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Inference Based on Type II Progressively Interval Censored from Inverse Flexible Weibull Distribution Using Different Simulation Methods

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Abstract This paper considers the estimation problem for inverse flexible Weibull model, when the lifetimes are collected under type-II progressive interval censoring. The maximum likelihood and the Bayes estimators for the two unknown parameters of the inverse flexible Weibull distribution are derived. Point estimation and confidence intervals based on maximum likelihood and bootstrap method are also proposed. Bayesian estimation for population parameter under type-II progressive interval censoring is studied via Markov Chain Monte Carlo (MCMC) simulation. To illustrate the proposed methods will discuss an example with the real data. Finally, comparing the two techniques through comparisons between the maximum likelihood using Monte Carlo simulation and bootstrap method on the one hand, and comparing them with the Bayes estimators using MCMC study on the other hand.

Keywords: *Inverse flexible Weibull distribution, progressive interval type-II censoring, bootstrap-t Algorithm, Bayesian and non-Bayesian approach, MCMC*

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1. Introduction

Assuming that n units are randomly selected at the beginning of study which will be terminated when there are m or more failed units. Under a type II progressive interval censored inspection scheme, that trial is terminated after the k^{th} inspection if the total number of failed units is equal to or exceeds m. Also; L_i , $i = 1, 2,...$ is the predetermined inspection times. Suppose that at the i^{th} inspection, d_i failed units Markov chain Monte Carlo (MCMC) are observed and r_i units are fixed removed from the test. In other words, d_i is the number of failed units between any two successive inspections L_{i-1} and L_i , where $L_0 = 0$. Thus, d_i is random variable with observed value pending on the outcomes of the study. Denote $\zeta_j = \sum_{i=1}^{j} d_i$, the test is terminated when $\zeta_{k-1} < m$ and $\zeta_k \geq m$, for the predetermined integer value m , $0 < m \leq n$.

There is shortage of topics for type II progressive interval censored. For instance, Xiang & Tse [10] treated the number of dropouts as a random variable and discussed a type II progressive interval censoring with random removals for Weibull distributed lifetime data. Ashour and Afify [1] considered the estimations of the parameters of exponentiated Weibull family with type II progressive interval censoring with random removals.

In Bayesian approach, It is too difficult to find integrate over the posterior distribution and the problem is that the integrals are usually impossible to evaluate analytically. But in MCMC technique provided a convenient and efficient way to sample from complex, high-dimensional statistical distributions. Recently, application of the MCMC method to the estimation of parameters or some other vital properties about statistical models is very common. Green et al. [5] using the MCMC method for estimating the three parameters Weibull distribution, and they showed that the MCMC method is better than the ML method, when given a proper prior distribution of the parameters. As a generalization of the two parameters Weibull model, Gupta et al. [6] gave a complete Bayesian analysis of the Weibull extension model using MCMC simulation and complete sample.

Bebbington et al. [2] shown that the flexible Weibull distribution is quite flexible, being able to model various ageing classes of lifetime distributions. So we can say that the flexible Weibull distribution is very important in several basic fields include engineering sciences, reliability, biological, demography and actuarial sciences. Also, El-Gohary et al. [3] introduced Inverse Flexible Weibull Extension Distribution.

A random variable X is said to have a flexible Weibull distribution with parameters $\Theta = (\lambda, \beta) > 0$ if $Y = 1/X$ then the random variable Y has the inverse flexible Weibull extension distribution, symbolically we write $Y \sim IFW(\lambda, \beta)$. The cumulative distribution function and the probability density function of Y are respectively given by

$$F(y;\lambda,\beta)=e^{-e^{\frac{\lambda}{y}-\beta y}}, \quad (1)$$

$$f(y;\lambda,\beta)=(\beta+\frac{\lambda}{y^2})e^{\frac{\lambda}{y}-\beta y}e^{-e^{\frac{\lambda}{y}-\beta y}}. \quad (2)$$

The main aim of this paper is evaluating the estimates the model under Type II progressive interval censored using both of bootstrap-t (Boot-t) and Monte Carlo simulation based on Classical estimation and Metropolis–Hastings algorithms based on Bayes estimation. In addition, we will assume the lifetime model which has inverse flexible Weibull distribution with two scale parameters. We assumed that the both scale parameters λ and β have gamma prior and they are independently distributed. We will evaluate performance some simulation experiments to see the behavior of the proposed Bayes estimators and compare their performances with the maximum likelihood estimators MLEs.

The rest of the paper is organized as follows. In the next section, bootstrap-t (Boot-t) based on Classical estimation are presented. In Section 3, we cover Bayesian estimation and MCMC technique. To illustrate the behavior of the proposed methods as well as evaluate the statistical performances of these estimates, we performed a real data analysis in section 4 with comparisons among estimators are investigated through Monte Carlo simulations in previous section and conclusions appear.

