American Journal of Applied Mathematics and Statistics, 2017, Vol. 5, No. 1, 8-10
Available online at http://pubs.sciepub.com/ajams/5/1/2

DOI:10.12691/ajams-5-1-2

Some Fixed Point Theorems on c-distance

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Abstract In this paper, we prove fixed point theorems on c-distance in ordered cone metric spaces. Our results are generalize, improve and extension of the recent work existing in the literature.

Keywords: *cone metric space, fixed point, common fixed point, coincidence point, c-distance, contractive mapping*

Cite This Article: K. Prudhvi, "Some Fixed Point Theorems on c-distance." *American Journal of Applied Mathematics and Statistics*, vol. 5, no. 1 (2017): 8-10. doi: 10.12691/ajams-5-1-2.

1. Introduction

Huang and Zhang [9] introduced the concept of a cone metric space, they replaced set of real numbers by an ordered Banach space and proved some fixed point theorems for contractive type conditions in cone metric spaces. Later on many authors have (for e.g., [1,2,3,5,6,8,12]) proved common fixed point theorems for different contractive type conditions in cone metric spaces. Some of the authors have studied fixed point theorems on partially ordered cone metric spaces (see, e.g., [3,4,11]). In 2011, Y. J. Cho, et. al. [7] introduced a concept of the c-distance in a cone metric spaces and proved some fixed point theorems in ordered cone metric spaces. In this paper, we obtained some fixed point theorems on c-distance in ordered cone metric spaces. Our results are improved and extended the results of Y. J. Cho, et. al. [7].

2. Preliminaries

2.1 [9] Definition

Let E be a real Banach space and θ denotes the zero element in E. P be a subset of E. The set P is called a cone if and only if:

(a). P is closed, non–empty and $P \neq \{\theta\}$;

(b). a, b$\in\mathbb{R}$, $a,b \geq 0$, x,y$\in$P implies ax+by$\in$P;

(c). $P \cap (-P) = \{\theta\}$.

2.2 [9] Definition

Let P be a cone in a Banach space E, define partial ordering '$\preccurlyeq$' with respect to P by $x \preccurlyeq y$ if and only if y-x$\in$P. We shall write $x \prec y$ to indicate $x \preccurlyeq y$ but $x \neq y$ while x<<y will stand for y-x$\in$int P, where int P denotes the interior of the set P. This cone P is called an order cone.

2.3 [9] Definition

Let E be a Banach space and $P \subset E$ be an order cone. The order cone P is called normal if there exists L>0 such that for all x, y$\in$E,

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leq \|y\|.$$

The least positive number L satisfying the above inequality is called the normal constant of P.

2.4 [9] Definition

Let X be a nonempty set of E. Suppose that the map d: $X \times X \rightarrow E$ satisfies:

(d1). $\theta \prec d(x,y)$ for all x, y$\in$X with $x \neq y$ and d(x, y) = 0 if and only if x = y;

(d2). d(x, y) = d(y, x) for all x, y$\in$X;

(d3). $d(x, y) \preccurlyeq d(x, z) + d(z, y)$ for all x, y, z$\in$X.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

It is clear that the concept of a cone metric space is more general than that of a metric space.

2.5 [9] Example

Let $E = \mathbb{R}^2$, $P = \{(x, y)\in E$ such that : $x, y \geq 0\} \subseteq \mathbb{R}^2$, $X = \mathbb{R}$ and d: $X \times X \rightarrow E$ such that $d(x , y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

2.6 [9] Definition

Let (X, d) be a cone metric space. Then $\{x_n\}$ is said to be

(i) a convergent sequence if for any c>>θ, there is a natural number N such that for all n>N, $d(x_n, x)$ <<c, for some fixed x in X. We denote this $x_n \rightarrow x$ (as $n \rightarrow \infty$).

(ii) a Cauchy sequence if for every c in E with c>>θ, there is a natural number N such that for all n, m>N, $d(x_n, x_m)$<<c.

(iii)a cone metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

2.7 [9] Lemma

Let (X, d) be a cone metric space and P be a normal cone with normal constant L. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X with $x_n \rightarrow x$ and $y_n \rightarrow y$. Then $d(x_n , y_n) \rightarrow d(x, y)$ as $n\rightarrow\infty$.

2.8 [7] Remark

(1) If E is real Banach space with a cone P and $a \preccurlyeq \lambda a$, where $a\in P$ and $0 < \lambda < 1$, then $a = \theta$.

(2) If $c \in$ int P, $\theta \preccurlyeq a_n$ and $a_n \rightarrow \theta$, then there exists a positive integer N such that a_n << c for all $n \geq N$.

The concept of c-distance introduced by Y. J. Cho, et. al. [7], which is a cone version of ω-distance of Kada et. al. [10].

