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Solving the Quantity Element Using New Numerical Techniques on the Discontinues Boundary Element Method

Hassan Ghassemi*, Alireza Ahani

Department of Maritime Engineering, Amirkabir University of Technology, Tehran, Iran
*Corresponding author: gasemi@aut.ac.ir

Abstract This paper deals with solving the quantity element using new numerical techniques on discontinues boundary element method (DBEM). The common practice in getting solution with BEM is using constant element and for that, in a Sub-parametric element, quantity has a constant value along the element and geometry discretization is supposed to have a linear variation. But using higher order (polynomial) distribution of quantity over elements could have a better description of physical process. For this, the corresponding discretized expressions based on new techniques are derived and used for solution of Laplace equation. Many results for the quantity elements are presented and discussed for the ellipse at various diameters and mesh numbers.

Keywords: *boundary element method, linear element, discontinuous element*

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1. Introduction

Increasing need for high performance numerical methods brings up solutions like boundary element method (BEM) that has major advantages compared to other solutions. One unique feature of BEM is that it uses only elements on boundary. This decreases number of computational elements which is more interesting when it comes to use the method in integrated simulation software that requires accuracy and speed at the same time.

Types of element in BEM include variety of choices regarding order of the polynomial that defines them. Studying linear element is the first step in implementing higher order elements in BEM. It will be said that deriving the equations in linear element complies with the concept of constant element in many ways. Since explaining and implementing constant element is more straightforward than higher order elements, they are rarely seen in contexts. In 1991, Ingham and Ritchie [1] used continuous linear and quadratic boundary element to solve Laplace equation with Dirichlet boundary condition and explained difficulties of the problem with that boundary condition, however discontinuous element has not this problem. After that, Tadeu (2000) [2] discussed on using discontinuous element both for linear and quadratic element in modeling a 3D elastic environment and concluded that using discontinuous element decreases error significantly. In 2004, Ali and Rajakumar [3] presented a specific formulation for a 2D linear element. Grecu and Vladimirescu (2005) [4] and Grecu et al. (2009) [5] researched on 3D usage of continuous linear boundary element in compressible medium and showed that linear element results a considerable accuracy even if gridding is not very fine. Recently Chen et al (2016) [6] has used discontinuous element for an acoustic problem.

As can be seen in Figure 1, the accuracy of methods in capturing the exact quantity are getting better from constant element to discontinuous linear element.

2. Governing Equation and Boundary Conditions

Here and in any higher order element, the center of element represents the element where the integral equations are derived. Therefore, constant element formulation and concept is used fundamentally.

The equation of problem is a 2D Laplace equation and boundary condition type is Newman that means gradient of quantity (u_n) is known on an ellipse with discretization shown in Figure 2. Geometry discretization is done by (1) [7]:

$$x_i = a\cos(\theta)\, y_i = b\sin(\theta)(i = 1, 2, \ldots, N). \qquad (1)$$

This type of boundary discretization makes the two curved ends of ellipse more refined and provides a better capturing over desired quantity.

The boundary element equation of Laplace in form of constant element is given according to references [8,9]:

$$\epsilon^i u^i = \sum_{j=1}^{N}\left(\int_{\Gamma_j} G u_n ds\right) - \sum_{j=1}^{N}\left(\int_{\Gamma_j} G_n u ds\right) \qquad (2)$$
$$(i = 1, 2, \ldots, N)$$