2. Bootstrap-t (Boot-t) Based on Classical Estimation

Classical estimation (MLEs) of the unknown parameters and approximate confidence intervals are presented. Also, the corresponding parametric bootstrap confidence intervals using Boot-t for the parameters are given in this section.

2.1. Classical Estimation

Xiang & Tse [10] point out that the $D=(d_1,d_2,...,d_k)$ where k is random and corresponds to the number of inspections before the termination of the experiment, the joint likelihood function of d_i and k, is given by

$$L(d_1,d_2,...,d_k,k;\Theta)$$
$$=C\prod_{i=1}^{k}(F(L_i)-F(L_{i-1}))^{d_i}(1-F(L_i))^{r_i}$$

where

$$C=\binom{n}{d_1}\binom{n-d_1-r_1}{d_2}\cdots\binom{n-\sum_{j=1}^{k-1}(d_j+r_j)}{d_k}$$

and

$$r_k=n-\sum_{j=1}^{k}d_j-\sum_{j=1}^{k-1}r_j. \quad (3)$$

Note that d_i and k are random variables in equation (3), to ensure that there at least m failed units at the end of the study, the number of units removed at each inspection time, r_i, is restricted to be any integer value between 0 and $n-m-\sum_{j=1}^{i-1}r_j$, thus, r_i would not be affected by d_j for all $j=1,2,...,i$.

Xiang & Tse [10] concluded that, the likelihood function under type II progressive censoring may be considered as a special case of equation (4) when all d_i's are fixed to be 1 and $L_i=y_{(i)}$, where $y_{(i)}$ is the i^{th} ordered survival time. By all previous condition, it reduces to the type II censored if $r_i=0$ for $i=1,2,...,m-1$ and $r_m=n-k$.

By taking logarithm in (3), the log likelihood function for type II progressive interval censored ignoring the normalized constant can be written as follows

$$l=\sum_{i=1}^{k}d_i\log\left[F(L_i)-F(L_{i-1})\right]+\sum_{i=1}^{k}d_i\log\left[1-F(L_i)\right] \quad (4)$$

where $l=logL(\Theta)$.

Thus, the maximum likelihood estimates $\hat{\lambda}$ and $\hat{\beta}$ can be obtained by maximizing (4) with respect to λ and β; that is, by simultaneously solving the estimating equations,

$$\sum_{i=1}^{k}\frac{d_i}{F(L_i)-F(L_{i-1})}\left(\frac{\partial F(L_i)}{\partial\hat{\lambda}}-\frac{\partial F(L_{i-1})}{\partial\hat{\lambda}}\right)+\sum_{i=1}^{k}\frac{r_i}{1-F(L_i)}\left(\frac{\partial\left[1-F(L_i)\right]}{\partial\hat{\lambda}}\right)=0 \quad (5)$$

and

$$\sum_{i=1}^{k}\frac{d_i}{F(L_i)-F(L_{i-1})}\left(\frac{\partial F(L_i)}{\partial\hat{\beta}}-\frac{\partial F(L_{i-1})}{\partial\hat{\beta}}\right)+\sum_{i=1}^{k}\frac{r_i}{1-F(L_i)}\left(\frac{\partial\left[1-F(L_i)\right]}{\partial\hat{\beta}}\right)=0. \quad (6)$$

To construct confidence intervals for the unknown parameters we need to compute the asymptotic matrix variance which obtained by inverting the Fisher information matrix $I(\lambda,\beta)$, in which elements are negatives of expected values of the second partial derivatives of the l. The first and second partial derivatives for $F(x)$ with respect to λ,β and the elements of the sample information matrix will be obtained in Appendix. The asymptotic normality of the MLEs can be used to compute the approximate confidence intervals (ACI) for parameters λ and β. Therefore, $(1-\gamma)$ 100% confidence intervals for parameters λ and β become

$$\hat{\lambda}\pm Z_{\gamma/2}\sqrt{Var(\hat{\alpha})} \text{ and } \hat{\beta}\pm Z_{\gamma/2}\sqrt{Var(\hat{\beta})}$$

where $Z_{\gamma/2}$ is percentile of the standard normal distribution with right-tail probability $\gamma/2$.

2.2. Percentile Bootstrap Algorithm (Boot-p)

We can increase information about the population value more than does a point estimate by using a parametric bootstrap interval. We propose to use confidence intervals based on the parametric bootstrap methods using bootstrap-t Algorithm (Boot-t) based on the idea of Hall [7]. The algorithms for estimating the confidence intervals using this method is illustrated as follows

1. Specify the values of n, m and L_i.
2. Specify the values of λ and β.

Step 1: set $i = 0$ and let $dsum = rsum = 0$.

