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Approximation for a Common Fixed Point for Family of Multivalued Nonself Mappings

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Abstract In this paper, we introduce Mann type iterative method for finite and infinite family of multivalued nonself and non expansive mappings in real uniformly convex Banach spaces. We extend the result to the class of quasi non expansive mappings in real Uniformity convex Banach spaces. We also extend for approximating a common fixed point for the class of multivalued, strictly pseudo contractive and generalized strictly pseudo contractive nonself mappings in real Hilbert spaces. We prove both weak and strong convergence results of the iterative method.

Keywords: *fixed point, nonself mapping, nonexpansive mapping, strictly pseudo contractive, generalized strictly pseudo contractive mappings, multivalued mapping, Mann type iterative method, uniformly convex Banach space*

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1. Introduction

Fixed point theory for multi-valued mappings becomes very interesting for numerous researchers of the field because of its many real world applications in convex optimization, game theory and differential inclusions. Multi-valued mappings are also important in solving critical points in optimal control and other problems (Agarwal et al [2] pp 188). In single valued case, for example in studying the operator equation $Au = 0$ (when the mapping A is monotone) if K is a subset of a Hilbert space H, then $A: K \to H$ is monotone mapping if $\langle Ax - Ay, x - y \rangle \geq 0, \forall x, y \in K$, Browder [5] introduced a new operator T defined by $T = I - A$, where I is the identity mapping on the Hilbert space H, the operator is called pseudo contractive operator and the solutions of $Au = 0$ are the fixed points of the pseudo contractive mapping T and vice versa. Consider a mapping $A: K \to H$ and the Variational inequality $\left\langle Ax^*, x - x^* \right\rangle \geq 0, \quad \forall x \in K$, in which the problem is to find $x^* \in K$ satisfying the in equality, this problem is the Variational inequality problem arises in convex optimization, differential inclusions.

Let $f: K \to \Re$ be convex, continuously differentiable function. Thus, $\left\langle \nabla f(x^*), x - x^* \right\rangle \geq 0 \ \forall x \in K$ is Variational inequality for $A = \nabla f$, this inequality is optimality condition for minimization problem $\min_{x \in K} f(x)$ which appears in many areas. An example of a monotone operator in optimization theory is the multi-valued mapping of the sub differential of the functional f, $\partial f: D(f) \subseteq H \to 2^H$ and is defined by

$$\partial f(x) = \left\{ g \in H : \langle x - y, g \rangle \leq f(x) - f(y) \right\}, \forall y \in K, \quad (1.1)$$

and $0 \in \partial f(x)$ satisfies the condition

$$\langle x - y, 0 \rangle = 0 \leq f(x) - f(y) \forall y \in K.$$

In particular, if $f: K \to \Re$ is convex, continuously differentiable function then $A = \nabla f$, the gradient is a sub differential which is single valued mapping and the condition $\nabla f(x) = 0$ is operator equation and $\langle \nabla f(x), x - y \rangle \geq 0$ is Variational in equality and both conditions are closely related to optimality conditions. Thus, finding fixed point or common fixed point for Multi valued mapping is important in many practical areas.

Let K be a non empty subset of a real normed space E , then $CB(E)$ denotes the set of non empty, closed and bounded subsets of E . We say K is proximal, if for every $x \in E$, there exists some $y \in K$ such that $\|x - y\| = \inf \left\{ \|x - z\|, \forall z \in K \right\}$. We denote the family of nonempty proximal bounded subsets of K by Prox(K). We observe that, in Hilbert spaces by projection theorem every non empty, closed and convex subset of H is proximal. Also Agarwal *et al* [2] presented that every nonempty, closed and convex subset of a uniformly convex Banach space is proximal. For A, B in $CB(E)$, we define the Housdorff distance between A and B in $CB(E)$ by

$$D(A,B) = Max\left\{\sup_{x\in B} d(x,A), \sup_{x\in A} d(x,B)\right\},$$

where $d(x,A) = \inf\{\|x-a\|, \forall a \in A\}$.

Kuratowski [19] presented that $(CB(E),D)$ is complete if E is complete.

A mapping $T: K \to 2^E$ is non self multivalued mapping in general and the set of fixed point of T is defined as $F = F(T) = \{x \in K : x \in Tx\}$.

As Chidume et al [8] proposed, we give the definition of multi valued version for contractive mappings on a non empty subset K of a real Banach spaces E which is a generalization of single valued case as follows.

Definition 1.1 The mapping $T: K \to 2^E$ is said to be

a) contraction, if there is $0 \le \alpha < 1$ such that $D(Tx,Ty) \le \alpha\|x-y\|$ for all $x, y \in K$.

b) L-Lipschitzian, if $D(Tx,Ty) \le L\|x-y\|$ for some $L > 0$ and for all $x, y \in K$.

c) nonexpansive, if. $D(Tx,Ty) \le \|x-y\|$ for all $x, y \in K$, when $L = 1$.

d) Quasi non expansive mapping if

$$F(T) \ne \emptyset \text{ and } D(Tx,Tp) \le \|x-p\| \quad \text{for all } p \in F(T), x \in K. \tag{1.2}$$

In real Hilbert space H, if K is nonempty subset of H $T: K \to CB(H)$ is said to be

e) Pseudo contractive, if

$$D^2(Tx,Ty) \le \|x-y\|^2 + \|(x-u-(y-v)\|^2, \quad u \in Tx, v \in Ty\ x, \text{ for all } y \in K.$$

f) Hemi contractive in real Hilbert space, if $F(T) \ne$ Ø and $D^2(Tx,Tp) \le \|x-y\|^2 + d^2(x,Tx)$ for all $p \in F(T), x \in D(T)$

g) k-strictly pseudo contractive mapping in Hilbert spaces, if there exists $k \in (0,1)$ such that

$$D^2(Tx,Ty) \le \|x-y\|^2 + k\|(x-u-(y-v)\|^2, \quad u \in Tx, v \in Ty \tag{1.3}$$

holds.

h) Demi contractive $F(T) \ne$ Ø and there exists $k \in (0,1)$ such that $D^2(Tx,Tp) \le \|x-p\|^2 + kd^2(x,Tx), p \in F(T)$, $x \in D(T)$ holds.

On the other hand, Chidume and Okpala [9] introduced generalized k-strictly pseudo contractive multivalued mapping which is defined as follow.

Definition 1.2 Let, K be a non empty subset of a real Hilbert space, and then the mapping $T: K \to CB(H)$ is said to be

a) generalized k –strictly pseudo contractive mapping if there exists $k \in (0,1)$ such that

$$D^2(Tx,Ty) \le \|x-y\|^2 + kD^2(x-Tx, y-Ty), \quad \forall x, y \in D(T) \tag{1.4}$$

holds;

b) Generalized Hemi contractive in real Hilbert space, if

$$F(T) \ne \emptyset \text{ and } D^2(Tx,Tp) \le \|x-p\|^2 + D^2(x,Tx) \quad \text{for all } p \in F(T), x \in D(T). \tag{1.5}$$

It can be seen that, the class of generalized k- strictly pseudo contractive mappings includes the class of k-strictly pseudo contractive mappings.

Thus, the class of contraction as well as non expansive mappings are subset of the class of Lipschitzian and the class of k strictly Pseudo contractive mappings and hence the generalized k-strictly pseudo contractive mappings. Furthermore, the class of quasi non expansive mappings includes the class of non expansive mappings. Thus, the class of k-generalized strictly pseudo contractive mappings is more general than the class of non expansive mappings and the class of strictly pseudo contractive mappings. The study of fixed points of non expansive and contractive types of Multi valued mappings is very important and more complex in its applications in convex optimization, optimal control theory, differential equations and others.

Example 1.1 Let $T: [0,\infty) \to 2^{(-\infty,\infty)}$ be given by $Tx = [-x, 0]$ for all $x \in [0,\infty)$.

Then, for all $x, y \in [0,\infty)$, $D(Tx,Ty) = |x-y|$ hence T is non expansive and non self mapping.

Example 1.2 Let $T: [0,1] \to 2^{\Re}$ be given by $Tx = \left\{0, 4 - \frac{4}{3}x\right\}$. Then T is nonself, multivalued, k-strictly pseudo contractive mapping but not non expansive type (see [35]) with $F(T) = \{0\}$.

Example 1.3 Let $T: [0,\infty) \to 2^{(-\infty,\infty)}$ be defined by $Tx = \left\{0, -\frac{4}{3}x\right\}$. $D(Tx,Ty) = \frac{4}{3}|x-y|$, thus

$$D^2(Tx,Ty) = \frac{16}{9}|x-y|^2 = |x-y|^2 + \frac{7}{9}|x-y|^2.$$

Then T is nonself which is not nonexpansive mapping.

Markin [23] was the first who presented the work on fixed points for multi-valued (nonexpansive) mappings by the application of Hausdorff metric and following his work, an extensive work was done by Nadler [24], since then existence of fixed points and their approximations for multi-valued contraction and nonexpansive mappings and their generalizations have been studied by several authors [1,3,4,8,10,14,19,20,21,24].

To mention a few, in 2005, Sastry and Babu [27] constructed Mann and Ishikawa-type iterations as given bellow

Let $T: K \to$ Prox (K) be a multi-valued mapping and let $p \in F(T) \ne$ Ø then, the sequence of Mann-type iterates given by

$$x_0 \in K, x_{n+1} = \alpha_n y_n + (1-\alpha_n)x_n,$$
$$\alpha_n \in [0,1], n \geq 0, y_n \in Tx_n \quad (1.6)$$
$$\text{such that } \|p - y_n\| = \|p - Tx_n\|, p \in F(T).$$

And the sequence of Ishikawa-type iterates

$$\begin{cases} x_0 \in K \\ y_n = (1-\beta_n)x_n + \beta_n u_n, u_n \in Tx_n \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n v_n, v_n \in Ty_n \end{cases}, \quad (1.7)$$
$$n \geq 1, \alpha_n, \beta_n \in [0,1]$$

such that

$$\|p - u_n\| = \|p - Tx_n\|, p \in F(T)$$
$$\text{and } \|p - v_n\| = \|p - Ty_n\|, p \in F(T) \quad (1.8)$$

and they proved strong convergence of the iterative methods to some points in $F(T)$ assuming that K is compact and a convex subset of a real Hilbert space H, T is nonexpansive mapping with $F(T) \neq \emptyset$ the parameters α_n, β_n satisfying certain nailed conditions.