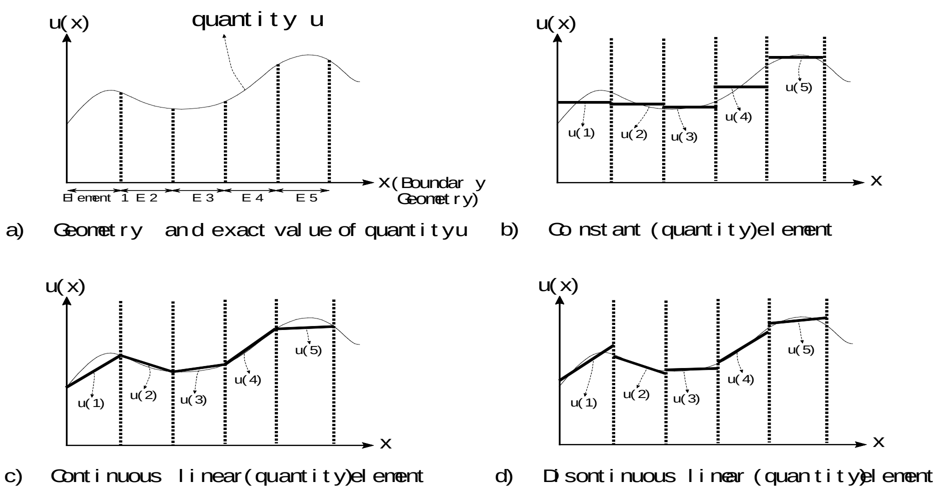

Figure 1. Comparison of the quantity (u) with different elements

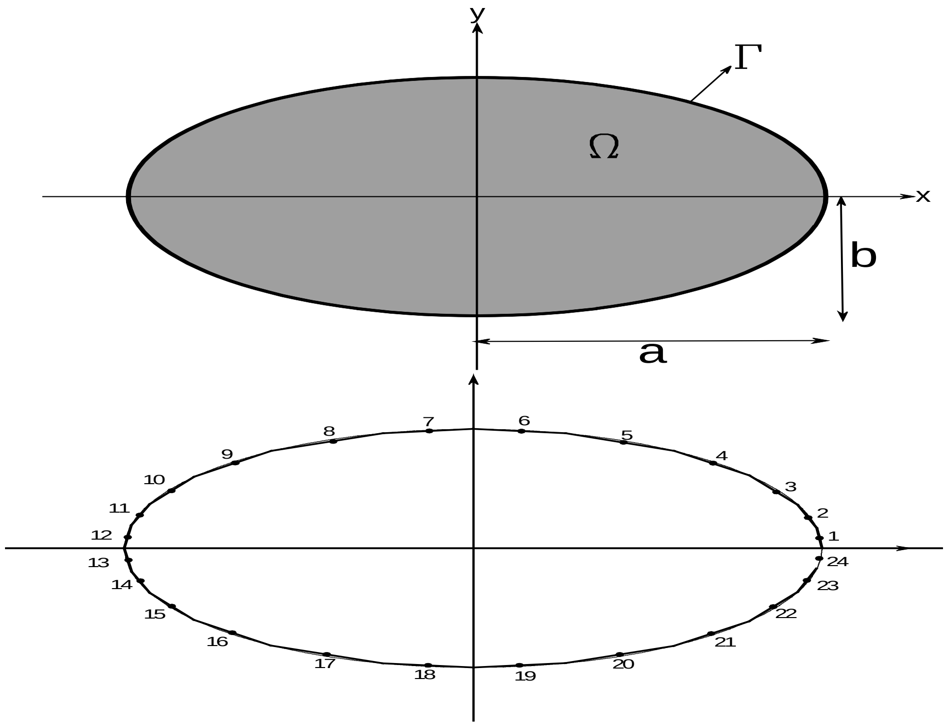

Figure 2. Problem domain (Left) and boundary discretization (Right)

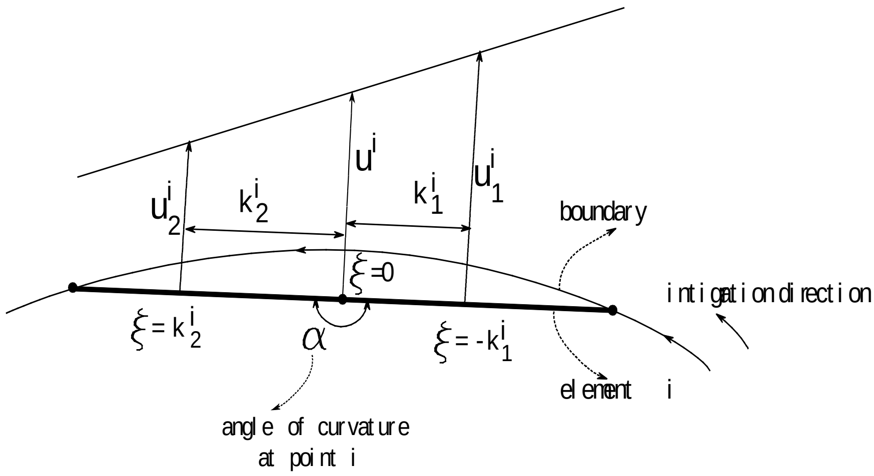

Figure 3. Distribution of quantity along an element

In which G is the Green function of Laplace equation ($G=\frac{1}{2\pi}\ln r$) and ϵ is an expression ($\epsilon^{i}=-\frac{\alpha^{i}}{2\pi}$) function of angle defining curvature of the element at point i (see Figure 3). Since element is linear, this angle (α) is π radians and value of ϵ is “-0.5” everywhere. In Eq. (2), u^{i} is the value of quantity at center of the element and subscript “n” means gradient in normal direction [10].

Direction of Integration convention is taken counterclockwise as is shown in Figure 3. This figure shows the place of center of element and two arbitrary chosen (control) points along it. Those are places that integration is going to be transferred to. Since distribution of quantity along the element is linear, it is correct to say:

$$u^{i}=\psi_1^{i}(\xi)|_{\xi=0}\,u_1^{i}+\psi_2^{i}(\xi)|_{\xi=0}\,u_2^{i}. \tag{3}$$

In which ψ_1^{i} means linear shape function for control point 1 in element i and could be written like this:

$$\begin{aligned}&\psi_1^{i}(\xi)=\frac{1}{k^{i}}(k_2^{i}-\xi),\ k^{i}=k_1^{i}+k_2^{i}\\&\psi_2^{i}(\xi)=\frac{1}{k^{i}}(k_1^{i}+\xi),\ 0\le k_1^{i},k_2^{i}\le 1.\end{aligned} \tag{4}$$

Similar expression (like (3)) could be written for term u_n^{j} in (2):

$$u_n^{j}=\psi_1^{j}(\xi)|_{\xi=0}\,u_{n1}^{j}+\psi_2^{i}(\xi)|_{\xi=0}\,u_{n2}^{j}. \tag{5}$$

Substituting (4) into (3) and using the expression of ϵ gives:

$$\epsilon^{i}u^{i}=\frac{-1}{2}\frac{k_2^{i}}{k^{i}}u_1^{i}-\frac{1}{2}\frac{k_1^{i}}{k^{i}}u_2^{i}. \tag{6}$$

Since Laplace Equation is a harmonic function and any answer of it, including the Green function, has linear property over definition domain, the Green function could be written as follow:

$$\begin{aligned}&G\left(p^{i},q\right)\\&=\psi_1^{i}(\xi)|_{\xi=0}\,G\left(p_1^{i},q\right)+\psi_2^{i}(\xi)|_{\xi=0}\,G\left(p_2^{i},q\right)\\&G_n\left(p^{i},q\right)\\&=\psi_1^{i}(\xi)|_{\xi=0}\,G_n\left(p_1^{i},q\right)+\psi_2^{i}(\xi)|_{\xi=0}\,G_n\left(p_2^{i},q\right).\end{aligned} \tag{7}$$

In these expressions p^{i} represents center of element and the other terms are defined in (4).

Substituting the Eqs. (5), (6) and (7) into Eq. (2) gives two simultaneous equations for each linear element i of boundary as (8).

$$\begin{aligned}&\frac{-1}{2}u_1^{i}+\sum_{j=1}^{N}\left[\int_{\Gamma_j}\left(\psi_1u_1^{j}+\psi_2u_2^{j}\right)G_n\left(p_1^{i},q\right)ds\right]\\&=\sum_{j=1}^{N}\left[\int_{\Gamma_j}(\psi_1u_{n1}^{j}+\psi_2u_{n2}^{j})G\left(p_1^{i},q\right)ds\right]\\&\frac{-1}{2}u_2^{i}+\sum_{j=1}^{N}\left[\int_{\Gamma_j}(\psi_1u_1^{j}+\psi_2u_2^{j})G_n\left(p_2^{i},q\right)ds\right]\\&=\sum_{j=1}^{N}\left[\int_{\Gamma_j}(\psi_1u_{n1}^{j}+\psi_2u_{n2}^{j})G\left(p_2^{i},q\right)ds\right].\end{aligned} \tag{8}$$

After writing these equations for all elements in boundary, dummy indices i and j will be replaced with “I” and “J”; the reason is, by decomposing one node in an element into two nodes, number of computational nodes in boundary is no longer similar to constant element and should be numbered again. In other words, in fully constant boundary elements it is $1\le i\le N$ and$1\le j\le N$, while in fully discontinues linear boundary elements it is $1\le I\le 2N$ and $1\le J\le 2N$.