2.9 [7] Definition

Let (X, d) be a cone metric space. Then a function q: $X \times X \to E$ is called a c-distance on X if the following are satisfied

(q1) $\theta \preccurlyeq q(x,y)$ for all $x, y \in X$;

(q2) $q(x, z) \preccurlyeq q(x, y) + q(y, z)$ for all $x, y, z \in X$;

(q3) for each $x \in X$ and $n \geq 1$, if $q(x, y_n) \preccurlyeq u$ for some $u = u_x \in P$, then $q(x, y) \preccurlyeq u$ whenever $\{y_n\}$ is a sequence in X convergent to a point $y \in X$.

(q4) for all $c \in E$ with $\theta << c$, there exists $e \in E$ with $0 << e$ such that $q(z, x) << e$ and $q(z, y) << e$ imply $q(x, y) << c$.

2.10 [7] Lemma

Let (X, d) be a cone metric space and q be a cone distance on X. Let $\{x_n\}$ and $\{y_n\}$be sequences in X and x, y, $z \in X$. Suppose that $\{u_n\}$ is a sequence in P converging to θ. Then the following are holds:

(1) If $q(x_n, y) \preccurlyeq u_n$ and $q(x_n, z) \preccurlyeq u_n$, then $y = z$.

(2) If $q(x_n, y_n) \preccurlyeq u_n$ and $q(x_n, z) \preccurlyeq u_n$, then $\{y_n\}$ converges to z.

(3) If $q(x_n, x_m) \preccurlyeq u_n$ for $m > n$, then $\{x_n\}$is a Cauchy sequence in X.

(4) If $q(y, x_n) \preccurlyeq u_n$, then $\{x_n\}$is a Cauchy sequence in X.

3. Fixed Point Theorems on c-distance

In this section we have extended the Theorem 3.1., and Theorem 3.3. of [7].

3.1. Theorem. Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X and f: $X \to X$ be a continuous and non-decreasing mapping with respect to $\sqsubseteq$. Suppose that the following assertions are hold:

(i) there exists a_1, a_2, a_3, a_4; $a_i > 0$ with $a_1 + a_2 + a_3 + a_4 < 1$ such that

$$q(fx,fy) \preceq a_1q(x,y) + a_2q(x,fx) + a_3q(y, fy) + a_4q(x,fy),$$

for all $x, y \in X$ with $y \sqsubseteq x$;

(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq f x_0$. Then f has a fixed point $x^* \in X$. If $v = fv$, then $q(v, v) = \theta$.

Proof: If $fx_0 = x_0$, then the proof is finished. Suppose that $fx_0 \neq x_0$ then we construct a sequence $\{x_n\}$in X by $x_n = f^n x_0 = fx_{n-1}$. Since f is non-decreasing with respect to $\sqsubseteq$, we obtain by the induction

$$x_0 \sqsubseteq fx_0 \sqsubseteq f^2x_0 \sqsubseteq \ldots \sqsubseteq f^n x_0 \sqsubseteq f^{n+1}x_0 \sqsubseteq \ldots$$

We have,

$$q(x_n, x_{n+1}) = q\left(f^{n-1}x_0, f^n x_0\right)$$
$$= q\left(f\left(f^{n-2}x_0\right), f\left(f^{n-1}x_0\right)\right)$$
$$\preceq a_1q\left(f^{n-2}x_0, f^{n-1}x_0\right) + a_2q\left(f^{n-2}x_0, f^{n-1}x_0\right)$$
$$+ a_3q\left(f^{n-1}x_0, f^n x_0\right) + a_4q\left(f^{n-2}x_0, f^{n-1}x_0\right)$$
$$= a_1q(x_{n-1}, x_n) + a_2q(x_{n-1}, x_n)$$
$$+ a_3q(x_n, x_{n+1}) + a_4q(x_{n-1}, x_{n+1}),$$
$$= a_1q(x_{n-1}, x_n) + a_2q(x_{n-1}, x_n)$$
$$+ a_3q(x_n, x_{n+1}) + a_4q(x_{n-1}, x_n) + a_4q(x_n, x_{n+1}),$$
$$= (a_1 + a_2 + a_4)q(x_{n-1}, x_n) + (a_3 + a_4)q(x_n, x_{n+1}).$$

And hence,

$$1-(a_3 + a_4)q(x_n, x_{n+1}). \preceq (a_1 + a_2 + a_4)q(x_{n-1}, x_n).$$

$$q(x_n, x_{n+1}). \preceq \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} q(x_{n-1}, x_n).$$

Put $\lambda = \dfrac{a_1 + a_2 + a_4}{1 - a_3 - a_4} < 1$.

$\Rightarrow q(x_n, x_{n+1}) \preccurlyeq \lambda\, q(x_{n-1}, x_n)$, for all $n \geq 1$.

