+a_2).$$

$$\left[\int_0^\infty \left[\lambda^{n+m+a_3-1}e^{-\lambda b_3-\sum_{i=1}^{n}\ln(1+\lambda x_i)-\sum_{j=1}^{m}\ln(1+\lambda y_j)} \left[(b_1+\sum_{i=1}^{n}\ln(1+\lambda x_i))^{n+a_1} \times (b_2+\sum_{j=1}^{m}\ln(1+\lambda y_j))^{m+a_2}\right]^{-1}\right] d\lambda\right]$$

Then the Bayes estimator of $R_{s,k}$ under squared error (SE) loss function is given by

$$\hat{R}_{s,k,B} = \int_0^\infty\int_0^\infty\int_0^\infty R_{s,k}\,\pi(\alpha_1,\alpha_2,\lambda \mid X,Y)\,d\alpha_1 d\alpha_2 d\lambda \quad (16)$$

It can be seen that, equation (16) cannot be reduced to a closed form. Hence, one can use Lindley's approximation method.

The simplest method to approximate is Lindley's [27] approximation method which approaches the ratio of the integrals as a whole and produces a single numerical result. If n is large, according to Lindley approximation, any ratio of the integrals of the form

$$I(x) = E[u(\theta_1,\theta_2,\theta_3)] = \frac{\int_{(\theta_1,\theta_2,\theta_3)} u(\theta_1,\theta_2,\theta_3)e^{L(\theta_1,\theta_2,\theta_3)+G(\theta_1,\theta_2,\theta_3)}d(\theta_1,\theta_2,\theta_3)}{\int_{(\theta_1,\theta_2,\theta_3)} e^{L(\theta_1,\theta_2,\theta_3)+G(\theta_1,\theta_2,\theta_3)}d(\theta_1,\theta_2,\theta_3)},$$

where

$$u(\theta) = u(\theta_1,\theta_2,\theta_3)$$

is a function of θ_1,θ_2 and θ_3, $L(\theta_1,\theta_2,\theta_3)$ is log of likelihood, $G(\theta_1,\theta_2,\theta_3)$ is log of joint prior of θ_1,θ_2 and θ_3, can be written as

$$I(x) = u(\theta_1,\theta_2,\theta_3) + (u_1a_1+u_2a_2+u_3a_3+a_4+a_5) + \frac{1}{2}[A(u_1\sigma_{11}+u_2\sigma_{12}+u_3\sigma_{13}) + B(u_1\sigma_{21}+u_2\sigma_{22}+u_3\sigma_{23}) + C(u_1\sigma_{31}+u_2\sigma_{32}+u_3\sigma_{33})],$$

where $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\theta}_3$ are the MLEs of θ_1,θ_2 and θ_3 respectively,

$$a_i = \rho_1\sigma_{i1}+\rho_2\sigma_{i2}+\rho_3\sigma_{i3}\ ,\ i=1,2,3,$$
$$a_4 = u_{12}\sigma_{12}+u_{13}\sigma_{13}+u_{23}\sigma_{23},$$
$$a_5 = \frac{1}{2}(u_{11}\sigma_{11}+u_{22}\sigma_{22}+u_{33}\sigma_{33})$$
$$A = \sigma_{11}L_{111}+2\sigma_{12}L_{121}+2\sigma_{13}L_{131}+2\sigma_{23}L_{231}+\sigma_{22}L_{221}+\sigma_{33}L_{331},$$
$$B = \sigma_{11}L_{112}+2\sigma_{12}L_{122}+2\sigma_{13}L_{132}+2\sigma_{23}L_{232}+\sigma_{22}L_{222}+\sigma_{33}L_{332},$$
$$C = \sigma_{11}L_{113}+2\sigma_{12}L_{123}+2\sigma_{13}L_{133}+2\sigma_{23}L_{233}+\sigma_{22}L_{223}+\sigma_{33}L_{333}$$

and subscripts 1,2,3 on right-hand sides refer to θ_1,θ_2 and θ_3 respectively, and

$$\rho_i = \frac{\partial\rho}{\partial\theta_i}, i=1,2,3,$$
$$u_i = \frac{\partial u(\theta_1,\theta_2,\theta_3)}{\partial\theta_i}, i=1,2,3,$$
$$u_{ij} = \frac{\partial^2 u(\theta_1,\theta_2,\theta_3)}{\partial\theta_i\partial\theta_j}, i,j=1,2,3,$$
$$L_{ij} = \frac{\partial^2 L(\theta_1,\theta_2,\theta_3)}{\partial\theta_i\partial\theta_j}, i,j=1,2,3,$$
$$L_{ijk} = \frac{\partial^2 L(\theta_1,\theta_2,\theta_3)}{\partial\theta_i\partial\theta_j\partial\theta_k}, i,j,k=1,2,3$$

and σ_{ij} is the $(i, j)^{th}$ element of the inverse of the matrix having elements $\{-L_{ij}\}$.