Step 2: $i = i + 1$

- Generate d_i as a binomial random variable with parameters $n - dsum - rsum$ and

$$p = \left(e^{-e^{\frac{\lambda}{L_{i-1}} - \beta L_{i-1}}} - e^{-e^{\lambda L_i - \frac{\beta}{L_i}}} \right) / \left(1 - e^{-e^{\frac{\lambda}{L_{i-1}} - \beta L_{i-1}}} \right)$$

- Calculate $r_i = floor\{p_i \times [n - dsum - rsum - d_i$, where p_i progressive schemes.

Step 3: Set $dsum = dsum + d_i$ and $rsum = rsum + r_i$.

Step 4: If $\zeta_{k-1} = \sum_{i=1}^{k-1} d_i < m$ and $\zeta_k = \sum_{i=1}^{k} d_i \geq m$, stop else go to step 2.

1. Compute the maximum likelihood estimates of the parameters $\hat{\lambda}$, $\hat{\beta}$ by solving the likelihood equations simultaneously in Appendix.
2. Using $\hat{\lambda}$ and $\hat{\beta}$ to generate a bootstrap sample k^*. Based on k^* compute the bootstrap estimate of λ and β using likelihood equations respectively, say $\hat{\lambda}^*$ and $\hat{\beta}^*$ and the following statistics

$$T_1^* = \frac{\sqrt{B}(\hat{\lambda}^* - \hat{\lambda})}{\sqrt{Var(\hat{\lambda}^*)}} \text{ and } T_2^* = \frac{\sqrt{B}(\hat{\beta}^* - \hat{\beta})}{\sqrt{Var(\hat{\beta}^*)}}$$

3. Where $Var(\hat{\lambda}^*)$ and $Var(\hat{\beta}^*)$ are obtained using the Fisher information matrix.
4. Repeat Step 4, B boot times.
5. For the T_1^* and T_2^* values obtained in step 4, determine the upper and lower bounds of the $100(1-\gamma)\%$ confidence interval bootstrap (CIB) of $\hat{\lambda}$ and $\hat{\beta}$ as follows: let $H(x) = P(T_i^* \leq x), i = 1,2,3$ be the cumulative distribution function of T_1^* and T_2^* for a given X, define

$$\left[\hat{\lambda}_{\text{Boot-t}}(\gamma), \hat{\lambda}_{\text{Boot-t}}(1-\gamma)\right]$$
$$\text{and } \left[\hat{\beta}_{\text{Boot-t}}(\gamma), \hat{\beta}_{\text{Boot-t}}(1-\gamma)\right].$$

3. Bayesian Estimation and MCMC Technique

In this section, we will focus to Bayesian approach using Markov chain Monte Carlo (MCMC) method to generate from the posterior distributions and in turn computing the Bayes estimators are developed.

3.1. Bayesian Estimation

In Bayesian scenario, we need to assume the prior distribution of the unknown model parameters to take into account uncertainty of the parameters. We consider the Bayesian estimation under the assumption that the random variables λ and β have an independent gamma prior distributions. Assumed that λ ~ Gamma(B,A) and β ~ Gamma(D,C), then, the joint prior density of λ and β can be written as

$$g(\lambda,\beta) \propto \lambda^{B-1}\beta^{D-1}e^{-(\lambda A+\beta C)} \quad (7)$$

Note that when $A = B = C = D = 0$, (we call it prior 0) they are the non-informative λ and β respectively. It follows from (1), (3) and (7) that the joint posterior density function of λ and β given x is thus

$$\pi^*(\lambda,\beta/d) \propto \frac{\left\{ \prod_{i=1}^{k} \left[e^{-e^{\frac{\lambda}{L_{i-1}} - \beta L_{i-1}}} - e^{-e^{\frac{\lambda}{L_i} - \beta L_i}} \right]^{d_i} e^{-r_i e^{\frac{\lambda}{L_i} - \beta L_i}} \lambda^{B-1}\beta^{D-1}e^{-(\lambda A+\beta C)} \right\}}{\int_0^\infty \int_0^\infty \left[L(d_1,d_2,\ldots,d_k,k;\lambda,\beta) \times g(\lambda,\beta) \right] d\lambda\, d\beta} \quad (8)$$

The Bayes estimate of any function of λ and β, say $U(\lambda,\beta)$, is

$$\tilde{U}(\lambda,\beta/d) \propto \frac{\iint_0^\infty \left\{ \prod_{i=1}^{k} \left[e^{-e^{\frac{\lambda}{L_{i-1}} - \beta L_{i-1}}} - e^{-e^{\frac{\lambda}{L_i} - \beta L_i}} \right]^{d_i} e^{-r_i e^{\frac{\lambda}{L_i} - \beta L_i}} \lambda^{B-1}\beta^{D-1}e^{-(\lambda A+\beta C)} \right\} d\lambda\, d\beta}{\int_0^\infty \int_0^\infty \left[L(d_1,d_2,\ldots,d_k,k;\lambda,\beta) \times g(\lambda,\beta) \right] d\lambda\, d\beta} \quad (9)$$