Panyanak [24], consequently, Song and Wang [32], with additional nailed condition $Tp = \{p\}, \forall p \in F(T)$ extended the result of Sastry and Babu [27] to more general spaces, uniformly convex Banach spaces, indeed, they proved the convergence results of Ishikawa-type iterative method. Moreover, Shahzad and Zegeye [29] extended the above results to multivalued quasi-nonexpansive mappings and removed the compactness assumption on K. They also constructed a new iterative scheme to relax the strong condition $Tp = \{p\}, \forall p \in F(T)$ in the Song and Wang [32], consequently, Djitte and Sene [4] constructed the Ishikawa type iterative method for multi-valued and Lipschitz pseudo contractive mapping, they also proved convergence with more restrictions. In addition, Chidume and Okpala[9] constructed iterative method of Mann and Ishikawa type for approximating fixed points for generalized k strictly pseudo contractive Multivalued mapping, later on Okpala[25] modified the iteration for three step Ishikawa iterative method for approximating fixed points for Hemi contractive mappings. However, all the above results were for self mappings, on the other hand, in practical areas, there are cases of which we must consider non self mapping or family of non self mappings.

For approximating fixed points of nonself single-valued mappings, several Mann and Ishikawa-type iterative schemes have been studied via projection for sunny nonexpansive retraction [16,19,22,29,30,31,33,30-40]. However, recently, Colao and Marino [12] presented that the computation for sunny non expansive retraction is costly and they proposed the method with lowering the requirement of metric projection. Motivated by the work of Colao and Marino [12] many authors presented iterative methods for approximating a fixed point and a common fixed point for both finite and infinite family of single valued mappings without the requirement of metric projection [34,35]. More recently, Tufa and Zegeye [37] introduced a Mann-type iterative scheme for approximating fixed points for multi-valued nonexpansive nonself single mapping in real Hilbert space, which generalizes the result of Colao and Marino [12] to the class of multivalued mappings and they proved convergence with the assumption that the mapping satisfies inward condition in the following theorem.

Definition 1.2 Let K be a nonempty subset of a real Banach space E, a mapping $T: K \to 2^E$ is said to be inward if for each $x \in K$,

$$Tx \subseteq IK(x) = \{x + c(w-x), c \geq 1, w \in K\}.$$

Example 1.3 Considering example 1.1, let $u \in Tx = [-x, 0]$. Then $u = t(-x) + (1-t)0,\ 0 \leq t \leq 1$, thus we have

$$u = u - x + x = x + (t(-x) - x)$$
$$= x + 2\left(\frac{-t}{2}x - \frac{x}{2} - \frac{x}{2} + \frac{x}{2}\right)$$
$$= x + 2\left(\frac{1-t}{2}x - x\right) = x + c(v - x),$$
$$c = 2 \geq 1, v = \frac{1-t}{2}x \in [0, \infty).$$

Hence, T is inward mapping, in fact, $F(T) = \{0\}$.

Thus, T is nonself, nonexpansive inward mapping.

Theorem TZ [37] (Tufa and Zegeye; Theorem 3.2) Let K be a nonempty, closed and convex subset of a real Hilbert H and let $T: K \to \text{Prox(H)}$ be an inward nonexpansive mapping with $F(T) \neq \emptyset$ and $Tp = \{p\}, \forall p \in F(T)$. Let $\{x_n\}$ be a sequence of Mann-type given by

$$x_1 \in K, x_{n+1} = \alpha_n x_n + (1-\alpha_n)u_n,$$
$$\alpha_n \in [0,1], n \geq 0, u_n \in Tx_n$$

such that $\|p - u_n\| = \|p - Tx_n\|$,

$$p \in F(T), \|u_n - u_{n+1}\| \leq D(Tx_n, Tx_{n+1}),$$

$$\alpha_1 = \max\left\{\frac{1}{2}, h_{u_1}(x_1)\right\}, \alpha_{n+1} = \max\left\{\alpha_n, h_{u_{n+1}}(x_{n+11})\right\},$$

$$h_{u_n}(x_n) = \inf\{\lambda \in [0,1] : \lambda x_n + (1-\lambda)u_n \in K\}.$$

Then, $\{x_n\}$ weakly converges to a fixed point of T. Moreover, if $\sum_{n=1} (1-\alpha_n) < \infty$ and K is strictly convex, then the convergence is strong.

It has been observed that, the existence of the sequence $\{u_n\}$ satisfying the condition $\|u_n - u_{n+1}\| \leq D(Tx_n, Tx_{n+1})$ is guaranteed by lemma 2.3 [17] which is stated in our preliminary section.

Authors [37] also extended the result for quasi-nonexpansive type mapping in a real uniformly convex Banach space E with some appropriate restrictions.

Definition 1.2 A uniformly convex space E is a normed space E for which for every $0 < \varepsilon < 2$, there is a $\delta > 0$, such that for every $x, y \in S = \{x \in E : \|x\| = 1\}$, if $\|x - y\| > \varepsilon (x \neq y)$, then $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$.

Hilbert spaces, the sequences space l_p, the Lebsgue space L_p ($1 < p < \infty$) are examples of Uniformly convex Banach spaces.

The above results so far discussed were applicable for a single non expansive or quasi non expansive mapping , on the other hand in many practical areas we may face family of mappings and a more general class of mappings the so called the class of strictly pseudo contractive mappings.

Thus, motivated by the ongoing research work, in particular, the result of Tuffa and Zegeye [37], our question is that, is it possible to approximate a common fixed point for the family of nonself, multivalued and non expansive and strictly pseudo contractive mappings in real Hilbert spaces and real uniformly convex Banach spaces?

Thus, it is the purpose of this paper to construct Mann type iterative method for approximating a common fixed point of both finite and infinite family of nonself, multivalued, nonexpansive mappings and quasi nonexpansive mappings as well and to extend the result to the class of strictly pseudo contractive mappings which is a positive answer to our question.

2. Preliminary Concepts

We use the following notations and definitions;

Definition 2.1 Let K be a non empty subset of a real Banach space E, and let $T: K \to 2^E$ be multivalued mapping, $I - T$ is demi closed at 0, if for any sequence $\{x_n\}$ in K converges weakly to p and $d(x_n, Tx_n) \to 0$, then $p \in Tp$. Moreover, $I - T$ is demi closed at 0 is strongly demi closed at 0, if for any sequence $\{x_n\}$ in K converges strongly to p and $d(x_n, Tx_n) \to 0$, then $d(p, Tp) = 0$.

Lemma 2.1 ([28], lemma 2.6) Let K be a nonempty, closed and convex subset of a real Hilbert space H and let $T: K \to$ Prox(H) be a nonexpansive multi-valued mapping. Then, $I - T$ is demi closed at zero.

Definition 2.2 A Banach space E is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in E, x_n converges weakly to some $x \in E$ implies

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in E, y \neq x$.

Definition 2.3 A sequence $\{x_n\}$ in K is said to be Fejer monotone with respect to a subset F of K, if $\forall x \in F, \|x_{n+1} - x\| \le \|x_n - x\|, \forall n$.

Lemma 2.2 [24] Let E be a real Banach space. Then, if $A, B \in CB(E)$) and $a \in A$, then for every $\gamma > 0$ there exists $b \in B$ such that $\|b - a\| \le D(A, B) + \gamma$.

Lemma 2.3 [17] Let E be a real Banach space. Then, if $A, B \in$ Prox (E) and $a \in A$, then there exists $b \in B$ such that $\|b - a\| \le D(A, B)$.

Lemma 2.4 (Xu [41]). Let $p > 1, R > 1$ be two fixed numbers and E is a real Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g:[0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)g(\|x-y\|)$$

for all $x, y \in B_R(0) = \{x \in X : \|x\| < R\}$ and $\lambda \in [0,1]$, where $W_p(\lambda) = \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$.

Lemma 2.5 [42] In real Hilbert space H, for all $x_i \in H$ and $\alpha_i \in [0,1]$ for such that $\sum_{i=1}^{n} \alpha_i = 1$ the equality

$$\left\|\sum_{i=1}^{n} \alpha_i x_i\right\|^2 = \sum_{i=1}^{n} \alpha_i \|x_i\|^2 - \sum_{1 \le i, i \le n} \alpha_i \alpha_j \|x_i - x_j\|^2$$

holds.

Lemma 2.6 (Browder [7], Ferreira-Oliveira [13]) Let E be a complete metric space and $K \subseteq E$ a nonempty subset. If $\{x_n\}$ is Fejer monotone with respect to K then $\{x_n\}$ is bounded. Furthermore, if a cluster point x of $\{x_n\}$ belongs to K then $\{x_n\}$ converges strongly to x. In the particular case of a Hilbert space, given the set of all weakly cluster points of $\{x_n\}$

$$\omega_w(x_n) = \{x : \exists x_{n_k} \to x, weakely\}$$

$\{x_n\}$ Converges weakly to a point $x \in K$ if and only if $\omega_w(x_n) \subseteq K$.

Lemma 2.7 (See, for example, Zeidler [43]pp 484) Let E be a real uniformly convex Banach space, $\{x_n\}, \{y_n\}$ in E be two sequences, if there exists a constant $r \ge 0$ such that

$$\limsup_{n\to\infty} \|x_n\| \le r, \limsup_{n\to\infty} \|y_n\| \le r$$
$$\text{and} \lim_{n\to\infty} \|\lambda_n x_n + (1-\lambda_n)y_n\| = r,$$

for $\{\lambda_n\} \subset [\varepsilon, 1-\varepsilon] \subset (0,1)$ for some $\varepsilon \in (0,1)$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.8 [8]: Let K be a nonempty subset of a real Hilbert space H and let $T: K \to CB(K)$ be a multivalued k-strictly pseudo contractive mapping. Then, T is Lipschitz with Lipchitz constant $\frac{1+\sqrt{k}}{1-\sqrt{k}}$.

Lemma 2.9 [38] Let H be a real Hilbert space. Suppose K is a closed, convex, nonempty subset of H. Assume that $T: K \to CB(K)$ is pseudo contractive multi-valued mapping with F(T) is non empty. Then, F(T) is closed and convex.

Lemma 2.10 [38] Let H be a real Hilbert space. Suppose K is a closed, convex, nonempty subset of H. Assume that $T: K \to CB(K)$ is Lipschitz pseudo contractive multi-valued mapping. Then $I-T$ is demi closed at zero.

Lemma 2.11 Let K be a nonempty subset of a real Hilbert space H and let $T:K\to$Prox(H) be a multivalued k-strictly pseudo contractive mapping. Then, T is Lipschitzian with Lipschitz constant $\frac{1+\sqrt{k}}{1-\sqrt{k}}$ and hence $I-T$ is demi closed at 0. (Proof can be done with lemma 2.3, lemma 2.8 and lemma 2.10).

Definition 2.4 Let F, K be two closed and convex nonempty sets in a Banach spaces E and $F\subset K$. For any sequence $\{x_n\}\subset K$ if $\{x_n\}$ converges strongly to an element $x\in\partial K\setminus F$, $x_n\neq x$ implies that $\{x_n\}$ is not Fejer-monotone with respect to the set $F\subset K$, we say the pair (F, K) satisfies S-condition.

Example Let $F=\{0\}\subset K=[-1,1]$. Then the pair (F,K) satisfies S- condition.