In our case, $(\theta_1, \theta_2, \theta_3) \equiv (\alpha_1, \alpha_2, \lambda)$ and

$$u = u(\alpha_1, \alpha_2, \lambda) = R_{s,k},$$

we have,

$$\rho_1 = \frac{a_1 - 1}{\alpha_1} - b_1, \rho_2 = \frac{a_2 - 1}{\alpha_2} - b_2, \rho_3 = \frac{a_3 - 1}{\lambda} - b_3$$

and L_{ij} can be obtained as follows for $i, j = 1,2,3$

$$L_{11} = -\frac{n}{\alpha_1^2}, \quad L_{22} = -\frac{m}{\alpha_2^2}$$

$$L_{13} = L_{31} = \sum_{i=1}^{n} \frac{x_i}{(1+\lambda x_i)},$$

$$L_{23} = L_{32} = \sum_{j=1}^{m} \frac{y_j}{(1+\lambda y_j)}$$

$$L_{33} = -\frac{n+m}{\beta^2} + \sum_{i=1}^{n} \frac{x_i^2}{(1+\lambda x_i)^2}[\alpha_1 + 1] + \sum_{j=1}^{m} \frac{y_j^2}{(1+\lambda y_j)^2}[\alpha_2 + 1]$$

and the values of L_{ijk} for i, j, k = 1, 2, 3

$$L_{111} = \frac{2n}{\alpha_1^3}, \quad L_{222} = \frac{2m}{\alpha_2^3},$$

$$L_{133} = L_{331} = \sum_{i=1}^{n} \frac{x_i^2}{(1+\lambda x_i)^2},$$

$$L_{233} = L_{332} = \sum_{j=1}^{m} \frac{y_j^2}{(1+\lambda y_j)^2},$$

$$L_{333} = \frac{2(n+m)}{\beta^3} - 2\sum_{i=1}^{n} \frac{x_i^3}{(1+\lambda x_i)^3}[\alpha_1 + 1] - 2\sum_{j=1}^{m} \frac{y_j^3}{(1+\lambda y_j)^3}[\alpha_2 + 1].$$

Since, $u = u(\alpha_1, \alpha_2, \lambda) = R_{s,k}$,

$$u_1 = \frac{\partial u}{\partial \alpha_1} = \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (-1)^{j+1} \frac{\alpha_2 (i+j)}{[\alpha_1 (i+j) + \alpha_2]^2}$$

$$u_2 = \frac{\partial u}{\partial \alpha_2} = \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (-1)^{j} \frac{\alpha_1 (i+j)}{[\alpha_1 (i+j) + \alpha_2]^2}$$

$$u_{11} = \frac{\partial^2 u}{\partial \alpha_1^2} = \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (-1)^{j} \frac{2\alpha_2 (i+j)^2}{[\alpha_1 (i+j) + \alpha_2]^3}$$

$$u_{22} = \frac{\partial^2 u}{\partial \alpha_2^2} = \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (-1)^{j+1} \frac{2\alpha_1 (i+j)}{[\alpha_1 (i+j) + \alpha_2]^3}$$

$$u_{12} = \frac{\partial^2 u}{\partial \alpha_1 \partial \alpha_2} = \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (-1)^{j+1} \frac{[\alpha_1 (i+j) - \alpha_2](i+j)}{[\alpha_1 (i+j) + \alpha_2]^3}$$

$u_3 = \frac{\partial u}{\partial \lambda} = 0$ and $u_{13} = u_{23} = u_{32} = u_{33} = 0$. Hence

$$a_4 = u_{12}\sigma_{12}, \quad a_5 = \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22})$$

$$A = \sigma_{11}L_{111} + \sigma_{33}L_{331},$$

$$B = \sigma_{22}L_{222} + \sigma_{33}L_{332},$$

$$C = 2\sigma_{13}L_{133} + 2\sigma_{23}L_{233} + \sigma_{33}L_{333}.$$

The quantities u_i, u_{ij}, σ_{ij} and L_{ijk}, $i, j, k = 1, 2, 3$ are evaluated by replacing $(\alpha_1, \alpha_2, \lambda)$ by $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\lambda})$.