It should be noted that for a quadratic element, number of equations is “3” (there are 3 simultaneous equations in (8)), for cubic element it is “4” and so on for any higher order element. Besides, this set of equations are just valid for those elements in which distribution of quantity is linear within; So if all elements on domain are linear, totally there are 2N equations to be solved, that means $1\le I\le 2N$ and $1\le J\le 2N$.

By simplifying the set of equations (8) and writing them for all elements, system of equations is expressed as (9).

$$[H]_{2N\times 2N}[u]_{2N\times 1}=[G]_{2N\times 2N}[u_n]_{2N\times 1}. \tag{9}$$

In which matrices $[H]$ and $[G]$ are consist of these elements:

$$\begin{aligned}&h_1^{ij}=\int_{-1}^{1}\psi_1(\xi)G_n ds,\ g_1^{ij}=\int_{-1}^{1}\psi_1(\xi)G ds,\\&h_2^{ij}=\int_{-1}^{1}\psi_2(\xi)G_n ds,\ g_2^{ij}=\int_{-1}^{1}\psi_2(\xi)G ds.\end{aligned} \tag{10}$$

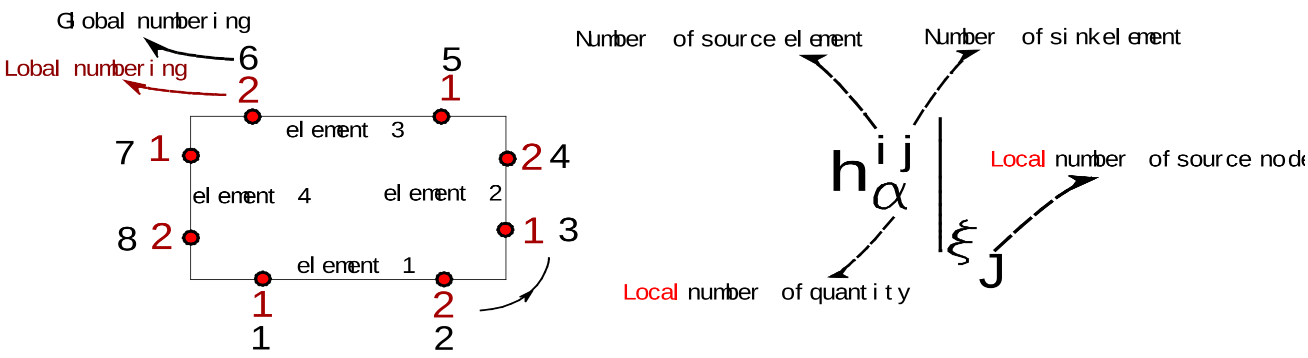

Figure 4. Notation and numbering

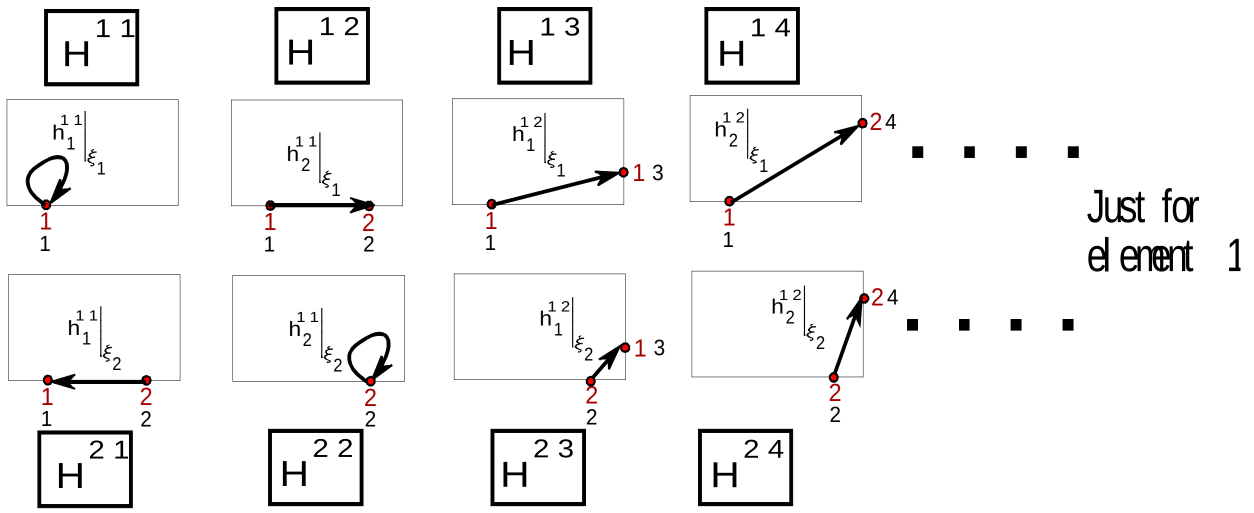

Figure 5. Method of filling for H matrix

Here G and G_n are defined as follow:

$$G=\frac{\ln r}{2\pi}\quad G_n=\frac{\hat{r}.\hat{n}}{2\pi r^2}. \tag{11}$$

Now a rectangle domain with four elements is considered to show how equations in (8) work; for this, first some notations are defined in Figure 4. Then some elements of matrix $[H]=H^{IJ}$ are presented in Figure 5.

Matrix $[G]=G^{IJ}$ could be completed in a same manner, except that "H" is replaced with "G", and "h" is replaced with "g".