By using binomial and exponential series for equation (8), the posterior conditional distribution for λ and β are

$$\pi^*(\lambda/d,\beta) \propto \prod_{i=1}^{k}\sum_{j=0}^{d_i}\sum_{l=0}^{\infty} \left[(-1)^i \frac{(-j.d_i.r_i)^l}{l!} \lambda^{B-1} e^{-\lambda\left\{A - l\left(\frac{1}{L_i} + \frac{1}{L_{i-1}}\right)\right\}} \right] \quad (10)$$
$$\pi^*(\beta/d,\lambda) \propto \prod_{i=1}^{k}\sum_{j=0}^{d_i}\sum_{l=0}^{\infty} \left[(-1)^i \frac{(-j.d_i.r_i)^l}{l!} \beta^{D-1} e^{-\beta\{C - l(L_i + L_{i-1})\}} \right]$$

respectively.

It is not possible to compute (9) analytically. The problem is that the integrals in (9) are usually impossible to evaluate analytically, and the numerical methods may fail. The MCMC method provides an alternative method for parameter estimation. In the following subsections, we propose using the MCMC technique to obtain Bayes estimates of the unknown parameters and construct the corresponding credible intervals.

3.2. MCMC Technique

Computer simulation of Markov chains in the space of parameter will depend on Markov chain Monte Carlo (MCMC) [see Gilks et al. [4]]. The Markov chains are defined in such a way that the posterior distribution in the given statistical inference problem is the asymptotic distribution. However, the posterior likelihood usually does not have a closed form for a given type II progressively interval censored data. Moreover, a numerical integration cannot be easily applied in this situation. The Metropolis – Hastings algorithm is a very general MCMC method first expansion by Metropolis et al. [9] and later extended by Hastings [8]. It is possible to use these algorithms by implement posterior simulation in essentially any problem which allows point wise evaluation of the prior distribution and likelihood function. It can be used to obtain random samples from any arbitrarily complicated target distribution of any dimension that is known up to a normalizing constant. In fact, Gibbs sampler is just a special case of the M-H algorithm.

Now, we propose the following scheme to generate λ and β from density functions and in turn obtain the Bayes estimates and the corresponding credible intervals.

0. Start with an $\lambda^{(0)} = \hat{\lambda}, \beta^{(0)} = \hat{\beta}$ and $M = \text{burn} - \text{in}$.
1. Set $t = 1$.
2. Generate $\lambda^{(t)}$ and $\beta^{(t)}$ from (10).
3. Set $t = t + 1$.
4. Repeats Steps 1-3 N times.
5. Obtain the Bayes estimates of λ and β with respect to the squared error loss function as

$$\tilde{\lambda} = \hat{E}(\lambda / x) = \frac{1}{N-M} \sum_{i=M+1}^{N} \lambda_i,$$

$$\tilde{\beta} = \hat{E}(\beta / x) = \frac{1}{N-M} \sum_{i=M+1}^{N} \beta_i,$$

6. To compute the credible intervals of λ and β order $\tilde{\lambda}_1, \dots, \tilde{\lambda}_{N-M}$ and $\tilde{\beta}_1, \dots, \tilde{\beta}_{N-M}$ as $\tilde{\lambda}_1 < \cdots < \tilde{\lambda}_{N-M}$ and $\tilde{\beta}_1 < \cdots < \tilde{\beta}_{N-M}$. Then the 100(1-γ)% symmetric credible intervals (SCI) of λ and β become:

$$\left[\tilde{\lambda}_{(N-M)\gamma/2}, \tilde{\lambda}_{(N-M)(1-\gamma/2)}\right]$$

$$\text{and} \left[\tilde{\beta}_{(N-M)\gamma/2}, \tilde{\beta}_{(N-M)(1-\gamma/2)}\right].$$

4. Numerical Results

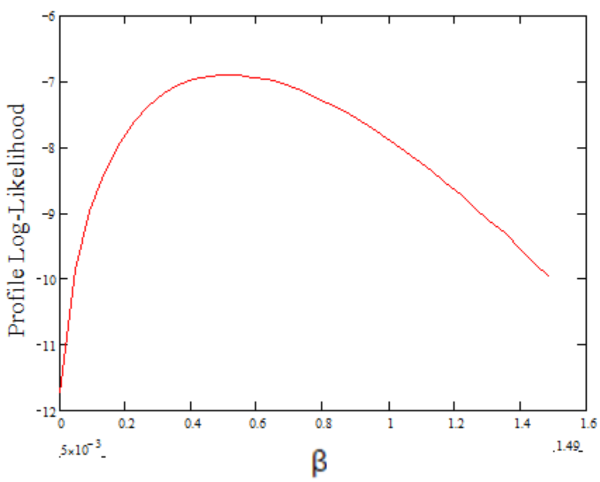
To illustrate the behavior of the proposed methods as well as evaluate the statistical performances of these estimates a numerical illustration is conducted where the performance of the different results obtained in the previous sections can't be compared theoretically. We reanalyze a real data set analyzed by Xiang and Tse [10]. Also, a simulations study is used to compare the performance of the different estimators, different confidence intervals using different parameter values and different schemes. In this section, the numerical study is carried out under type II progressive interval censored with unknown parameters. All of computations were performed using MATHCAD program version 2007.