Definition 2.5. Let $\{T_n\}_{n=1}^{\infty}:K\to prox(E)$ be sequence of mappings with nonempty common fixed point set $F=\bigcap_{n=1}^{\infty}F(T_n)$. Then, the family $\{T_n\}_{n=1}^{\infty}$ is said to be uniformly weakly closed if for any convergent sequence $\{x_n\}\subset K$ such that $\lim_{n\to\infty}d(x_n,T_nx_n)=0$, then the weak cluster Points of $\{x_n\}\subset K$ belong to F.

Lemma 2.12 [9]: Let K be a nonempty subset of a real Hilbert space H and $T:K\to CB(K)$ be a multivalued generalized k-strictly pseudo contractive mapping. Then, T is Lipschitz with Lipschitz constant $\frac{1+\sqrt{k}}{1-\sqrt{k}}$ and F(T) is closed and convex.

Lemma 2.13 [9] Let K be a nonempty and closed subset of a real Hilbert space H and let $T:K\to$CB(K) be a multivalued generalized k-strictly pseudo contractive mapping. Then, T is Lipschitzian with Lipschitz constant $\frac{1+\sqrt{k}}{1-\sqrt{k}}$ and $I-T$ is strongly demi closed at 0.

Definition 2.6 Let K be a nonempty and closed subset of a real Hilbert space H. Then a map $T:K\to$CB(H) is said to be Hemi compact, if for any sequence $\{x_n\}$ in K such $d(x_n,Tx_n)\to 0$, then there exists a sub sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to p in K.

Remark: Any mapping on a compact domain is Hemi compact.

Lemma2.15 [36] Let $\{a_n\}$ be a sequence of non negative real numbers such that $a_{n+1}\leq a_n+\delta_n, \sum_{n=1}^{\infty}\delta_n<\infty$, then $\{a_n\}$ converges and if in addition the sequence $\{a_n\}$ has a subsequence which converges to 0, then the original sequence $\{a_n\}$ converges to 0.

The following lemma can be found in [9].

Lemma 2.16 [9] Let E be a normed linear space, $A,B\in CB(E)$ and $x,y\in E$. Then, the following hold;

a) $D(x+A,x+B)=D(A,B);$;
b) $D(-A,-B)=D(A,B);$
c) $D(x+A,y+B)\leq\|x-y\|+D(A,B);$
d) $D(\{x\},A)=\sup_{a\in A}\{\|x-a\|\};$
e) $D(\{x\},A)=D(0,x-A).$

Consequently, from (d) the following was obtained [9]

Lemma 2.17 [9] Let K be a non empty and closed subset of a real Hilbert space H and let $T:K\to CB(H)$ be generalized k- strictly pseudo contractive mapping. Then, for any given $\{x_n\}$ in K there exists $u_n\in Tx_n$ such that

$$D^2(\{x_n\},Tx_n)\leq\|x_n-u_n\|^2+\frac{1}{n^2}.$$

In particular, if Tx_n is proximal, there exists

$$u_n\in Tx_n,\ni\|u_n-x_n\|^2=D^2(x_n,Tx_n)\leq\|x_n-u_n\|^2+\frac{1}{n^2}.$$

3. Main Results

Let $T_1,T_2,...T_K:K\to$Prox(E) be family of non self and multivalued mappings on a non-empty closed, convex subset K of a real uniformly convex Banach space E, our objective is to introduce an iterative method for common fixed point of the family and determine conditions for convergence of the iterative method. We use the condition that mappings to be inward instead of metric projection, which is computationally expensive in many cases, and we prove both weak and strong convergence of the iterative method. Thus, we shall have the following lemma.

Lemma 3.1 Let K be a nonempty, closed and convex subset of a real Banach space E, $T_1,T_2,...T_N:K\to CB(E)$ or Prox(E) be multivalued mappings, $u_k\in T_kx$ .Define $h_{u_k}:K\to\Re$ by

$$h_{u_k}(x)=\inf\{\lambda\in[0,1]:\lambda x+(1-\lambda)u_k\in K\}.$$

Then for any $x\in K$, the following hold:

1) $h_{u_k}(x)\in[0,1]$ and $h_{u_k}(x)=0$ if and only if $u_k\in K$;
2) If $\beta\in[h_{u_k}(x),1]$, then $\beta x+(1-\beta)u_k\in K$;
3) If T_k is inward mapping $h_{u_k}(x)<1$;
4) If $u_k\notin K$, then $h_{u_k}(x)x+(1-h_{u_k}(x))u_k\in\partial K$, where ∂K is the boundary of K.

The proof of this lemma follows from lemma 3.1 of Takele and Reddy [32] Calo and Mariao [12] and Tuffa and Zegeye [37].

Theorem 3.2: Let $T_1,T_2,...T_N:K\to$Prox(H) be family of, non self, multi valued, nonexpansive and inward mappings on a non-empty, closed and convex subset K of a real Hilbert space H, with $F=\bigcap_{k=1}^{N}F(T_k)$ non empty, $T_k=T_{k(ModN)+1}$, for all $p\in F=\bigcap_{k=1}^{N}F(T_k)$, $T_K(p)=\{p\}$. Let $\{x_n\}$ be a sequence of Mann type defined by the iterative method given by

$$\begin{cases} x_1\in K, u_1\in T_1x_1, \alpha_1=\max\{\alpha, h_{u_1}(x_1)\}, \alpha>0, \\ x_{n+1}=\alpha_n x_n+(1-\alpha_n)u_n, u_n\in T_nx_n \\ \ni \|u_n-u_{n+1}\|\le D(T_nx_n, T_{n+1}x_{n+1}), \\ \alpha_{n+1}=\max\{\alpha_n, h_{u_{n+1}}(x_{n+1})\}, \\ h_{u_n}(x_n)=\inf\{\lambda\ge 0:\lambda x_n+(1-\lambda)u_n\in K\} \end{cases}$$

is well-defined and if $\{\alpha_n\}\subseteq[\varepsilon,1-\varepsilon]\subset(0,1)$ for some $\varepsilon>0$, then the sequence $\{x_n\}$ converges weakly some element p of $F=\bigcap_{k=1}^{N}F(T_k)$. Moreover, if $\sum_{n=1}^{\infty}(1-\alpha_n)<\infty$ and (F,K) satisfies S-condition, then the convergence is strong.

Proof: By lemma 3.1 $\{x_n\}$ is well-defined and is in K, thus, to prove the theorem first we prove $\{x_n\}$ is fejer monotone with respect to F, to do so, let $p\in F$, then we have the following inequality;

$$\begin{aligned} \|x_{n+1}-p\| &= \|\alpha_n x_n+(1-\alpha_n)u_n-p\| \\ &\le \alpha_n\|x_n-p\|+(1-\alpha_n)\|u_n-Tp\| \\ &\le \alpha_n\|x_n-p\|+(1-\alpha_n)D(T_nx_n,T_np) \\ &= \alpha_n\|x_n-p\|+(1-\alpha_n)\|x_n-p\|=\|x_n-p\|. \end{aligned} \tag{3.1}$$

Thus, the sequence $\{x_n\}$ is fejer monotone with respect to F.

Since $\|x_n-p\|$ is decreasing and bounded below it converges, and hence $\{x_n\}$ and $\{u_n\}$ are bounded.

That is,

$$\begin{aligned} \|u_n\| &= \|u_n-p+p\|\le\|u_n-p\|+\|p\| \\ &\le D(T_nx_n,T_np)\le\|x_n-p\|+\|p\|\le M \end{aligned}$$

for some $M\ge 0$.

Also, we have the following inequality,

$$\begin{aligned} &\|x_{n+1}-p\|^2 \\ &=\|\alpha_n x_n+(1-\alpha_n)u_n-p\|^2 \\ &=\alpha_n\|x_n-p\|^2+(1-\alpha_n)\|u_n-Tp\|^2 \\ &-\alpha_n(1-\alpha_n)\|x_n-u_n\|^2 \\ &\le\alpha_n\|x_n-p\|^2+(1-\alpha_n)D^2(T_nx_n,T_np) \\ &-\alpha_n(1-\alpha_n)\|x_n-u_n\|^2 \\ &\le\alpha_n\|x_n-p\|^2+(1-\alpha_n)\|x_n-p\|^2 \\ &-\alpha_n(1-\alpha_n)\|x_n-u_n\|^2 \\ &=\|x_n-p\|^2-\alpha_n(1-\alpha_n)\|x_n-u_n\|^2. \end{aligned} \tag{3.2}$$

Suppose $\{\alpha_n\}\subseteq[\varepsilon,1-\varepsilon]$, then

$$\begin{aligned} &\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)\|x_n-u_n\|^2 \\ &\le\sum_{n=1}^{\infty}(\|x_n-p\|^2-\|x_{n+1}-p\|^2)<\infty. \end{aligned} \tag{3.3}$$

Hence, $\lim_{n\to\infty}\|x_n-u_n\|=0$ and

$$\|x_{n+1}-x_n\|=(1-\alpha_n)\|x_n-u_n\|,$$

which implies that,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\alpha_n)\|x_n-u_n\|=0.$$

Thus, by induction and triangle inequality, we have

$$\lim_{n\to\infty}\|x_{n+i}-x_n\|=0 \text{ for all } 0\le i\le N. \tag{3.4}$$

Thus,

$$\lim_{n\to\infty}\|x_n-u_{n+i}\|\le\lim_{n\to\infty}\|x_n-x_{n+i}\|+\lim_{n\to\infty}\|x_{n+i}-u_{n+i}\|=0.$$

Thus, by definition of infimum and d we have $d(x_n,T_{n+i}x_{n+i})\le\|x_n-u_{n+i}\|\to 0$ as $n\to\infty$.

$$\begin{aligned} &0\le d(x_n,T_{n+i}x_n) \\ &\le\|x_n-x_{n+i}\|+d(x_{n+i},T_{n+i}x_{n+i}) \\ &+D(T_{n+i}x_{n+i},T_{n+i}x_n)\to 0, n\to\infty. \end{aligned} \tag{3.5}$$

Thus, $\lim_{n\to\infty}d(x_n,T_{n+i}x_{n+i})=0$. Since $\{x_n\}$ is bounded, it has a convergent subsequence $\{x_{n_j}\}$ such that $x_{n_j}\to x$ weakly, since K is closed and convex, $x\in K$, and $n+i=l(\bmod N)+1$ for some $l\in\{1,2,...N\}$ and for each $l\in\{1,2,...N\}$ there is some $0\le i\le N$ such that $n+i=l(\bmod N)+1$.

Thus, $d(x_n,T_lx_n)\to 0$ as $n\to\infty$.. $d(x_{n_j},T_lx_{n_j})\to 0$. Since T_l is demi closed, we have, $x\in F(T_l)$ and since l is arbitrary, we have $x\in\bigcap_{k=1}^{N}F(T_k)$.