It should be noted that in derived integral form of solution for Laplace equation, there is no volume integral term to be accounted for inner volume of domain, unlike equations such as Poisson equation [11] or Helmholtz with source term [12]. In these equations, an additional method is implemented to deal with these terms named Dual Reciprocity Boundary Element Method (DRBEM) [11] that transforms the volume integral into boundary integral. This method is being studied by Zakerdoost and Ghassemi [13].

3. Calculations of the Integrals

Here method of calculating integrals in (10) is described. For integrals of "h", according to Figure 6, in the case of linear geometry element, where index i is equal to j ($i=j$), integral gives zero answer considering the term $\hat{r}.\hat{n}$ in (11); But for other conditions the integral is calculated straightforward with Gaussian quadrature integral.

It should be noted that for other forms of geometry (for example Quadratic geometry element in Figure 6), where "$i=j$", a special care should be taken because $\hat{r}.\hat{n}\neq 0$ and the integral should be solved (one of these cares is using Gaussian quadrature integral [14]), besides, there is one Cauchy-type singularity point due to the presence of r^2 multiplier in denominator (see G_n in (11)).

For the integrals of "g", where "$i=j$", a logarithmic singularity is definitely introduced in all geometry elements (see (11)), and should be considered. There are 3 choices here to deal with these integrals. They are Analytical integration, Numerical integration and Integration by extracting the singularity. In order to take account the accuracy, Analytical integration is chosen.

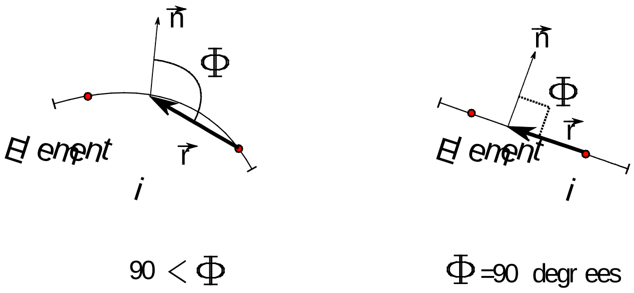

Figure 6. Linear geometry element (right) Quadratic geometry element (left)

Integrand should be analytic on integral domain and this suggests that the integral domain should be broken down into integrals with analytic integrand inside (see (12)). This is the basic idea behind using Analytical integration.

$$\begin{aligned}\frac{4\pi}{l_j}g_1^{ii}&=\int_{-1}^{1}\psi_1(\xi)\ln r\,d\xi\\&=\int_{-1}^{\xi_J}\psi_1(\xi)\ln r\,d\xi+\int_{\xi_J}^{1}\psi_1(\xi)\ln r\,d\xi.\end{aligned}\tag{12}$$

Since term "$\ln r$" is singular in node "ξ_J", the integral is broke down into these two integrals (see (12)). These integrals are then solved analytically by using integration by part and change of variables. On account of briefness, details are not brought here, but the final expressions to get the answer of these singular integrals (g_1^{ii}, g_2^{ii}) are presented by reference [7]:

$$\begin{aligned}\frac{4\pi}{l_j}g_1^{ii}&=\frac{1}{k^i}\left(k_2^i-\xi_J\right)\left(1+\xi_J\right)\left\{\ln\left[\frac{l_i}{2}\left(1+\xi_J\right)\right]-1\right\}\\&+\frac{1}{k^i}\left(1+\xi_J\right)^2\left\{\frac{1}{2}\ln\left[\frac{l_i}{2}\left(1+\xi_J\right)\right]-\frac{1}{4}\right\}\\&+\frac{1}{k^i}\left(k_2^i-\xi_J\right)\left(1-\xi_J\right)\left\{\ln\left[1-\xi_J\right]-1\right\}\\&-\frac{1}{k^i}\left(1-\xi_J\right)^2\left\{\frac{1}{2}\ln\left[\frac{l_i}{2}\left(1-\xi_J\right)\right]-\frac{1}{4}\right\}.\end{aligned}\tag{13}$$

And for g_2^{ii} there is [7]:

$$\begin{aligned}\frac{4\pi}{l_j} g_2^{ii} = &\frac{1}{k^i}\left(k_1^i + \xi_J\right)(1+\xi_J)\left\{\ln\left[\frac{l_i}{2}(1+\xi_J)\right]-1\right\} \\ &-\frac{1}{k^i}(1+\xi_J)^2\left\{\frac{1}{2}\ln\left[\frac{l_i}{2}(1+\xi_J)\right]-\frac{1}{4}\right\} \\ &+\frac{1}{k^i}\left(k_1^i + \xi_J\right)(1-\xi_J)\left\{\ln\left[\frac{l_i}{2}(1-\xi_J)\right]-1\right\} \\ &+\frac{1}{k^i}(1-\xi_J)^2\left\{\frac{1}{2}\ln\left[\frac{l_i}{2}(1-\xi_J)\right]-\frac{1}{4}\right\}.\end{aligned} \tag{14}$$

Depend on place of the source point, ξ_J would be $-k_1^i$ or k_2^i in these equations. "l_j" is length of linear element.

4. Numerical Results

The case study to verify this method is an ellipse that is described in Figure 2. The solution is then compared to exact solution in different ratios of "a/b" and different number of gridding. It is considered that $k_1^i = k_2^i = 0.5$in all elements on boundary for convenience. The exact solution for this problem is:

$$u = x^2 - y^2 + C, u_n = \frac{2\left(b^2x^2 - a^2y^2\right)}{\sqrt{b^4x^2 + a^4y^2}}. \tag{15}$$

To have a unique solution, boundary condition for one point (that is the end point of gridding) is set to Dirichlet (value of u is calculated from (15)) instead of Newman and the arbitrary constant C is set to "2.0".

Numerical results are presented in Figure 7 ~ Figure 18. In those Figures, different conditions for a/b and number of boundary gridding (N) is presented. The continuous form of solution is actually $k_1^i = k_2^i = 1.0$ that is a particular condition for discontinuous element.