4.1. Real Data

In the first subsection, we will rely on re-analyzed the real data which was originally analyzed by Xiang and Tse [10]. The data was obtained an experiment which was conducted to assess the toxicity of substance to animals. Forty mice were selected, every week a blood sample was collected from each of them, and the number of mice that showed evidence of toxicity was recorded. During the course of study, some mice which had to be removed from the study because they had developed other diseases, which made them unfit for the study. The data collected in the study are summarized in the following table:

	Week					
	1	2	3	4	5	6
d_i	7	3	3	4	6	3
r_i	1	0	2	2	2	

Before computing the MLEs, we get the MLEs of λ and β from equations 5 and 6 respectively. On the hand, for fixed β, the MLE of λ can be obtained as function in β as $\hat{\lambda}(\beta)$, By Substituting $\hat{\lambda}(\beta)$ in (4) and from other hand, for fixed λ, the MLE of β can be obtained as function in λ as $\hat{\beta}(\lambda)$, By Substituting $\hat{\beta}(\lambda)$ in (4); we can plot the profile log likelihood of β and λ as follows



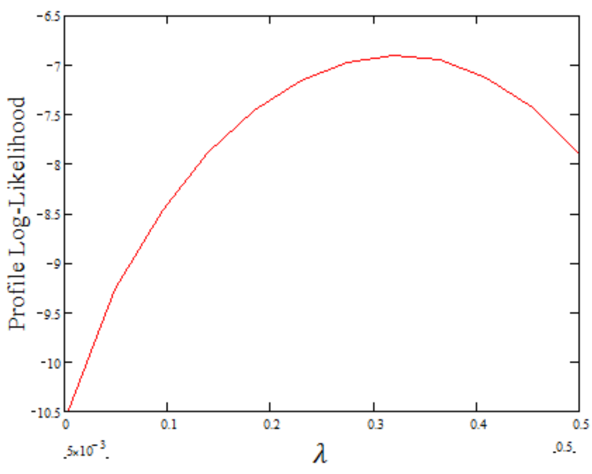


Figure 1. Profile log likelihood of β and λ

It is noted from Figure 1 that the likelihood equations have a unique solution, so we suggest using $\beta = 0.52$ and $\lambda = 0.33$ as initial values to start the iteration to obtain the MLEs of β and λ. The maximum likelihood estimates are $\hat{\lambda} = 5.2$ and $\hat{\beta} = 2.2$, and the corresponding 95% confidence intervals are (4.988, 5.734) and (2.084, 2.577) respectively. Based on the bootstrap sample of size 1000, the bootstrap estimates are $\hat{\lambda}^* = 5.341$ and $\hat{\beta}^* = 2.482$, and the corresponding 95% confidence intervals bootstrap are (5.201, 5.511) and (2.227, 2.670). We assume the non-informative priors, because we have no prior information about the unknown parameters. Based on the MCMC samples of size 10000, the Bayes estimates under the squared error loss function are $\tilde{\lambda} = 5.211$ and $\tilde{\beta} = 2.340$, and the corresponding 95% symmetric credible intervals (5.014, 5.467) and (2.201, 2.470).

The analysis of the previous real data set demonstrates the importance and usefulness of type II progressive Interval censored and inferential procedures based on them. From the previous example, we observed that the predetermined time number of inspection and number of failures plays an important role for the estimation of the unknown parameters and the corresponding confidence intervals. Also, it can be seen that the performance of the different methods for estimation are quite close to each other. However, the MLEs and Bayes estimators under squared error loss function with respect to the non-informative priors are the closest.

4.2. Simulation Study

The simulation study is conducted by considering different values of sample sizes $n = 20, 30, 50$, different effective number of failure $m = 5, 10, 15$, and by choosing $(\lambda, \beta) = (1.5, 3)$ in all the cases, also used generate type II progressive interval censored data under each of the four progressive schemes with withdraw probabilities denoted as p_1, p_2, p_3 and p_4 and depend on k. All the progressive schemes used for the study are defined as follows:

$$p_1 = (0.25, 0.25, 0.25, \ldots, 1)$$
$$p_2 = (0.5, 0.5, 0.5, \ldots, 1)$$
$$p_3 = (0, 0, 0, \ldots, 1)$$
$$p_4 = (0.25, 0, 0, 0, \ldots, 1)$$

Where censoring in p_1 is lighter for the all intervals and p_2 is heavier for the all intervals.While p_3 are the conventional interval censoring where there are no removals prior to the experiment termination and the censoring in p_4 only occurs at the left-most and the right-most.