Since H satisfies opial's condition and $\|x_n-x\|$ is convergent, we get $x_n\to x$ weakly.

Thus, the sequence $\{x_n\}$ converges weakly some element p of $F=\bigcap_{k=1}^{N}F(T_k)$.

Moreover, if $\sum_{n=1}^{\infty}(1-\alpha_n)<\infty$, then

$$\sum_{n=1}^{\infty}\|x_{n+1}-x_n\|=\sum_{n=1}^{\infty}(1-\alpha_n)\|x_n-u_n\|<M\sum_{n=1}^{\infty}(1-\alpha_n)<\infty.$$

Hence, the sequence $\{x_n\}$ is strongly Cauchy, thus it is Cauchy and converges to some element $x \in K$.

Moreover, since T_n is inward, then $h_{u_n}(x) < 1$, hence for every $\beta_n \in [h_{u_n}(x), 1)$, we have $\beta_n x + (1-\beta_n)u_n \in K$, in particular, since $\lim\limits_{n\to\infty} \alpha_n = 1$, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim\limits_{j\to\infty} h_{u_{n_j}}(x_{n_j}) = 1$, $\frac{j}{j+1} h_{u_{n_j}}(x_{n_j})x_{n_j} + (1 - \frac{j}{j+1} h_{u_{n_j}}(x_{n_j}))u_{n_j} \notin K$, whose limit is $x \in K$. Thus $x \in \partial K$, and since the pair (F, K) satisfies S- condition $x \in F$.

Thus $\{x_n\}$ converges strongly to some element $p \in \bigcap_{k=1}^{N} F(T_k)$.

Theorem 3.3: Let $T_1, T_2, \ldots T_N : K \to$ Prox (E) be family of non self, multi valued, nonexpansive and inward mappings on a non-empty, closed and convex subset K of a real Uniformly convex Banach space E, satisfying opial's condition with $F = \bigcap_{k=1}^{N} F(T_k)$ non empty, $T_k = T_{k(ModN)+1}$, for all $p \in F = \bigcap_{k=1}^{N} F(T_k)$, for each k, $T_K(p) = \{p\}$, and suppose $I - T_k$ is demi closed at 0, let $\{x_n\}$ be a sequence of Mann type defined by the iterative method,

$$\begin{cases} x_1 \in K, u_1 \in T_1 x_1, \alpha_1 = \max\left\{\alpha, h_{u_1}(x_1)\right\}, \alpha > 0, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)u_n, u_n \in T_n x_n, \\ \ni \|u_n - u_{n+1}\| \le D(T_n x_n, T_{n+1}x_{n+1}), \\ \alpha_{n+1} = \max\left\{\alpha_n, h_{u_{n+1}}(x_{n+1})\right\}, \\ h_{u_n}(x_n) = \inf\left\{\lambda \ge 0 : \lambda x_n + (1-\lambda)u_n \in K\right\}. \end{cases}$$

Then the sequence is well-defined and if $\{\alpha_n\} \subseteq [\varepsilon, 1-\varepsilon] \subset (0,1)$ for some $\varepsilon > 0$, and E satisfies opial's condition, then the sequence $\{x_n\}$ converges weakly some element p of $F = \bigcap_{k=1}^{N} F(T_k)$. Moreover, if $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$ and (F,K) satisfies S-condition, then the convergence is strong.

Proof: By lemma 3.1 $\{x_n\}$ is well-defined and is in K, thus to prove the theorem, first we prove $\{x_n\}$ is fejer monotone with respect to F, to do so, let $p \in F$, then we have the following in equality;

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n x_n + (1-\alpha_n)u_n - p\| \\ &\le \alpha_n\|x_n - p\| + (1-\alpha_n)\|u_n - Tp\| \\ &\le \alpha_n\|x_n - p\| + (1-\alpha_n)D(T_n x_n, T_n p) \\ &= \alpha_n\|x_n - p\| + (1-\alpha_n)\|x_n - p\| = \|x_n - p\|. \end{aligned} \tag{3.6}$$

Thus, the sequence $\{x_n\}$ is fejer monotone with respect to F.

Since $\|x_n - p\|$ is decreasing and bounded below, thus it converges, and hence $\{x_n\}$ and $\{u_n\}$ are bounded.

That is,

$$\begin{aligned} \|u_n\| &= \|u_n - p + p\| \le \|u_n - p\| + \|p\| \\ &\le D(T_n x_n, T_n p) \le \|x_n - p\| + \|p\| \le M \end{aligned}$$

for some $M \ge 0$.

Suppose $\{\alpha_n\} \subseteq [\varepsilon, 1-\varepsilon]$, then

$$\begin{aligned} &\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)\|x_n - u_n\|^2 \\ &\le \sum_{n=1}^{\infty}(\|x_n - p\|^2 - \|x_{n+1} - p\|^2) < \infty \end{aligned}$$

can be shown by Lemma 2.4, Xu [38] since E is uniformly convex Banach space , for $p > 1, R > 1$ real numbers, there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that $\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)g(\|x - y\|)$ for all $x, y \in B_R(0) = \{x \in E : \|x\| < R\}$ and $\lambda \in [0,1]$, where $W_p(\lambda) = \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$. (3.7)

Since $\{x_n\}$ is bounded R can be chosen so that $\{x_n\} \subseteq B_R(0)$. If $p = 2 > 1$, we have the inequality $\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - W_2(\lambda)g(\|x - y\|)$.

Thus, for $\lambda = \alpha_n, x = x_n, y = u_n$ we get

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ &= \|\alpha_n x_n + (1-\alpha_n)u_n - p\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|u_n - T_n p\|^2 \\ &-\alpha_n(1-\alpha_n)g\|x_n - u_n\| \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)D^2(T_n x_n, T_n p) \\ &-\alpha_n(1-\alpha_n)g\|x_n - u_n\| \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\ &-\alpha_n(1-\alpha_n)g\|x_n - u_n\| \\ &= \|x_n - p\|^2 - \alpha_n(1-\alpha_n)g\|x_n - u_n\|. \end{aligned} \tag{3.8}$$

Which implies

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)g\|x_n - u_n\| \le \sum_{n=1}^{\infty}\|x_n - p\|^2 - \|x_{n+1} - p\|^2 < \infty,$$

cancellation of terms and convergence of $\|x_n - p\|$ with $0 < \alpha_n < 1$ and hence, $W_2(\alpha_n) = \alpha_n(1-\alpha_n) \ge \varepsilon^2 > 0$ for some $\varepsilon > 0$ we get $\sum_{n=1}^{\infty} g\|x_n - u_n\| < \infty$,

Since g is continuous, strictly increasing, and convex function $g\|x_n - u_n\| \to 0$ as $n \to \infty$.

Also by lemma 2.7 [40] $\|x_n - u_n\| \to 0$ as $n \to \infty$.

Thus, $\|x_{n+1} - x_n\| = (1-\alpha_n)\|x_n - u_n\|$, which implies that, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\alpha_n)\|x_n - u_n\| = 0$.

Thus, by induction and triangle inequality, we have $\lim_{n\to\infty} \|x_{n+i} - x_n\| = 0$ for all $0 \le i \le N$.

Thus,

$$\begin{aligned}&\lim_{n\to\infty} \|x_n - u_{n+i}\| \\ &\le \lim_{n\to\infty} \|x_n - x_{n+i}\| + \lim_{n\to\infty} \|x_{n+i} - u_{n+i}\| = 0.\end{aligned} \tag{3.9}$$

Thus, by definition of infimum and d, we have $d(x_n, T_{n+i}x_{n+i}) \le \|x_n - u_{n+i}\| \to 0$, as $n \to \infty$.

$$\begin{aligned}&0 \le d(x_n, T_{n+i}x_n) \\ &\le \|x_n - x_{n+i}\| + d(x_{n+i}, T_{n+i}x_{n+i}) \\ &+ D(T_{n+i}x_{n+i}, T_{n+i}x_n) \to 0, n \to \infty.\end{aligned}$$

Thus, $\lim_{n\to\infty} d(x_n, T_{n+i}x_n) = 0$. Since $\{x_n\}$ is bounded, it has a convergent subsequence $\{x_{n_j}\}$ such that $x_{n_j} \to x$ weakly, since K is closed and convex, $x \in K$ and $n+i = l(\mathrm{mod}\, N)+1$ for some $l \in \{1,2,...N\}$ and for each $l \in \{1,2,...N\}$ there is some $0 \le i \le N$ such that, $n+i = l(\mathrm{mod}\, N)+1$. Thus, $d(x_n, T_l x_n) \to 0$ as $n \to \infty$.

Thus, $d(x_{n_j}, T_l x_{n_j}) \to 0$ as $n \to \infty$, since T_l is demi closed, we have, $x \in F(T_l)$ and since l is arbitrary, we have $x \in \bigcap_{k=1}^{N} F(T_k)$.

Since E satisfies opial's condition and $\|x_n - x\|$ is convergent, we get $x_n \to x$ weakly.

Thus, the sequence $\{x_n\}$ converges weakly to some element p of $F = \bigcap_{k=1}^{N} F(T_k)$.

Moreover, if $\sum_{n=1}^{\infty} (1-\alpha_n) < \infty$, then

$$\sum_{n=1}^{\infty} \|x_{n+1} - x_n\| = \sum_{n=1}^{\infty} (1-\alpha_n)\|x_n - u_n\| < M \sum_{n=1}^{\infty} (1-\alpha_n) < \infty.$$

Hence, the sequence $\{x_n\}$ is strongly Cauchy, thus it is Cauchy and converges to some element $x \in K$.

Moreover, since T_n is inward, then $h_{u_n}(x) < 1$, hence for every $\beta_n \in [h_{u_n}(x), 1)$, we have $\beta_n x + (1-\beta_n)u_n \in K$, in particular, since $\lim_{n\to\infty} \alpha_n = 1$, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} h_{u_{n_j}}(x_{n_j}) = 1$, $\frac{j}{j+1} h_{u_{n_j}}(x_{n_j}) x_{n_j} + (1 - \frac{j}{j+1} h_{u_{n_j}}(x_{n_j})) u_{n_j} \notin K$, whose limit is $x \in K$. Thus $x \in \partial K$, and since the pair (F, K) satisfies S- condition $x \in F$.

Thus $\{x_n\}$ converges strongly to some element $p \in \bigcap_{k=1}^{N} F(T_k)$.