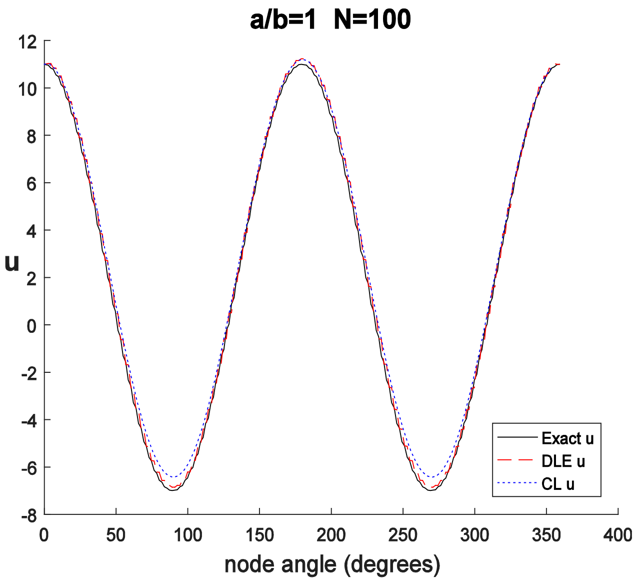

Figure 7. Comparison of the quantity value (u) for a/b=1and element number 100

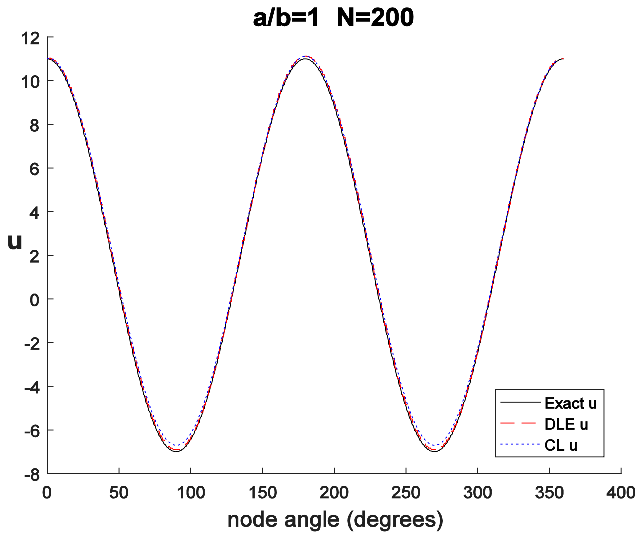

Figure 8. Comparison of the quantity value (u) for a/b=1 and element number 200

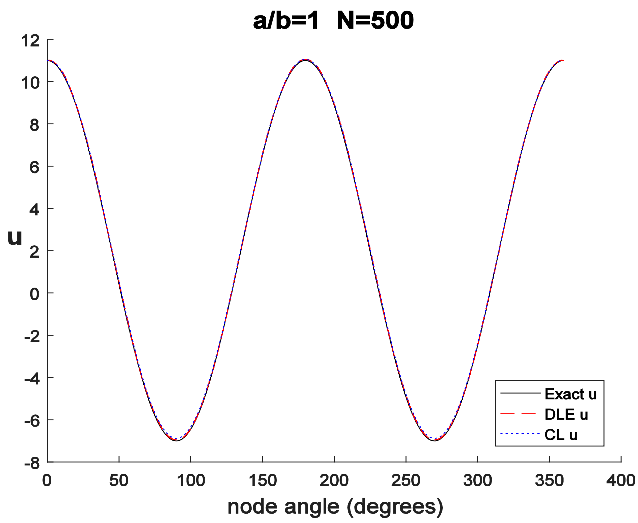

Figure 9. Comparison of the quantity value (u) for a/b=1 and element number 500

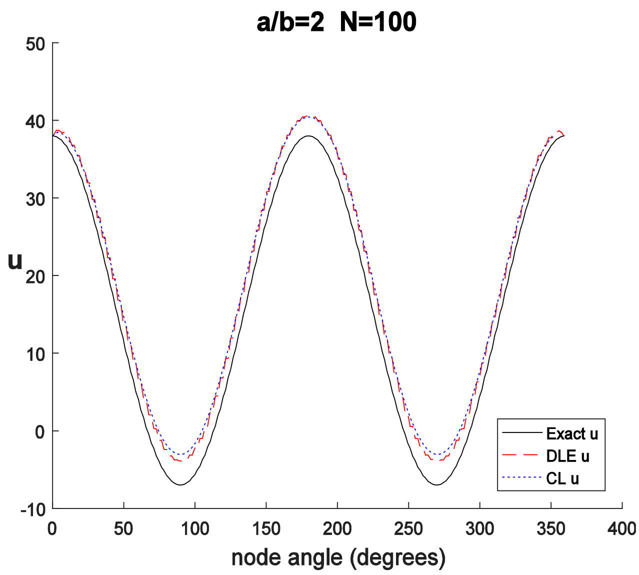

Figure 10. Comparison of the quantity value (u) for a/b=2 and element number 100

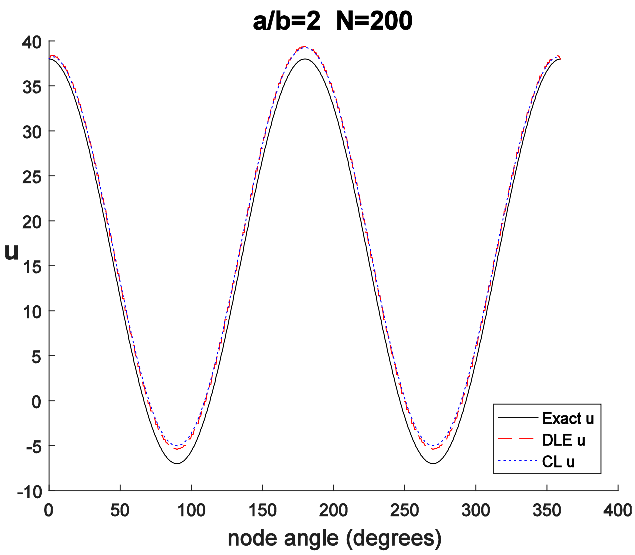

Figure 11. Comparison of the quantity value (u) for a/b=2 and element number 200

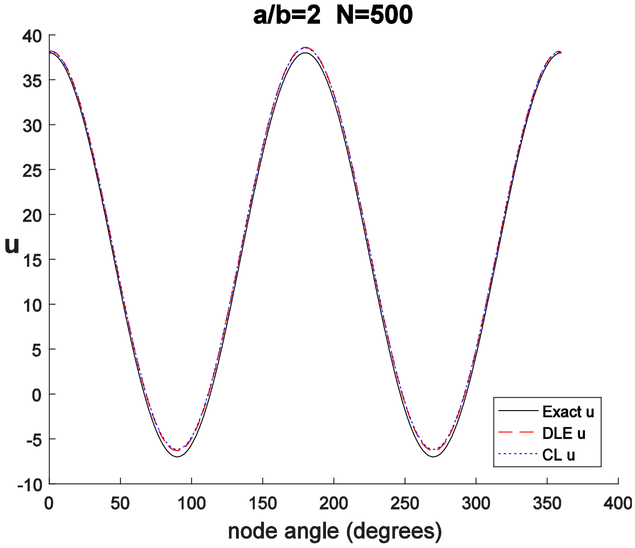

Figure 12. Comparison of the quantity value (u) for a/b=2 and element number 500

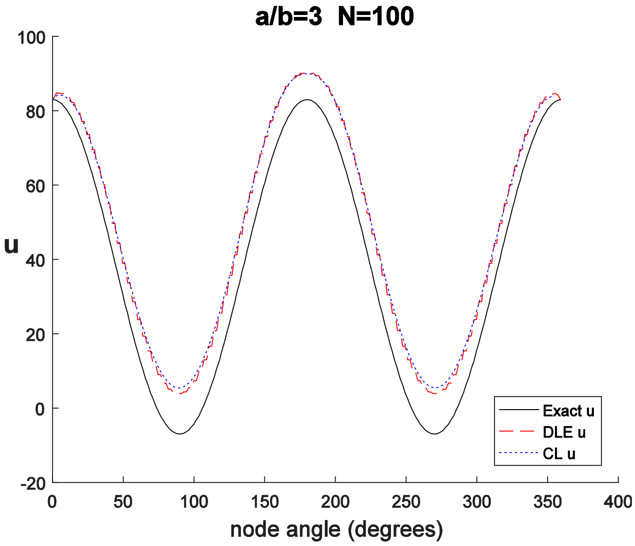

Figure 13. Comparison of the quantity value (u) for a/b=3 and element number 100

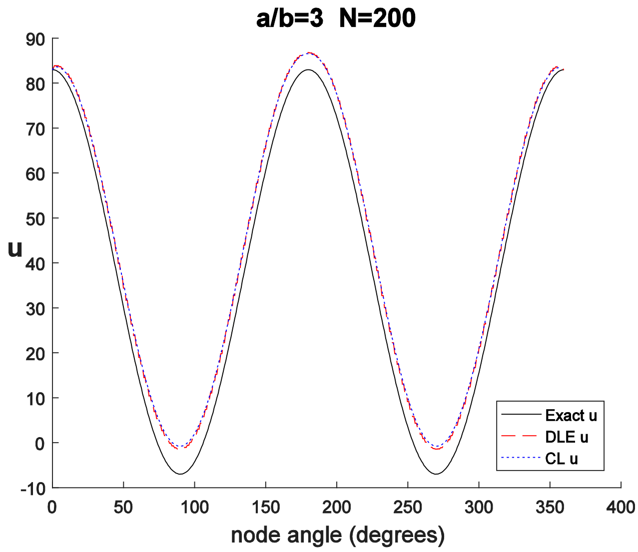

Figure 14. Comparison of the quantity value (u) for a/b=3 and element number 200

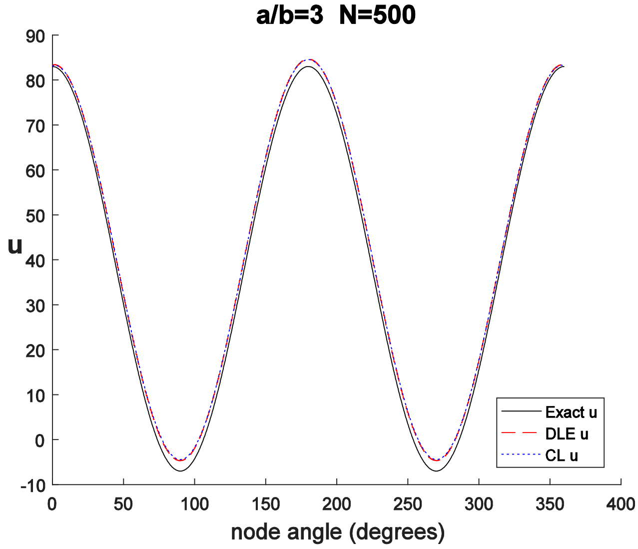