Table 1. The average values (AVE),mean square error (MSE), variance(VAR), bias andlength of 95% ACI (LACI)of the MLEs using Monte Carlo simulation

n	m		λ p_1	λ p_2	λ p_3	λ p_4	β p_1	β p_2	β p_3	β p_4
20	5	AVE	1.309	1.340	1.354	1.346	2.863	2.898	2.882	2.928
		MSE	0.069	0.071	0.080	0.099	0.013	0.083	0.076	0.054
		VAR	0.005	0.001	0.001	0.008	0.009	0.012	0.007	0.009
		BIAS	0.069	0.071	0.080	0.099	0.013	0.083	0.076	0.054
		LACI	0.400	0.382	0.390	0.399	0.298	0.357	0.501	0.376
	10	AVE	1.431	1.464	1.464	1.482	2.867	2.913	2.924	2.949
		MSE	0.060	0.033	0.022	0.078	0.077	0.060	0.047	0.096
		VAR	0.001	0.001	0.007	0.008	0.010	0.005	0.002	0.012
		BIAS	0.060	0.033	0.022	0.078	0.077	0.060	0.047	0.096
		LACI	0.392	0.416	0.401	0.410	0.400	0.506	0.595	0.407
	15	AVE	1.416	1.444	1.473	1.448	2.882	2.924	2.925	2.951
		MSE	0.009	0.034	0.024	0.098	0.059	0.078	0.010	0.068
		VAR	0.001	0.003	0.003	0.005	0.011	0.012	0.002	0.007
		BIAS	0.009	0.034	0.024	0.098	0.058	0.078	0.010	0.068
		LACI	0.419	0.414	0.448	0.405	0.494	0.270	0.588	0.155

Table 2. The average values (AVE), mean square error (MSE), variance (VAR), bias and length of 95% ACI (LACI) of the MLEs using Monte Carlo simulation

n	m		λ p_1	λ p_2	λ p_3	λ p_4	β p_1	β p_2	β p_3	β p_4
30	5	AVE	1.292	1.405	1.433	1.509	2.801	2.801	2.762	2.825
		MSE	0.090	0.062	0.049	0.012	0.010	0.016	0.002	0.010
		VAR	0.007	0.000	0.001	0.000	0.012	0.009	0.001	0.012
		BIAS	0.090	0.062	0.049	0.012	0.009	0.016	0.002	0.009
		LACI	0.475	0.459	0.464	0.462	0.535	0.238	0.325	0.535
	10	AVE	1.382	1.405	1.440	1.535	2.817	2.817	2.822	2.863
		MSE	0.071	0.004	0.069	0.064	0.012	0.071	0.086	0.012
		VAR	0.004	0.008	0.002	0.007	0.004	0.009	0.001	0.004
		BIAS	0.071	0.004	0.069	0.064	0.012	0.070	0.086	0.012
		LACI	0.381	0.418	0.378	0.373	0.520	0.365	0.356	0.520
	15	AVE	1.524	1.564	1.575	1.624	2.820	2.822	2.829	2.884
		MSE	0.053	0.075	0.002	0.038	0.097	0.071	0.005	0.097
		VAR	0.006	0.005	0.005	0.005	0.010	0.000	0.009	0.010
		BIAS	0.053	0.075	0.002	0.038	0.097	0.071	0.005	0.097
		LACI	0.374	0.369	0.400	0.356	0.413	0.168	0.420	0.413

Table 3. The average values (AVE), mean square error (MSE), variance (VAR), bias and length of 95% ACI (LACI) of the MLEs using Monte Carlo simulation

n	m		λ				β			
			p_1	p_2	p_3	p_4	p_1	p_2	p_3	p_4
		AVE	1.256	1.290	1.463	1.578	2.989	3.019	3.028	3.045
		MSE	0.048	0.081	0.012	0.085	0.058	0.045	0.055	0.023
	5	VAR	0.006	0.000	0.004	0.008	0.005	0.003	0.006	0.000
		BIAS	0.048	0.081	0.012	0.085	0.058	0.045	0.055	0.023
		LACI	0.432	0.419	0.448	0.379	0.368	0.515	0.310	0.188
		AVE	1.391	1.480	1.520	1.579	2.992	3.027	3.045	3.113
		MSE	0.096	0.007	0.037	0.022	0.006	0.089	0.077	0.001
50	10	VAR	0.002	0.010	0.004	0.009	0.000	0.004	0.008	0.002
		BIAS	0.096	0.006	0.037	0.022	0.006	0.089	0.077	0.001
		LACI	0.420	0.417	0.400	0.367	0.202	0.627	0.279	0.422
		AVE	1.559	1.567	1.572	1.619	3.037	3.046	3.075	3.142
		MSE	0.099	0.037	0.021	0.054	0.075	0.011	0.096	0.045
	15	VAR	0.008	0.008	0.005	0.006	0.006	0.004	0.010	0.006
		BIAS	0.099	0.037	0.021	0.054	0.075	0.011	0.096	0.045
		LACI	0.324	0.345	0.258	0.285	0.172	0.131	0.115	0.151