Theorem 3.4 Let K be a convex, closed and nonempty subset of a real Hilbert space H and let $\{T_n\}_{n=1}^{\infty} : K \to \mathrm{Pr}\,ox(H)$ be a uniformly weakly closed, countable family of non self, multi valued and nonexpansive mappings with $F = \bigcap_{k=1}^{\infty} F(T_k)$ is non empty and for all $p \in F = \bigcap_{k=1}^{\infty} F(T_k)$, $T_n p = \{p\}$ .Let $\{x_n\}$ be a sequence defined by the Mann type iterative method,

$$\begin{cases} x_1 \in K, u_1 \in T_1 x_1, \alpha_1 = \max\{\alpha, h_{u_1}(x_1)\}, \alpha > 0, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) u_n, \\ u_n \in T_n x_n \ni \|u_n - p\| \le D(T_n x_n, p), \\ \|u_n - u_{n+1}\| \le D(T_n x_n, T_{n+1} x_{n+1}), \\ \alpha_{n+1} = \max\{\alpha_n, h_{u_{n+1}}(x_{n+1})\}, \\ h_{u_n}(x_n) = \inf\{\lambda \ge 0 : \lambda x_n + (1-\lambda) u_n \in K\}. \end{cases}$$

Then, the sequence $\{x_n\}$ is well-defined and if $\{\alpha_n\} \subseteq [\varepsilon, 1-\varepsilon] \subset (0,1)$ for some $\varepsilon > 0$, then the sequence $\{x_n\}$ converges weakly some element p of $F = \bigcap_{k=1}^{\infty} F(T_k)$.

Moreover, if $\sum_{n=1}^{\infty} (1-\alpha_n) < \infty$, and (F,K) satisfies S-condition, then the convergence is strong.

Proof, let $p \in F$, and by lemma 2.3 [17] there is a sequence $\{u_n\}$, $u_n \in T_n x_n$ satisfying

$$\|u_n - u_{n+1}\| \le D(T_n x_n, T_{n+1} x_{n+1}),$$

thus we have the following in equality;

$$\begin{aligned}&\|x_{n+1} - p\| = \|\alpha_n x_n + (1-\alpha_n) u_n - p\| \\ &\le \alpha_n \|x_n - p\| + (1-\alpha_n)\|u_n - Tp\| \\ &\le \alpha_n \|x_n - p\| + (1-\alpha_n) D(T_n x_n, T_n p) \\ &= \alpha_n \|x_n - p\| + (1-\alpha_n)\|x_n - p\| = \|x_n - p\|.\end{aligned} \tag{3.11}$$

Thus $\{x_n\}$ is fejer monotone with respect to F.

Since $\|x_n - p\|$ is decreasing and bounded below, it converges, and hence $\{x_n\}$ and $\{u_n\}$ are bounded.

That is,

$$\|u_n\| = \|u_n - p + p\| \le \|u_n - p\| + \|p\|$$
$$\le D(T_n x_n, T_n p) \le \|x_n - p\| + \|p\| \le M$$

for some $M \ge 0$.

Also, we have the following inequality,

$$\begin{aligned}
&\|x_{n+1} - p\|^2 = \|\alpha_n x_n + (1-\alpha_n)u_n - p\|^2 \\
&= \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|u_n - Tp\|^2 \\
&-\alpha_n(1-\alpha_n)\|x_n - u_n\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)D^2(T_n x_n, T_n p) \\
&-\alpha_n(1-\alpha_n)\|x_n - u_n\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&-\alpha_n(1-\alpha_n)\|x_n - u_n\|^2 \\
&= \|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - u_n\|^2 .
\end{aligned} \tag{3.12}$$

Suppose $\{\alpha_n\} \subseteq [\varepsilon, 1-\varepsilon]$, then

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)\|x_n - u_n\|^2$$
$$\le \sum_{n=1}^{\infty} (\|x_n - p\|^2 - \|x_{n+1} - p\|^2) < \infty.$$

Hence, $\lim_{n\to\infty} \|x_n - u_n\| = 0.$

Thus, by definition of infimum and d we have $d(x_n, T_n x_n) \le \|x_n - u_n\| \to 0,$ as $n \to \infty$.

Since $\{x_n\}$ is bounded, it has a convergent subsequence $\{x_{n_j}\}$ such that $x_{n_j} \to x$ weakly, since K is closed and convex, $x \in K$, $d(x_{n_j}, T_{n_j} x_{n_j}) \to 0$, since $\{T_n\}_{n=1}^{\infty}$ is uniformly weakly closed, $x \in F$, that is, $x \in \bigcap_{k=1}^{\infty} F(T_k)$.

Since H satisfies opial's condition and $\|x_n - x\|$ is convergent, we get $x_n \to x$ weakly.

Thus, the sequence $\{x_n\}$ converges weakly to some element p of $F = \bigcap_{k=1}^{\infty} F(T_k)$.

Moreover, if $\sum_{n=1}^{\infty} (1-\alpha_n) < \infty,$ then

$$\sum_{n=1}^{\infty} \|x_{n+1} - x_n\| = \sum_{n=1}^{\infty} (1-\alpha_n)\|x_n - u_n\|$$
$$< M \sum_{n=1}^{\infty} (1-\alpha_n) < \infty.$$

Hence, the sequence $\{x_n\}$ is strongly Cauchy, thus it is Cauchy and converges to some element $x \in K.$

Moreover, since T_n is inward, then $h_{u_n}(x) < 1$, hence for every $\beta_n \in [h_{u_n}(x), 1)$, we have $\beta_n x + (1-\beta_n)u_n \in K$, in particular, since $\lim_{n\to\infty} \alpha_n = 1$, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} h_{u_{n_j}}(x_{n_j}) = 1$, $\frac{j}{j+1} h_{u_{n_j}}(x_{n_j}) x_{n_j} + (1 - \frac{j}{j+1} h_{u_{n_j}}(x_{n_j})) u_{n_j} \notin K$, whose limit is $x \in K$. Thus $x \in \partial K$, and since the pair (F, K) satisfies S- condition $x \in F$.

Thus $\{x_n\}$ converges strongly to some element $p \in \bigcap_{k=1}^{\infty} F(T_k)$.

Theorem 3.5 Let K be a convex, closed and nonempty subset of a real Uniformly convex Banach space E satisfying opial's condition and let $\{T_n\}_{n=1}^{\infty} : K \to Prox(E)$ be a uniformly weakly closed, countable family of non self, multi valued and nonexpansive(quasi non expansive) mappings with $F = \bigcap_{k=1}^{\infty} F(T_k)$ is non empty and for all $p \in F = \bigcap_{k=1}^{\infty} F(T_k)$, $T_n p = \{p\}$. Let $\{x_n\}$ be a sequence defined by the Mann type iterative method

$$\begin{cases}
x_1 \in K, u_1 \in T_1 x_1, \alpha_1 = \max\{\alpha, h_{u_1}(x_1)\}, \alpha > 0, \\
x_{n+1} = \alpha_n x_n + (1-\alpha_n)u_n, u_n \in T_n x_n, \\
\ni \|u_n - u_{n+1}\| \le D(T_n x_n, T_{n+1} x_{n+1}), \\
\alpha_{n+1} = \max\{\alpha_n, h_{u_{n+1}}(x_{n+1})\}, \\
h_{u_n}(x_n) = \inf\{\lambda \ge 0 : \lambda x_n + (1-\lambda)u_n \in K\}.
\end{cases}$$

Then, the sequence $\{x_n\}$ is well-defined and if $\{\alpha_n\} \subseteq [\varepsilon, 1-\varepsilon] \subset (0,1)$ for some $\varepsilon > 0$, and E satisfies opial's condition, then the sequence $\{x_n\}$ converges weakly some element p of $F = \bigcap_{k=1}^{\infty} F(T_k)$.

Moreover, if $\sum_{n=1}^{\infty} (1-\alpha_n) < \infty$, and (F,K) satisfies S-condition , then the convergence is strong.

Proof can be made in similar way as theorem 3.3 and 3.4.

Theorem 3.6 Let K be a strictly convex, closed and nonempty subset of a real Hilbert space H and let $T: K \to \Pr ox(H)$ be a non self, multi valued and k-strictly pseudo contractive and inward mapping with $F = F(T)$ is non empty and for each $x \in K$, Tx is closed and $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined by the iterative method,

$$\begin{cases} x_1 \in K, u_1 \in Tx_1, \alpha_1 = \max\left\{\alpha, h_{u_1}(x_1)\right\}, k < \alpha < 1, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)u_n, u_n \in Tx_n \\ \ni \|u_n - u_{n+1}\| \le D(Tx_n, Tx_{n+1}), \\ \alpha_{n+1} = \max\left\{\alpha_n, h_{u_{n+1}}(x_{n+1})\right\}, \\ h_{u_n}(x_n) = \inf\left\{\lambda \ge 0 : \lambda x_n + (1-\lambda)u_n \in K\right\}. \end{cases}$$

Then, the sequence $\{x_n\}$ is well-defined and the sequence $\{x_n\}$ converges weakly to some element p of $F = F(T)$. Moreover, if $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$, then the convergence is strong.

Proof. By lemma 3.1 $\{x_n\}$ is well-defined and is in K, thus, to prove the theorem first we prove $\{x_n\}$ is fejer monotone with respect to F, to do so, let $p \in F$, then the following holds;

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)u_n - p\|^2 \\ &= \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - u_n\|^2 \\ &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)D^2(Tx_n, Tp) \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - u_n\|^2 \\ &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)[\|x_n - p\|^2 + k\|x_n - u_n\|^2] \\ &= \|x_n - p\|^2 - (1-\alpha_n)(a_n - k)\|x_n - u_n\|^2 \\ &\le \|x_n - p\|^2. \end{aligned} \tag{3.13}$$

Thus $\{x_n\}$ is fejer monotone with respect to F.

Since $\|x_n - p\|$ is decreasing and bounded below, it converges, and hence $\{x_n\}$ and since T is Lipschitzian by lemma 2.8 [8] $\{u_n\}$ are bounded.

That is,

$$\begin{aligned} \|u_n\| &= \|u_n - p + p\| \le \|u_n - p\| + \|p\| \\ &\le D(Tx_n, Tp) + p \le \frac{1+\sqrt{k}}{1-\sqrt{k}}\|x_n - p\| + \|p\| \le M \end{aligned}$$

for some $M \ge 0$.