Figure 15. Comparison of the quantity value (u) for a/b=3 and element number 500

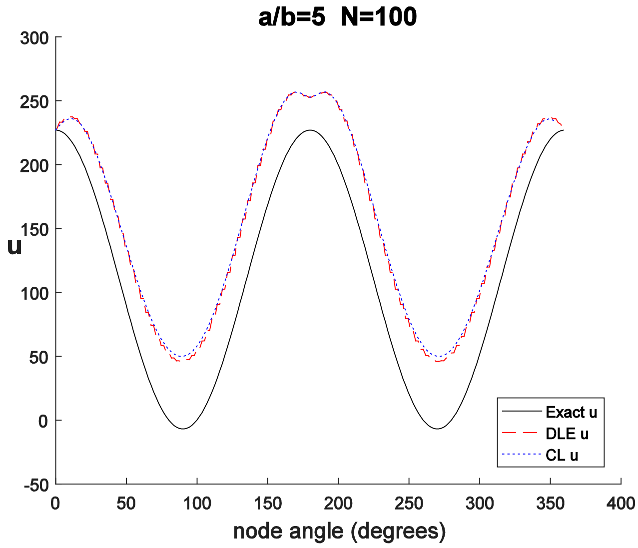

Figure 16. Comparison of the quantity value (u) for a/b=5 and element number 100

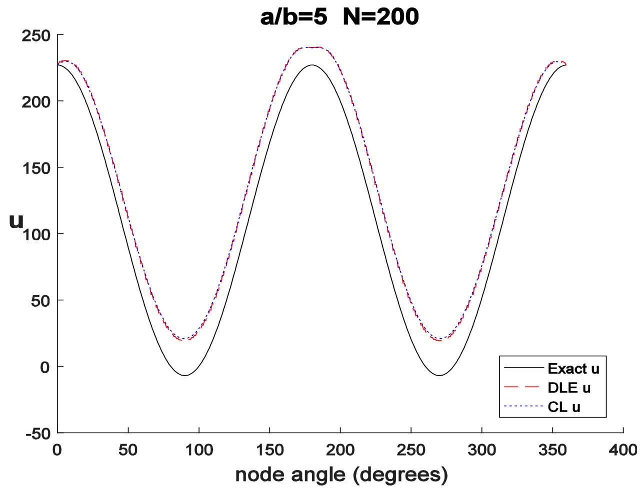

Figure 17. Comparison of the quantity value (u) for a/b=5 and element number 200

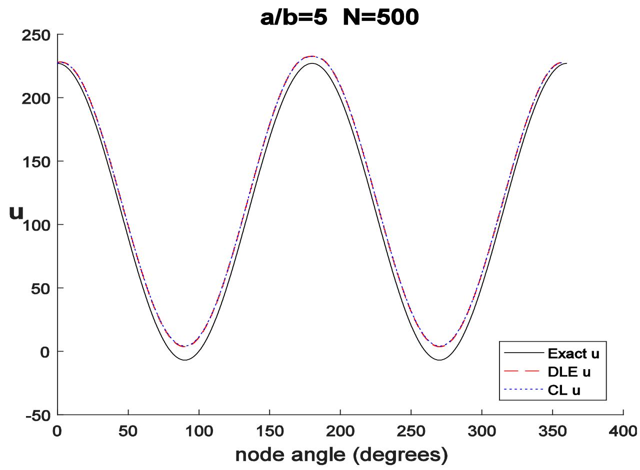

Figure 18. Comparison of the quantity value (u) for a/b=5 and element number 500

In order to have a better understanding of effects from these parameters, "a/b" and N, errors are extracted in 2 critical angles of gridding ("90°" and "180°") in all conditions and normalized by dividing them to the worst observed value condition for error ("$a/b = 5$" and "$N = 100$") plotted in Figure 19 and Figure 20. In these two figures, error with value "1.0" corresponds to the mentioned worst condition.

The other point is, in all conditions, CL and DLE both of them give overestimate of exact value, especially with increase in ratio of "a/b", that mean tending to a flat plate. In the opposite, with tending this shape to a circle ("$a/b = 1$"), both of them give a good approximation.

The error in angles near "90°" and "270°" is mostly due to large length of gridding there. On the other hand, in angles of around "180°", error is due to high curvature of geometry there that makes capturing the gradient of quantity hard at the zone.

Here a question may be raised which is why this error (error in angles of around "180°") is not repeated in angles near "0°" and "360°"? The answer is that in order to make answer unique, the last node (near "360°") value is fixed and this propagates a suitable accuracy near around. Table 1 shows possibilities for choosing k_1^i, k_2^i and their effects.

Table 1. Possibilities for k_1^i, k_2^i

Values for k_1^i, k_2^i	Remark
$k_1^i = 0.0$, $k_2^i = 0.0$	Represents constant element
$k_1^i = 1.0$, $k_2^i = 1.0$	Represents continues linear element
$k_1^i = 0.5$, $k_2^i = 0.5$	It is used over all elements in this study
$0 \le k_1^i \le 1$ $0 \le k_2^i \le 1$	They could be modified on each element for better capturing the quantity