Repeating this process, we get that

$$q(x_n, x_{n+1}) \preceq \lambda^n q(x_0, x_1). \tag{1}$$

Let $m > n$, then it follows from (1) that

$$q(x_n, x_m) \preceq q(x_n, x_{n+1}) + \ldots + q(x_{m-1}, x_m)$$
$$\preceq (\lambda^n + \ldots + \lambda^{m-1})\, q(x_0, x_1),$$
$$\preceq \frac{\lambda^n}{1-\lambda} q(x_0, x_1) \to q \text{ as } n \to \infty \text{ since } \lambda < 1.$$

Thus by Lemma (2.10) shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \to x$ as $n \to \infty$. Finally, the continuity of f and $f(f^n x_0) = f^{n+1}(x_0) \to x^*$ implies that $f x^* = x^*$. Thus we prove that x^* is a fixed point of f.

Suppose that $v = fv$. Then we have

$$q(v,v) = q(fv,fv)$$
$$\preceq a_1q(v,v) + a_2q(v,fv) + a_3q(v,fv) + a_4q(v,fv),$$
$$\preceq a_1q(v,v) + a_2q(v,v) + a_3q(v,v) + a_4q(v,v),$$
$$\preceq (a_1 + a_2 + a_3 + a_4)q(v,v).$$

Since, $(a_1 + a_2 + a_3 + a_4) < 1$.

We have $q(v, v) = \theta$.

This completes the proof.

3.2. Theorem. Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space and P is a normal cone with normal constant L. Let q be a c-distance on X and f: $X \to X$ is a non-decreasing mapping with respect to $\sqsubseteq$. Suppose that the following assertions are hold:

(i) there exists $a_1, a_2, a_3, a_4, a_5 > 0$ with $a_1 + a_2 + 2a_3 + a_4 + a_5 < 1$ such that

$$q(fx,fy) \preceq a_1q(x,y) + a_2q(x,fx) + a_3q(y,fy) + a_4q(x,fy) + a_5q(y,fx),$$

for all $x, y \in X$ with $y \sqsubseteq x$;

(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq f x_0$.

(iii) $\inf\{\|q(x, y)\| + \|q(x, fx)\| : x \in X\} > 0$, for all $y \in X$ with $y \neq fy$, then f has a fixed point $x^* \in X$. If $v = fv$, then $q(v, v) = \theta$.

Proof: If we take $x_n = f^n x_0$ in the proof of Theorem 3.1, then we have

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \ldots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \ldots$$

Moreover, $\{x_n\}$ converges to a point $x^* \in X$ and $q(x_n, x_m) \preceq \frac{\lambda^n}{1-\lambda} q(x_0, x_1)$, for all $m > n \geq 1$, where $\lambda = \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} < 1.$

By (q_3), we have $q\left(x_n, x^*\right) \preceq \frac{\lambda^n}{1-\lambda} q(x_0, x_1)$, for all $n \geq 1$.

Since P is normal cone with normal constant L, we have

$$\|q(x_n, x_m)\| \preceq L \frac{\lambda^n}{1-\lambda} \|q(x_0, x_1)\|, \text{ for all } m > n > 1. \quad (2)$$

And

$$\left\|q\left(x_n, x^*\right)\right\| \preceq L \frac{\lambda^n}{1-\lambda} \|q(x_0, x_1)\|, \text{ for all } n \geq 1. \quad (3)$$

If $x^* \neq fx^*$, then by the hypothesis, (2) and (3) with $m = n + 1$, we have

$$\begin{aligned} 0 &< \inf\left\{\left\|q\left(x, x^*\right)\right\| + \|q(x, fx)\| : x \in X\right\} \\ &\leq \inf\left\{\left\|q\left(x_n, x^*\right)\right\| + \|q(x_n, x_{n+1})\| : n \geq 1\right\} \\ &\leq \inf\left\{L \frac{\lambda^n}{1-\lambda} \|q(x_0, x_1)\| + L \frac{\lambda^n}{1-\lambda} \|q(x_0, x_1)\| : n \geq 1\right\} \\ &= 0. \end{aligned}$$

This is a contradiction.

Therefore, we have $x^* = f x^*$. Suppose that $v = fv$ holds, then from the above Theorem 3.1 we can easily prove $q(v, v) = \theta$.

This completes the proof.

3.3. Remark. If we choose $a_4 = 0$ in the above Theorem 3.1, then we get the Theorem 3.1 of [7].

3.4. Remark. If we choose $a_4 = a_5 = 0$ in the above Theorem 3.2 , then we get the Theorem 3.2 of [7].

3.5. Conclusion. In this paper, we have extended the results of Y. J. Cho, et. al. [7]

Acknowledgements

The author is grateful to the reviewer to suggest improve this paper.

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