Table 4. The average values (AVE), mean square error (MSE), variance (VAR), bias and length of 95%CIB (LCIB) of the MLEs using Bootstrap method

n	m		λ				β			
			p_1	p_2	p_3	p_4	p_1	p_2	p_3	p_4
		AVE	1.204	1.210	1.229	1.272	2.661	2.690	2.753	2.826
		MSE	0.003	0.088	0.031	0.078	0.056	0.100	0.001	0.113
	5	VAR	0.008	0.012	0.001	0.006	0.029	0.024	0.001	0.029
		BIAS	0.003	0.088	0.031	0.078	0.055	0.099	0.001	0.113
		LCIB	0.465	0.512	0.389	0.403	0.237	0.254	0.338	0.282
		AVE	1.216	1.225	1.255	1.274	2.690	2.708	2.769	2.847
		MSE	0.077	0.041	0.071	0.024	0.117	0.015	0.031	0.093
20	10	VAR	0.012	0.007	0.004	0.003	0.032	0.028	0.020	0.028
		BIAS	0.077	0.041	0.070	0.024	0.116	0.014	0.030	0.092
		LCIB	0.421	0.405	0.410	0.405	0.325	0.169	0.158	0.187
		AVE	1.219	1.268	1.289	1.293	2.697	2.735	2.787	2.849
		MSE	0.062	0.020	0.039	0.056	0.109	0.022	0.067	0.008
	15	VAR	0.004	0.009	0.008	0.010	0.005	0.028	0.018	0.013
		BIAS	0.062	0.020	0.039	0.056	0.109	0.021	0.066	0.008
		LCIB	0.426	0.471	0.421	0.435	0.274	0.179	0.275	0.275

Table 5. The average values (AVE), mean square error (MSE), variance (VAR), bias and length of 95% CIB (LCIB) of the MLEs using Bootstrap method

n	m		λ				β			
			p_1	p_2	p_3	p_4	p_1	p_2	p_3	p_4
		AVE	1.200	1.209	1.221	1.244	2.658	2.681	2.779	2.832
		MSE	0.001	0.028	0.044	0.023	0.088	0.070	0.055	0.040
	5	VAR	0.004	0.011	0.008	0.002	0.005	0.002	0.021	0.003
		BIAS	0.001	0.028	0.044	0.023	0.088	0.070	0.055	0.040
		LCIB	0.514	0.492	0.439	0.381	0.271	0.238	0.292	0.154
		AVE	1.207	1.244	1.224	1.279	2.666	2.720	2.797	2.833
		MSE	0.094	0.050	0.063	0.077	0.075	0.002	0.091	0.080
30	10	VAR	0.005	0.005	0.010	0.006	0.030	0.005	0.009	0.006
		BIAS	0.094	0.050	0.063	0.077	0.074	0.002	0.091	0.080
		LCIB	0.491	0.457	0.366	0.545	0.239	0.197	0.326	0.160
		AVE	1.242	1.254	1.273	1.293	2.668	2.766	2.799	2.835
		MSE	0.033	0.068	0.053	0.065	0.112	0.044	0.055	0.117
	15	VAR	0.004	0.004	0.010	0.007	0.011	0.002	0.031	0.019
		BIAS	0.033	0.068	0.052	0.065	0.112	0.044	0.054	0.116
		LCIB	0.489	0.448	0.496	0.386	0.219	0.348	0.166	0.308

Table 6. The average values (AVE), mean square error (MSE), variance (VAR), bias and length of 95% CIB (LCIB) of the MLEs using Bootstrap method

n	m		λ				β			
			p_1	p_2	p_3	p_4	p_1	p_2	p_3	p_4
		AVE	1.218	1.223	1.236	1.242	2.668	2.691	2.732	2.776
		MSE	0.090	0.082	0.100	0.094	0.113	0.016	0.004	0.003
	5	VAR	0.004	0.007	0.010	0.001	0.008	0.009	0.033	0.007
		BIAS	0.090	0.082	0.100	0.094	0.113	0.016	0.003	0.003
		LCIB	0.397	0.401	0.485	0.463	0.265	0.185	0.216	0.173
		AVE	1.225	1.254	1.276	1.279	2.683	2.715	2.735	2.780
		MSE	0.080	0.050	0.099	0.060	0.091	0.022	0.022	0.121
50	10	VAR	0.002	0.011	0.001	0.009	0.031	0.007	0.016	0.004
		BIAS	0.080	0.049	0.099	0.060	0.090	0.022	0.022	0.121
		LCIB	0.491	0.441	0.540	0.469	0.276	0.201	0.287	0.167
		AVE	1.229	1.268	1.276	1.283	2.691	2.717	2.776	2.781
		MSE	0.031	0.030	0.062	0.075	0.009	0.008	0.020	0.052
	15	VAR	0.003	0.010	0.000	0.004	0.004	0.033	0.024	0.014
		BIAS	0.031	0.030	0.062	0.075	0.009	0.007	0.019	0.052
		LCIB	0.506	0.355	0.472	0.462	0.236	0.349	0.270	0.185