We also have the following inequality;

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)u_n - p\|^2 \\ &= \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - u_n\|^2 \\ &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)D^2(Tx_n, Tp) \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - u_n\|^2 \\ &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)[\|x_n - p\|^2 + k\|x_n - u_n\|^2] \\ &= \|x_n - p\|^2 - (1-\alpha_n)(a_n - k)\|x_n - u_n\|^2. \end{aligned}$$

Thus,

$$\begin{aligned} &\sum_{n=1}^{\infty}(\alpha_n - k)(1-\alpha_n)\|x_n - u_n\|^2 \\ &\le \sum_{n=1}^{\infty}(\|x_n - p\|^2 - \|x_{n+1} - p\|^2) < \infty. \end{aligned} \tag{3.14}$$

Suppose $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, since $\alpha_n > k$, there exists $\varepsilon > 0$, such that $\alpha_n - k > \varepsilon$, thus $\sum_{n=1}^{\infty}(\alpha_n - k)(1-\alpha_n) = \infty$, hence $\liminf_{n\to\infty}\|x_n - u_n\| = 0$, also from the method of proof of Mariano and Trombetta [22] it can be seen $\{\|x_n - u_n\|\}$ is decreasing as $\|x_n - x_{n+1}\| = (1-\alpha_n)\|x_n - u_n\|$ and

$$\begin{aligned} &\|x_{n+1} - u_{n+1}\|^2 = \|\alpha_n x_n + (1-\alpha_n)u_n - u_{n+1}\|^2 \\ &= \|\alpha_n(x_n - u_{n+1}) + (1-\alpha_n)(u_n - u_{n+1}\|^2 \\ &= \alpha_n \|x_n - x_{n+1} + x_{n+1} - u_{n+1}\|^2 + (1-\alpha_n)\|u_n - u_{n+1}\|^2 \\ &-\alpha_n(1-\alpha_n)\|x_n - u_n\|^2 \\ &\le \alpha_n \begin{bmatrix} \|x_n - x_{n+1}\|^2 + \|x_{n+1} - u_{n+1}\|^2 \\ +2\langle x_n - x_{n+1}, x_{n+1} - u_{n+1}\rangle \end{bmatrix} \\ &-\alpha_n(1-\alpha_n)\|x_n - u_n\|^2 + (1-\alpha_n)D^2(Tx_n, Tx_{n+1}) \\ &\le \alpha_n \begin{bmatrix} (1-\alpha_n)^2\|x_n - u_n\|^2 + \|x_{n+1} - u_{n+1}\|^2 \\ +2(1-\alpha_n)\langle x_n - u_n, x_{n+1} - u_{n+1}\rangle \end{bmatrix} \\ &-\alpha_n(1-\alpha_n)\|x_n - u_n\|^2 \\ &+(1-\alpha_n)[(1-\alpha_n)^2\|x_n - u_n\|^2 \\ &+k[\|x_n - u_n\|^2 + \|(x_{n+1} - u_{n+1}\|^2 - 2\langle x_n - u_n, x_{n+1} - u_{n+1}\rangle] \\ &\le (\alpha_n + k(1-\alpha_n))\|x_{n+1} - u_{n+1}\|^2 \\ &+(1-\alpha_n)(1 + k - 2\alpha_n)\|x_n - u_n\|^2 \\ &+2(\alpha_n - k)(1-\alpha_n)\|x_n - u_n\|\|x_{n+1} - u_{n+1}\| \end{aligned}$$

Letting $\gamma_n = \|x_n - u_n\|$, we have the following;

$$\left(\frac{\gamma_{n+1}}{\gamma_n}\right)^2 - 2\frac{(\alpha_n - k)}{1-k}\frac{\gamma_{n+1}}{\gamma_n} \le \frac{1 + k - 2\alpha_n}{1-k},$$

solving the inequality we get $\frac{\gamma_{n+1}}{\gamma_n} \le 1$, which gives $\{\ \gamma_n\ \} = \{\ \|x_n - u_n\|\ \}$ is decreasing and hence converges to $\liminf_{n\to\infty}\|x_n - u_n\| = 0$, thus $\lim_{n\to\infty}\|x_n - u_n\| = 0$, as a result, $0 \le \lim_{n\to\infty} d(x_n, Tx_n) \le \lim_{n\to\infty}\|x_n - u_n\| = 0$ which implies that $d(x_n, Tx_n) \to 0$ as $n \to \infty$.

On the other hand, since the sequence $\{x_n\}$ is bounded, it has a weakly convergent subsequence $\{x_{n_j}\}$ such that

$x_{n_j} \to x$ weakly, since K is closed and convex $x \in K$, since $I-T$ is demi closed at 0, $x \in Tx$.

Since, every Hilbert space satisfies opial's condition, $x_n \to x$ weakly for some $x \in Tx$.

Moreover, if $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$, then

$$\sum_{n=1}^{\infty}\|x_{n+1}-x_n\| = \sum_{n=1}^{\infty}(1-\alpha_n)\|x_n-u_n\| < M'\sum_{n=1}^{\infty}(1-\alpha_n) < \infty.$$

Hence, the sequence $\{x_n\}$ is strongly Cauchy, thus it is Cauchy and converges to some element $x \in K$.

Moreover, since T is inward, then $h_{u_n}(x) < 1$, hence for every $\beta_n \in [h_{u_n}(x),1)$, we have $\beta_n x + (1-\beta_n)u_n \in K$, in particular, since $\lim_{n\to\infty}\alpha_n = 1$, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} h_{u_{n_j}}(x_{n_j}) = 1$,

$$\frac{j}{j+1}h_{u_{n_j}}(x_{n_j})x_{n_j} + (1-\frac{j}{j+1}h_{u_{n_j}}(x_{n_j}))u_{n_j} \notin K$$

whose limit is $x \in K$ thus, $x \in \partial K$. The continuity of Lipschitz mapping T gives

$$\|u_n - u_{n+1}\| \le D(Tx_n, Tx_{n+1}) \le \frac{1+\sqrt{k}}{1-\sqrt{k}}\|x_n - x_{n+1}\|,$$

thus, there is $u \in H$ such that $u_n \to u$ as $n \to \infty$. Since T is continuous and with each Tx is closed the following holds; $d(u_n,Tx) \le d(u_n,Tx_n) + D(Tx_n,Tx) \to 0$ as $n \to \infty$, $u \in Tx$, thus, for all $\beta \in [h_u(x),1)$ we have $\beta x + (1-\beta)u \in K$, as a result it can be shown that $\beta x + (1-\beta)u \in \partial K$. Since K is strictly convex, in similar fashion (see [37]) it can be seen that $\beta x + (1-\beta)u = x$, hence $u = x \in Tx$. Thus, the sequence $\{x_n\}$ converges strongly to some element $p \in F(T)$.

Theorem 3.7 Let K be a strictly convex, closed and nonempty subset of a real Hilbert space H and let $T: K \to \Pr ox(H)$ be a non self, multi valued and generalized k-strictly pseudo contractive and inward mapping with $F = F(T)$ is non empty and for all $p \in F(T)$, $Tp = \{p\}$ for each $x \in K$, Tx is closed. Let $\{x_n\}$ be a by the iterative sequence defined method,

$$\begin{cases} x_1 \in K, u_1 \in Tx_1, \alpha_1 = \max\{\alpha, h_{u_1}(x_1)\}, \alpha > k, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)u_n, u_n \in Tx_n, \\ \ni \|u_n - u_{n+1}\| \le D(Tx_n, Tx_{n+1}), \\ D^2(\{x_n\}, Tx_n) \le \|x_n - u_n\| + \frac{1}{n^2}, \\ \alpha_{n+1} = \max\left\{\alpha_n, h_{u_{n+1}}(x_{n+1}), 1-\frac{1}{(n+1)^2}\right\}; \\ h_{u_n}(x_n) = \inf\{\lambda \ge 0 : \lambda x_n + (1-\lambda)u_n \in K\}. \end{cases}$$

Then, the sequence $\{x_n\}$ is well-defined and the sequence $\{x_n\}$ converges strongly to some element p of $F = F(T)$.

Proof. By lemma 3.1 $\{x_n\}$ is well-defined and is in K.

Let $p \in F$. Since $\|u_n - p\| \le D(Tx_n, Tp)$ and by lemma 2.6 we have the following inequality;

$$\begin{aligned} \|x_{n+1}-p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)u_n - p\|^2 \\ &= \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|u_n-p\|^2 \\ &\quad -\alpha_n(1-\alpha_n)\|x_n-u_n\|^2 \\ &\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)D^2(Tx_n,Tp) \\ &\quad -\alpha_n(1-\alpha_n)\|x_n-u_n\|^2 \\ &\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)[\|x_n-p\|^2 \\ &\quad + kD^2(x_n - Tx_n, 0] - (1-\alpha_n)a_n\|x_n-u_n\|^2 \qquad (3.15) \\ &= \|x_n-p\|^2 + (1-\alpha_n)k\left[D^2(\{x_n\}, Tx_n\right] \\ &\quad -\alpha_n(1-\alpha_n)\|x_n-u_n\|^2 \\ &\le \|x_n-p\|^2 + (1-\alpha_n)k\left[\|x_n-u_n\|^2 + \frac{1}{n^2}\right] \\ &\quad -\alpha_n(1-\alpha_n)\|x_n-u_n\|^2 \\ &\le \|x_n-p\|^2 + \frac{(1-\alpha_n)k}{n^2}. \end{aligned}$$

Thus by lemma 2.15 we have the sequence $\{\|x_n - p\|\}$ converges to some $r \ge 0$.

Thus, the sequence $\{x_n\}$ and hence $\{u_n\}$ are bounded.

Since $\sum_{n=1}^{\infty}(1-\alpha_n) < \sum_{n=1}^{\infty}\frac{1}{(n+1)^2} < \infty$, then we have $\sum_{n=1}^{\infty}\|x_{n+1}-x_n\| = \sum_{n=1}^{\infty}(1-\alpha_n)\|x_n-u_n\| < M'\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$ for some $M \ge 0$.

Hence, the sequence $\{x_n\}$ is strongly Cauchy, thus it is Cauchy and converges to some element $x \in K$.

Moreover, since T is inward, then $h_{u_n}(x) < 1$, hence for every $\beta_n \in [h_{u_n}(x),1)$, we have $\beta_n x + (1-\beta_n)u_n \in K$. Since $\lim_{n\to\infty}\alpha_n = 1$, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} h_{u_{n_j}}(x_{n_j}) = 1$,

$$\frac{j}{j+1}h_{u_{n_j}}(x_{n_j})x_{n_j} + (1-\frac{j}{j+1}h_{u_{n_j}}(x_{n_j}))u_{n_j} \notin K$$

whose limit is $x \in K$, thus $x \in \partial K$. Since T is Lipschitz mapping $\|u_n - u_{n+1}\| \le D(Tx_n, Tx_{n+1}) \le \frac{1+\sqrt{k}}{1-\sqrt{k}}\|x_n - x_{n+1}\|$, hence $\{u_n\}$ is Cauchy sequence, thus, there is $u \in H$ such that $u_n \to u$ as $n \to \infty$. Since T Lipchitz continuous we have $d(u_n,Tx) \le d(u_n,Tx_n) + D(Tx_n,Tx) \to 0$ as $n \to \infty$, since Tx is closed, $u \in Tx$, hence for all $\beta \in [h_u(x),1)$, we have $\beta x+(1-\beta)u \in K$,as a result it can be shown that $\beta x+(1-\beta)u \in \partial K$. Since K is strictly convex, in similar fashion (see [37]) it can be seen that, $u = x \in Tx$. Thus, the sequence $\{x_n\}$ converges strongly to some element $p \in F(T)$.