Then, the Bayes estimator of $R_{s,k}$ is

$$\hat{R}_{s,k}^{B} = R_{s,k} + (u_1 a_1 + u_2 a_2 + a_4 + a_5) + \frac{1}{2}[A(u_1\sigma_{11} + u_2\sigma_{12}) + B(u_1\sigma_{21} + u_2\sigma_{22}) + C(u_1\sigma_{31} + u_2\sigma_{32})]$$

5. Simulation Study

Simulation study consists of estimating multicomponent stress-strength reliability when the sample is generated from generalized Pareto distribution under SRS and RSS using ML and Bayesian approaches. The comparison of the estimates are done though mean squared error criteria based on 100000 random samples of size n, m, m_x, m_y, r_x and r_y for the strength and stress populations, with different parameter values. The values of $(\alpha_1, \alpha_2, \lambda)$ used for comparison are (1,1,0.6), (1,1.5,0.5) and (0.5,0.2,0.5). The corresponding true values of stress-strength reliability for s-out-of-k system with $(s, k) = (1,3)$ are 0.75, 0.8476, 0.4747 and that for $(s, k) = (2,4)$ are 0.6, 0.7229, 0.3315. The Bayesian estimates under squared error loss function using gamma prior are $a_1 = 9$, $a_2 = 4$, $a_3 = 1$, $b_1 = 3$, $b_2 = 2$, $b_3 = 1$ (prior1) and $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $b_1 = 1$, $b_2 = 1$, $b_3 = 1$ (prior2).

From the simulation study, it is observed the MSEs for the estimates decreases as the sample size increases in all the cases. The Bayes estimates of the $R_{s,k}$ under the squared error loss function have the smaller MSEs. It is also be seen that when comparing the maximum likelihood estimates under SRS and RSS, RSS performs better as it has smaller MSE when compared to SRS.

Table 1. MLEs under SRS and RSS, Bayes estimators and MSEs for the estimators of $R_{s,k}$

$\alpha_1=1,\ \alpha_2=1, \lambda=0.6, r_x=5, r_y=5,\ prior\ 1$										
(s, k)	$R_{s,k}$	n	m	(m_x, m_y)	$\hat{R}_{s,k}^{Msrs}$	$\hat{R}_{s,k}^{Mrss}$	$\hat{R}_{s,k}^{B}$	MSE($\hat{R}_{s,k}^{Msrs}$)	MSE($\hat{R}_{s,k}^{Mrss}$)	MSE($\hat{R}_{s,k}^{B}$)
(1, 3)	0.75	10	10	(2,2)	0.7721	0.7595	0.7616	0.0092	0.0063	0.0042
		10	15	(2,3)	0.7682	0.7584	0.7592	0.0067	0.0042	0.0023
		10	25	(2,5)	0.7596	0.7536	0.7523	0.0046	0.0037	0.0019
		15	25	(3,5)	0.7542	0.7526	0.7521	0.0023	0.0019	0.0011
		25	25	(5,5)	0.7536	0.7502	0.7511	0.0009	0.0006	0.0002
(2, 4)	0.6	10	10	(2,2)	0.6909	0.6834	0.6796	0.0082	0.0069	0.0063
		10	15	(2,3)	0.6889	0.6491	0.6376	0.0061	0.0053	0.0050
		10	25	(2,5)	0.6646	0.6382	0.6315	0.0047	0.0035	0.0026
		15	25	(3,5)	0.6414	0.6212	0.6185	0.0036	0.0022	0.0019
		25	25	(5,5)	0.6084	0.6020	0.6007	0.0029	0.0016	0.0011

Table 2. MLEs under SRS and RSS, Bayes estimators and MSEs for the estimators of $R_{s,k}$

$\alpha_1=1,\ \alpha_2=1.5, \lambda=0.5, r_x=5, r_y=5,\ prior\ 2$										
(s, k)	$R_{s,k}$	n	m	(m_x, m_y)	$\hat{R}_{s,k}^{Msrs}$	$\hat{R}_{s,k}^{Mrss}$	$\hat{R}_{s,k}^{B}$	MSE($\hat{R}_{s,k}^{Msrs}$)	MSE($\hat{R}_{s,k}^{Mrss}$)	MSE($\hat{R}_{s,k}^{B}$)
(1, 3)	0.8476	10	10	(2,2)	0.8592	0.8493	0.8477	0.0098	0.0090	0.0081
		10	15	(2,3)	0.8671	0.8496	0.8461	0.0076	0.0024	0.0014
		10	25	(2,5)	0.8484	0.8479	0.8462	0.0028	0.0016	0.0012
		15	25	(3,5)	0.8482	0.8452	0.8471	0.0016	0.0013	0.0010
		25	25	(5,5)	0.8480	0.8477	0.8460	0.0010	0.0009	0.0005
(2, 4)	0.7229	10	10	(2,2)	0.7318	0.7297	0.7277	0.0098	0.0090	0.0085
		10	15	(2,3)	0.7291	0.7276	0.7267	0.0065	0.0057	0.0029
		10	25	(2,5)	0.7273	0.7244	0.7229	0.0032	0.0015	0.0013
		15	25	(3,5)	0.7244	0.7233	0.7206	0.0016	0.0010	0.0009
		25	25	(5,5)	0.7240	0.7230	0.7222	0.0012	0.0008	0.0004