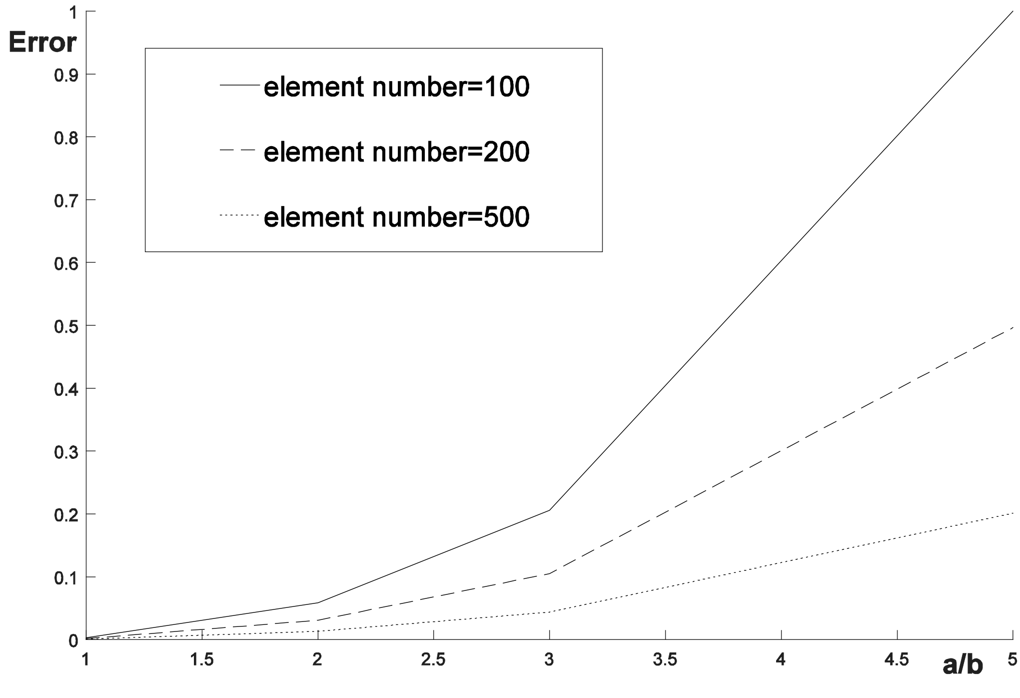

Figure 19. Percentage errors of DLE in 90° angle

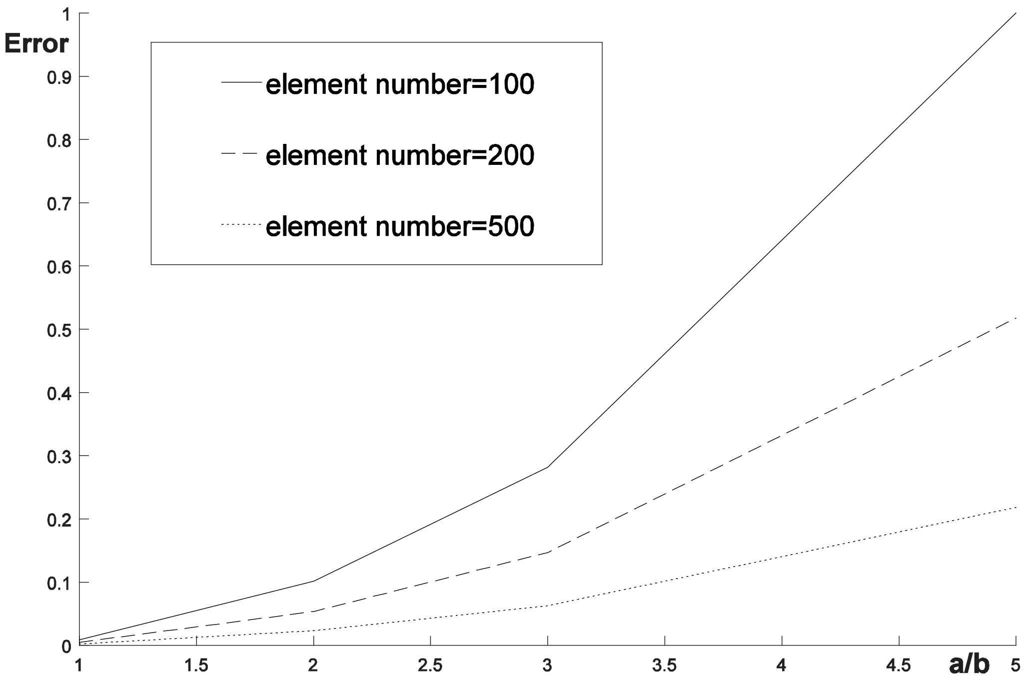

Figure 20. Percentage errors of DLE in 180° angle

5. Conclusions

Laplace equation has been solved by continuous and discontinuous element of the BEM on the domain of ellipse with different diameters and different element numbers. Based on the numerical results, the following conclusions can be drawn:

1. The accuracy of methods in capturing the exact quantity are obtained by discontinuous elements relative to the continous elements.
2. Continuous linear elements and discontinuous linear elements (CL and DLE, respectively) are presented compared. With increasing the “a/b” error is increased due to the sharp of the ellipse at the leading and trailing.
3. The values of k_1^i, k_2^i (as given in Table 1) may effect on the results. They improve the accuracy in this method [6,15]. The future work that is our intent is to employ discontinuous elements on some parts of boundary (with critical gradient of quantity) in combination with constant element on the others (in which quantity is sensed nearly constant) that improves the ratio of accuracy over time (of solution).

References

[1] Ingham, D.B, Ritchie, J.A, Taylor, C.M, "The linear boundary element solution of Laplace's equation with Dirichlet boundary conditions," *Mathematical and Computer Modelling*, 15(5), 195-204, 1991.

[2] Tadeu, A, António, J, "Use of constant, linear and quadratic boundary elements in 3D wave diffraction analysis," *Engineering Analysis with Boundary Elements*, 24(2), 131-144, 2000.

[3] Ali, A, Rajakumar, C, *The Boundary Element Method: Applications in Sound and Vibration*, A.A. Balkema Publishers, 2004.

[4] Grecu, L, "A solution of the boundary integral equation of the three-dimensional airfoil in subsonic flow using linear boundary elements," *ROMAI*, 1(2), 101-107, 2005.

[5] Grecu, L, Vladimirescu, I, "BEM with Linear Boundary Elements for Solving the Problem of the 3D Compressible Fluid Flow around Obstacles," in *Proceedings of the International Multi Conference of Engineers and Computer Scientists (IMECS)*, Hong Kong, 2009.

[6] Chen, L, Chen, H, Zheng, C, Marburg, S, "Structural–acoustic sensitivity analysis of radiated sound power using a finite element/ discontinuous fast multipole boundary element scheme," *International Journal for Numerical Methods in Fluids*, 82(12), 858-878, 2016.

[7] Katsikadelis, J, *Boundary elements: Theory and Application*, Athens: ELSEVIER, 2002.

[8] Pozrikidis, C, *A practical guide to boundary element methods with the software library BEMLIB*, Florida: CRC Press, 2002.

[9] Ghassemi, H, Panahi, S, Kohansal, A. R, "Solving the Laplace’s Equation by the FDM and BEM, Using Mixed Boundary Conditions," *American Journal of Applied Mathematics and Statistics*, 4(2), 37-42, 2016.

[10] Bergman, S, Schiffer, M, *Kernel functions and elliptic differential equations in mathematical physics*, New York: Academic Press Inc. Publishers, 1953.

[11] Zakerdoost, H, Ghassemi, H, "Dual Reciprocity Boundary Element Method for Steady State Convection-Diffusion-Radiation problems," *International Journal of Partial Differential Equations and Applications*, 2(4), 68-71, 2014.

[12] Ghassemi, H, Fazelifar, S, Nadery, A, "DRBEM Applied to the 3D Helmholtz Equation and Its Particular Solutions with Various Radial Basis Functions," *International Journal of Partial Differential Equations and Applications*, 4(1), 1-6, 2016.

[13] Zakerdoost, H, Ghassemi, H, Iranmanesh, M, "Solution of Boundary Value Problems Using Dual Reciprocity Boundary Element Method," *Advances in Applied Mathematics and Mechanics*, 9(3), 680-697, 2017.

[14] Chen, G, Zhou, J, *Boundary element methods with applications to nonlinear problems*, 2nd ed., Springer Science & Business Media, 2010.

[15] Sauter, S.A, Schwab, C, *Boundary element methods*, Berlin, Springer-Verlag Berlin Heidelberg, 2011.