Theorem 3.8 Let K be a strictly convex, closed and nonempty subset of a real Hilbert space H and let $T : K \to CB(H)$ be a non self, multi valued and generalized k-strictly pseudo contractive and inward mapping with $F = F(T)$ is non empty and for all $p \in F(T)$, $Tp = \{p\}$. Let $\{x_n\}$ be a by the iterative sequence defined method,

$$\begin{cases} x_1 \in K, u_1 \in Tx_1, \alpha_1 = \max\left\{\alpha, h_{u_1}(x_1)\right\}, \\ \alpha > k, x_{n+1} = \alpha_n x_n + (1-\alpha_n)u_n, \\ u_n \in Tx_n, \gamma_n \in [0,\infty), \sum_{n=1}^{\infty}\gamma_n < \infty \\ \ni \|u_n - u_{n+1}\| \le D(Tx_n, Tx_{n+1}) + \gamma_n, \\ D^2(\{x_n\}, Tx_n) \le \|x_n - u_n\| + \frac{1}{n^2}, \\ \alpha_{n+1} = \max\left\{\alpha_n, h_{u_{n+1}}(x_{n+1}), 1 - \frac{1}{(n+1)^2}\right\}, \\ h_{u_{n+1}}(x_n) = \inf\{\lambda \ge 0 : \lambda x_n + (1-\lambda)u_n \in K\}. \end{cases}$$

Then, the sequence $\{x_n\}$ is well-defined and the sequence $\{x_n\}$ converges strongly to some element p of $F = F(T)$.

Proof. By lemma 3.1 $\{x_n\}$ is well-defined and is in K.

Let $p \in F$. Then, applying lemma 2.5 and lemma 2.16 we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)u_n - p\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\ &\quad -\alpha_n(1-\alpha_n)\|x_n - u_n\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)D^2(Tx_n, Tp) \\ &\quad -\alpha_n(1-\alpha_n)\|x_n - u_n\|^2 + \gamma_n \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\|x_n - p\|^2 \\ &\quad + kD^2(x_n - Tx_n, 0] - (1-\alpha_n)a_n\|x_n - u_n\|^2 + \gamma_n \\ &= \|x_n - p\|^2 + (1-\alpha_n)k\left[D^2(\{x_n\}, Tx_n\right] \\ &\quad -\alpha_n(1-\alpha_n)\|x_n - u_n\|^2 + \gamma_n \\ &\le \|x_n - p\|^2 + (1-\alpha_n)k\left[\|x_n - u_n\|^2 + \frac{1}{n^2}\right] \\ &\quad -\alpha_n(1-\alpha_n)\|x_n - u_n\|^2 + \gamma_n \\ &\le \|x_n - p\|^2 + \frac{(1-\alpha_n)k}{n^2} + (1-\alpha_n)(k-\alpha_n)\|x_n - u_n\|^2 \\ &\le \|x_n - p\|^2 + \frac{(1-\alpha_n)k}{n^2} + \gamma_n. \end{aligned} \tag{3.16}$$

Thus by lemma 2.15 we have the sequence $\{\|x_n - p\|\}$ converges to some $r \ge 0$.

Thus, the sequence $\{x_n\}$ and hence $\{u_n\}$ are bounded.

Since $\sum_{n=1}^{\infty}(1-\alpha_n) < \sum_{n=1}^{\infty}\frac{1}{(n+1)^2} < \infty$, then we have $\sum_{n=1}^{\infty}\|x_{n+1} - x_n\| = \sum_{n=1}^{\infty}(1-\alpha_n)\|x_n - u_n\| < M'\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$.

Hence, the sequence $\{x_n\}$ is strongly Cauchy, thus it is Cauchy and converges to some element $x \in K$.

Moreover, since T is inward, then $h_{u_n}(x) < 1$, hence for every $\beta_n \in [h_{u_n}(x),1)$, we have $\beta_n x + (1-\beta_n)u_n \in K$.

Since $\lim_{n\to\infty}\alpha_n = 1$, there is a subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} h_{u_{n_j}}(x_{n_j}) = 1$,

$$\frac{j}{j+1}h_{u_{n_j}}(x_{n_j})x_{n_j} + (1 - \frac{j}{j+1}h_{u_{n_j}}(x_{n_j}))u_{n_j} \notin K$$

whose limit is $x \in K$.

Since T is Lipschitz mapping

$$\|u_n - u_{n+1}\| \le D(Tx_n, Tx_{n+1}) \le \frac{1+\sqrt{k}}{1-\sqrt{k}}\|x_n - x_{n+1}\|,$$

hence $\{u_n\}$ is Cauchy sequence, thus, there is $u \in H$ such that $u_n \to u$ as $n \to \infty$. Since T is Lipschitz continuous we have

$$d(u_n, Tx) \le d(u_n, Tx_n) + D(Tx_n, Tx) \to 0 \text{ as } n \to \infty.$$

Since Tx is closed, $u \in Tx$, hence for all $\beta \in [h_u(x),1)$, we have $\beta x+(1-\beta)u \in K$,as a result it can be shown that $\beta x+(1-\beta)u \in \partial K$. Since K is strictly convex, in similar fashion (see [37]) it can be seen that, $u = x \in Tx$ .Thus, the sequence $\{x_n\}$ converges strongly to some element $p \in F(T)$.

Theorem 3.9 Let K be a strictly convex, closed and nonempty subset of a real Hilbert space H and let $T_i : K \to CB(H), i = 1,2,...$ be a non self, multi valued and generalized k-strictly pseudo contractive and inward mapping with $F = \bigcap_{i=1}^{N} F(T_i)$ is non empty and for all i , $T_i p = \{p\}$. Let

$$u_i^n \in T_i x_n \ni \left\|x_n - u_i^n\right\|^2 \le D^2(x_n, T_i x_n) \le \left\|x_n - u_i^n\right\|^2 + \frac{1}{n^2}$$

$$\& \left\|u_i^{n+1} - u_i^n\right\| \le D(T_i x_{n+1}, T_i x_n) + \gamma_n, \sum_{n=1}^{\infty}\gamma_n < \infty.$$

Let $\left\{\delta_i^n\right\} \subset (0,1)$ such that $\liminf_n \delta_i^n > 0$ and $\sum_{i=1}^{N}\delta_i^n = 1$. Let $\{x_n\}$ be a sequence defined by the iterative method,

$$\begin{cases} x_1 \in K, \alpha_1 = \max\{\alpha, h_{u_1}(x_1)\}, k < \alpha < 1, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)u_n, u_n = \sum_{i=1}^{N} \delta_i^n u_i^n, \\ \alpha_{n+1} = \max\{\alpha_n, h_{u_{n+1}}(x_{n+1})\}, \\ h_{u_n}(x_n) = \inf\{\lambda \geq 0 : \lambda x_n + (1-\lambda)u_n \in K\}. \end{cases}$$

Suppose T_1 is hemi compact and $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, then $\{x_n\}$ converges to some $p \in \bigcap_{i=1}^{N} F(T_i)$. And if K is strictly convex, $\sum_{n=1}^{\infty}\left\|\delta_i^{n+1} - \delta_i^n\right\| < \infty$ and $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$, then $\{x_n\}$ converges to some $p \in \bigcap_{i=1}^{N} F(T_i)$.

Proof. First, we see that, for any $x \in K$, since each T_i is inward, then $h_{u_n}(x) < 1$, indeed, for $u_n = \sum_{i=1}^{n} \delta_i^n u_i^n$, $u_i^n = x + c_i^n(w_i^n - x),\ c_i^n \geq 1 \ \&\ w_i^n \in K$, we have

$$u_n = \sum_{i=1}^{n} \delta_i^n u_i^n = \sum_{i=1}^{N} \delta_i^n x + \sum_{i=1}^{n} c_i^n \left(\frac{\sum_{i=1}^{N} w_i^n}{\sum_{i=1}^{N} c_i^n} - x) \right)$$

$$= x + c^n(w_n - x), c^n \geq 1, w_n \in K.$$

Thus, $h_{u_n}(x) < 1 - \frac{1}{c^n} < 1$. Let $p \in \bigcap_{i=1}^{N} F(T_i)$.

Thus, applying lemma 2.5 and lemma 2.16 we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\alpha_n x_n + (1-\alpha_n)u_n - p\|^2 \\
&= \|\alpha_n(x_n - p) + (1-\alpha_n)(u_n - p)\|^2 \\
&= \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\|x_n - u_n\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left\|\sum_{i=1}^{N} \delta_i^n u_i^n - p\right\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\left\|x_n - \sum_{i=1}^{N} \delta_i^n u_i^n\right\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\begin{bmatrix} \sum_{i=1}^{N} \delta_i^n \|u_i^n - p\|^2 \\ - \sum_{1 \leq i < j \leq N} \delta_i^n \delta_j^n \|u_i^n - u_j^n\|^2 \end{bmatrix} \\
&-\alpha_n(1-\alpha_n)\begin{bmatrix} \sum_{i=1}^{N} \delta_i^n \|x_n - u_i^n\|^2 \\ - \sum_{1 \leq i < j \leq N} \delta_i^n \delta_j^n \|u_i^n - u_j^n\|^2 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\sum_{i=1}^{N} \delta_i^n D^2(T_i p, T_i x_n) \\
&\quad - (1-\alpha_n)\sum \delta_i^n \delta_j^n \|u_i^n - u_j^n\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\sum_{i=1}^{N} \delta_i^n \|x_n - u_i^n\|^2 \\
&\quad + \alpha_n(1-\alpha_n)\sum \delta_i^n \delta_j^n \|u_i^n - u_j^n\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\sum \delta_i^n \{\|x_n - p\|^2 \\
&\quad + kD^2(x_n, T_i x_n)\} - (1-\alpha_n)\sum \delta_i^n \delta_j^n \|u_i^n - u_j^n\| \\
&\quad - \alpha_n(1-\alpha_n)\sum \delta_i^n \|x_n - u_i^n\|^2 \\
&\quad + \alpha_n(1-\alpha_n)\sum \delta_i^n \delta_j^n \|u_i^n - u_j^n\|^2 \\
&\leq \|x_n - p\|^2 - (1-\alpha_n)(\alpha_n - k)\sum_{i=1}^{N} \delta_i^n \|x_n - u_i^n\|^2 \\
&\quad - (1-\alpha_n)^2 \sum \delta_i^n \delta_j^n \|u_i^n - u_j^n\|^2 + \frac{(1-\alpha_n)k}{n^2} \qquad (3.17) \\
&\leq \|x_n - p\|^2 + \frac{k}{n^2}.
\end{aligned}$$