Table 3. MLEs under SRS and RSS, Bayes estimators and MSEs for the estimators of $R_{s,k}$

$\alpha_1=0.5,\ \alpha_2=0.2, \lambda=0.5, r_x=5, r_y=5,\ prior\ 1$										
(s, k)	$R_{s,k}$	n	m	(m_x, m_y)	$\hat{R}_{s,k}^{Msrs}$	$\hat{R}_{s,k}^{Mrss}$	$\hat{R}_{s,k}^{B}$	MSE($\hat{R}_{s,k}^{Msrs}$)	MSE($\hat{R}_{s,k}^{Mrss}$)	MSE($\hat{R}_{s,k}^{B}$)
(1, 3)	0.4747	10	10	(2,2)	0.4858	0.4812	0.4792	0.0123	0.0092	0.0089
		10	15	(2,3)	0.4796	0.4772	0.4765	0.0111	0.0082	0.0079
		10	25	(2,5)	0.4779	0.4765	0.4769	0.0110	0.0036	0.0030
		15	25	(3,5)	0.4781	0.4731	0.4749	0.0066	0.0018	0.0014
		25	25	(5,5)	0.4763	0.4722	0.4714	0.0014	0.0010	0.0009
(2, 4)	0.3315	10	10	(2,2)	0.3471	0.3416	0.3411	0.0092	0.0082	0.0072
		10	15	(2,3)	0.3479	0.3392	0.3408	0.0089	0.0062	0.0052
		10	25	(2,5)	0.3381	0.3364	0.3341	0.0082	0.0044	0.0035
		15	25	(3,5)	0.3362	0.3346	0.3326	0.0065	0.0026	0.0021
		25	25	(5,5)	0.3317	0.3310	0.3307	0.0056	0.0019	0.0011

6. Data Analysis

In this section, we present a real data which was originally reported by Badar and Priest [28]. The data represents the strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 20 mm (Data sets I) and 10 mm (Data set II), with sample sizes n = 69 and m = 63 respectively.

Data set I:

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382,2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585

Data set II:

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020

For the above data sets, we fit the generalized Pareto model and also checked the validity of the model using Kolmogorov-Smirnov (K-S) test for each data set. It was found that for data set I and II, the K-S distanced are 0.0479 and 0.0986 with the corresponding p value are 0.5638 and 0.6425 respectively. From the result, it shows that generalized Pareto distribution fits better for the data sets. The maximum likelihood estimate and Bayes estimate, based on the parameters $\hat{\alpha}_1 = 0.46909,\ \hat{\alpha}_2 = 2.7786$ and are obtained as $\hat{R}_{1,3}^{Msrs} = 0.5953,\ \hat{R}_{1,3}^{B} = 0.4498$ under prior1 and $\hat{R}_{1,3}^{Msrs} = 0.5847,\ \hat{R}_{1,3}^{B} = 0.5634$ under prior 2. For s =2 and k = 4 MLE and Bayes estimators are $\hat{R}_{2,4}^{Msrs} = 0.4245, \hat{R}_{1,3}^{B} = 0.4285$ under prior1 and $\hat{R}_{1,3}^{Msrs} = 0.4591,\ \hat{R}_{1,3}^{B} = 0.4587$ under prior2.

7. Conclusion

The main aim of this article, is to study the multicomponent system reliability which has k independent and identical strength components and each component exposed to a common random stress by assuming both strength and stress variables follow generalized Pareto distribution. The reliability of the system is estimated using maximum Likelihood under SRS and RSS scheme and Bayes approaches. The performance of these estimates are compared using MSEs, the results show that the MLE has greater MSE when compared to Bayes estimates. The ML estimates under RSS have lesser MSE than SRS. However, as sample size increases, MSEs of both the approaches, i.e., SRS and RSS are close to each other.

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