Thus, by lemma 2.15 we have $\{\|x_n - p\|\}$ converges to some $r \geq 0$, hence the sequence $\{x_n\}$, $\{u_i^n\}$ and $\{u_n\}$ are bounded. From (3.17) we have

$$\begin{aligned}
&(1-\alpha_n)(\alpha_n - k)\delta_i^n \|x_n - u_i^n\|^2 \\
&\leq (1-\alpha_n)(\alpha_n - k)\sum_{i=1}^{N} \delta_i^n \|x_n - u_i^n\|^2 \\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \frac{k}{n^2}.
\end{aligned}$$

Since $\liminf_n \delta_i^n = m > 0$, for some $m > 0$, we have

$$\begin{aligned}
&(1-\alpha_n)(\alpha_n - k)m\|x_n - u_i^n\|^2 \\
&\leq (1-\alpha_n)(\alpha_n - k)\sum_{i=1}^{N} \delta_i^n \|x_n - u_i^n\|^2 \\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \frac{k}{n^2}.
\end{aligned}$$

Case 1 suppose $\sum_{i=1}^{\infty}(1-\alpha_n) = \infty$ and T_1 is hemi compact, since $\alpha_n - k > 0$, let by Archimedean property of real numbers $\alpha_n - k = \varepsilon > 0,$ we have $\sum_{i=1}^{\infty}(1-\alpha_n)(\alpha_n - k) = \infty$ and

$$\sum (1-\alpha_n)(\alpha_n-k)m\left\|x_n-u_i^n\right\|^2$$
$$\leq \sum (1-\alpha_n)(\alpha_n-k)\sum_{i=1}^{N}\delta_i^n\left\|x_n-u_i^n\right\|^2 \qquad (3.18)$$
$$\leq \sum(\|x_n-p\|^2-\|x_{n+1}-p\|^2+\frac{k}{n^2})<\infty.$$

Thus, for each i, $\liminf_{n\to\infty}\left\|x_n-u_i^n\right\|=0$, hence there exists a subsequence $\left\{x_{n_k}-u_i^{n_k}\right\}$ of $\left\{x_n-u_i^n\right\}$ such that $\lim_{k\to\infty}\left\|x_{n_k}-u_i^{n_k}\right\|=0$, thus as $k\to\infty$. Since is hemi compact there exists a subsequence $\left\{x_{n_j}\right\}$ of $\left\{x_{n_k}\right\}$ such that $x_{n_j}\to q\in K$. Moreover, if we take $u_i^{n_j}\in T_i x_{n_j}$ satisfying $\left\|x_{n_j}-u_i^{n_j}\right\|\leq d(x_{n_j},T_i x_{n_j})+\frac{1}{j}$, and lipschitz property T_i of we have

$$\begin{aligned}
d(q,T_i q)&\leq\left\|q-x_{n_j}\right\|+\left\|x_{n_j}-u_i^{n_j}\right\|+d(u_i^{n_j},T_i q)\\
&\leq\left\|q-x_{n_j}\right\|+d(x_{n_j},T_i x_{n_j})+\frac{1}{j}+D(T_i x_{n_j},T_i q)\\
&\leq\left\|x_{n_j}-q\right\|+d(x_{n_j},T_i x_{n_j})\\
&\quad+\frac{1}{j}+\frac{1+\sqrt{k}}{1-\sqrt{k}}\left\|x_{n_j}-q\right\|\to 0, j\to\infty.
\end{aligned} \qquad (3.19)$$

Thus, $d(q,T_i q)=0$, hence $q\in F(T_i)$, since the result is true for any i, $q\in\bigcap_{i=1}^{N}F(T_i)$.

Since for any $q\in\bigcap_{i=1}^{N}F(T_i)$, $\{\|x_n-q\|\}$ converges, hence the sequence $\{x_n\}$ converges strongly to $q\in\bigcap_{i=1}^{N}F(T_i)$.

Case 2. Suppose K is strictly convex and $\sum_{n=1}^{\infty}(1-\alpha_n)<\infty$, then

$$\sum_{n=1}^{\infty}\|x_{n+1}-x_n\|=\sum_{n=1}^{\infty}(1-\alpha_n)\|x_n-u_n\|$$
$$<M'\sum_{n=1}^{\infty}(1-\alpha_n)<\infty.$$

Hence, the sequence $\{x_n\}$ is strongly Cauchy, thus it is Cauchy and converges to some element $x\in K$.

Moreover, since T_i is inward, then $h_{u_n}(x)<1$, hence for every $\beta_n\in[h_{u_n}(x),1)$, we have $\beta_n x+(1-\beta_n)u_n\in K$. Since $\lim_{n\to\infty}\alpha_n=1$, there is a subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$ such that $\lim_{j\to\infty}h_{u_{n_j}}(x_{n_j})=1$,

$$\frac{j}{j+1}h_{u_{n_j}}(x_{n_j})x_{n_j}+(1-\frac{j}{j+1}h_{u_{n_j}}(x_{n_j}))u_{n_j}\notin K$$

whose limit is $x\in K$, thus, $x\in\partial K$.

Since T_i is Lipschitz mapping

$$\left\|u_i^n-u_i^{n+1}\right\|\leq D(T_i x_n,T_i x_{n+1})\leq\frac{1+\sqrt{k}}{1-\sqrt{k}}\|x_n-x_{n+1}\|,$$

hence $\left\{u_i^n\right\}$ is Cauchy sequence, thus, there is $u_i\in H$ such that $u_i^n\to u_i$ as $n\to\infty$. Since $\left\{\delta_i^n\right\}$ is strongly Cauchy, it converges, hence there exists $\delta_i>0$ such that $\delta_i^n\to\delta_i$, let $u=\sum_{i=1}^{N}\delta_i u_i$, then we have

$$\|u_n-u\|\leq\sum_{i=1}^{N}\delta_i^n\left\|u_i^n-u_i\right\|+\|u_i\|\left|\delta_i^n-\delta_i\right|\to 0, n\to\infty. \quad (3.20)$$

T_i is Lipschitz continuous we have

$$d(u_i^n,T_i x)\leq d(u_i^n,T_i x_n)+D(T_i x_n,T_i x)\to 0$$

as $n\to\infty$, since $T_i x$ is closed, $u_i\in T_i x$, hence for all $\beta\in[h_u(x),1)$, we have $\beta x+(1-\beta)u\in K$, as a result it can be shown that $\beta x+(1-\beta)u\in\partial K$. Since K is strictly convex, in similar fashion (see [37]) it can be seen that, $u_i=x\in T_i x$. Thus, the sequence $\{x_n\}$ converges strongly to some element $p\in\bigcap_{i=1}^{N}F(T_i)$.

Remark: In the above discussions, if we consider family of k_i strictly pseudo contractive or generalized k_i strictly pseudo contractive mappings we can use $k=\max\{k_i\}$ in theorem 3.9.

Example 3.1 Now we give an example of sequence of multivalued mappings $\{T_n\}_{n=1}^{\infty}$.

Let $T_n:[0,1]\to prox(\Re)$ be defined by $T_n x=\left[\frac{-x}{n},0\right]$.

Then $D(T_n x,T_n y)=\frac{1}{n}|x-y|\leq|x-y|$.

Thus, T_n is nonexpansive multivalued nonself mapping. For each $x\in[0,1]$ let $u\in T_n x$ $u\in\left[\frac{-x}{n},0\right]$, then $u=t\left(\frac{-x}{n}\right)+(1-t)0,\ 0\leq t\leq 1$, and $u=u-x+x$, thus

$$u=x+\frac{n+1}{n}\left[\frac{(1-t)x}{n+1}-x\right]=x+c(v-x),$$
$$c=\frac{n+1}{n}\geq 1\ \&\ v=\frac{(1-t)}{n+1}x\in[0,1],$$

hence T_n is inward mapping.

Thus, the sequence of mappings satisfies the condition of the theorem 3.2 thus, the algorithm converges to a unique common fixed point $p \in F = \{0\}$, we also see that $F = \{0\} \subset [0,1] = K$ and the pair (F,K) satisfies S-condition. We see the first four iterates as;

Let $x_1 = 1$, $\alpha_0 = \frac{1}{2}$. Then $T_1 x_1 = [-1,0]$, taking $u_1 = \frac{-1}{2}$, thus, $h_{u_1}(x_1) = \frac{1}{3}$, thus $\alpha_2 = \frac{1}{2}$, $x_2 = \frac{1}{4}$ and $T_2 x_2 = \left[\frac{-1}{8}, 0\right]$, taking $u_2 \in \left[\frac{-1}{8}, 0\right]$ such that

$$|u_2 - u_1| \le D(T_2 x_2, T_1 x_1) = \left|x_1 - \frac{x_2}{2}\right| = \frac{7}{8},$$

say $u_2 = \frac{-1}{8}$, we get $h_{u_2}(x_2) = \frac{1}{3}$, $\alpha_2 = \frac{1}{2}$, $x_3 = \frac{1}{16}$ and $T_3 x_3 = \left[\frac{-1}{48}, 0\right]$, in the same fashion taking $u_3 = \frac{-1}{48}$ we get $h_{u_3}(x_3) = \frac{1}{4}$, $\alpha_3 = \frac{1}{2}$ and $x_4 = \frac{1}{48}$.

Remark: Let $T_1 = T_2 =, .. = T_N = T : K \to \text{Prox(H)}$ be non self, multi valued, nonexpansive and inward mapping on a non-empty, closed and convex subset K of a real Hilbert space H, with $F = F(T)$ non empty, for all $p \in F = F(T)$, $T(p) = \{p\}$. Let $\{x_n\}$ be a sequence of Mann type defined by the iterative method

$$\begin{cases} x_1 \in K, u_1 \in Tx_1, \alpha_1 = \max\{\alpha, h_{u_1}(x_1)\}, \\ \alpha > 0, x_{n+1} = \alpha_n x_n + (1-\alpha_n)u_n, u_n \in Tx_n, \\ \ni \|u_n - u_{n+1}\| \le D(Tx_n, Tx_{n+1}), \\ \alpha_{n+1} = \max\{\alpha_n, h_{u_{n+1}}(x_{n+1})\}, \\ h_{u_n}(x_n) = \inf\{\lambda \ge 0 : \lambda x_n + (1-\lambda)u_n \in K\} \end{cases}$$

is well-defined and if $\{\alpha_n\} \subseteq [\varepsilon, 1-\varepsilon] \subset (0,1)$ for some $\varepsilon > 0$, then the sequence $\{x_n\}$ converges weakly to some element p of $F = F(T)$. Moreover, if $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$, and (F,K) satisfies S-condition, then the convergence is strong.

4. Conclusion

Our theorems extend many results in literature, in particular, our theorems [3.2-3.5] extend the result of Tufa and Zegeye [33] to a common fixed point for the family of non expansive mappings. We also extend the result of [9] and [25] to approximation for a fixed point and a common fixed point for family of more general class of mappings, the so called generalized k-strictly pseudo contractive nonself mappings.

Authors' Contributions

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

Competing Interests

The authors declare that they have no competing interests.

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