Table 7. The average values (AVE), mean square error (MSE), variance (VAR), bias and length of 95% SCI (LSCI) of the Bayes estimates using MCMC

n	m		λ				β			
			p_1	p_2	p_3	p_4	p_1	p_2	p_3	p_4
		AVE	1.378	1.406	1.412	1.457	2.867	2.885	2.891	2.904
		MSE	0.070	0.078	0.047	0.018	0.008	0.009	0.006	0.001
	5	VAR	0.005	0.010	0.005	0.003	0.001	0.000	0.019	0.013
		BIAS	0.070	0.077	0.047	0.018	0.008	0.009	0.005	0.001
		LSCI	0.273	0.251	0.237	0.157	0.169	0.119	0.122	0.178
		AVE	1.381	1.414	1.446	1.540	2.869	2.925	3.012	3.026
		MSE	0.023	0.075	0.002	0.057	0.008	0.002	0.007	0.018
20	10	VAR	0.008	0.003	0.006	0.011	0.016	0.013	0.009	0.008
		BIAS	0.023	0.075	0.002	0.057	0.008	0.002	0.007	0.018
		LSCI	0.201	0.233	0.230	0.223	0.118	0.131	0.141	0.196
		AVE	1.389	1.469	1.528	1.543	2.888	2.986	3.027	3.032
		MSE	0.061	0.068	0.080	0.075	0.015	0.012	0.012	0.001
	15	VAR	0.003	0.008	0.011	0.001	0.015	0.000	0.006	0.009
		BIAS	0.061	0.068	0.080	0.075	0.014	0.012	0.012	0.001
		LSCI	0.219	0.187	0.264	0.185	0.130	0.120	0.109	0.120

Table 8. The average values (AVE), mean square error (MSE), variance (VAR), bias and length of 95% SCI (LSCI) of the Bayes estimates using MCMC

n	m		λ				β			
			p_1	p_2	p_3	p_4	p_1	p_2	p_3	p_4
		AVE	1.385	1.539	1.567	1.588	2.997	3.000	3.034	3.119
		MSE	0.055	0.021	0.070	0.048	0.011	0.012	0.016	0.011
	5	VAR	0.011	0.005	0.006	0.004	0.005	0.009	0.019	0.011
		BIAS	0.055	0.021	0.070	0.048	0.011	0.012	0.015	0.011
		LSCI	0.185	0.228	0.248	0.232	0.199	0.143	0.151	0.173
		AVE	1.521	1.548	1.574	1.623	3.006	3.039	3.100	3.129
		MSE	0.040	0.051	0.014	0.061	0.020	0.011	0.009	0.005
30	10	VAR	0.001	0.007	0.011	0.000	0.003	0.001	0.003	0.002
		BIAS	0.040	0.051	0.014	0.061	0.020	0.011	0.009	0.005
		LSCI	0.270	0.275	0.196	0.207	0.103	0.208	0.144	0.203
		AVE	1.558	1.563	1.584	1.652	3.035	3.066	3.108	3.188
		MSE	0.033	0.065	0.023	0.049	0.017	0.004	0.010	0.008
	15	VAR	0.001	0.000	0.008	0.005	0.016	0.010	0.014	0.012
		BIAS	0.033	0.065	0.023	0.049	0.017	0.004	0.010	0.008
		LSCI	0.161	0.264	0.219	0.170	0.139	0.121	0.140	0.208

Table 9. The average values (AVE), mean square error (MSE), variance (VAR), bias and length of 95% SCI (LSCI) of the Bayes estimates using MCMC

n	m		λ				β			
			p_1	p_2	p_3	p_4	p_1	p_2	p_3	p_4
50	5	AVE	1.544	1.401	1.524	1.381	3.020	3.030	3.061	3.068
		MSE	0.033	0.008	0.028	0.020	0.008	0.006	0.008	0.001
		VAR	0.009	0.001	0.007	0.001	0.002	0.003	0.001	0.007
		BIAS	0.033	0.008	0.028	0.020	0.008	0.006	0.008	0.001
		LSCI	0.160	0.258	0.238	0.233	0.143	0.142	0.159	0.124
	10	AVE	1.583	1.457	1.530	1.567	3.001	3.016	3.047	3.076
		MSE	0.047	0.019	0.042	0.068	0.008	0.009	0.009	0.008
		VAR	0.008	0.011	0.011	0.006	0.007	0.004	0.008	0.001
		BIAS	0.046	0.019	0.042	0.068	0.008	0.009	0.009	0.008
		LSCI	0.175	0.184	0.178	0.207	0.101	0.112	0.117	0.155
	15	AVE	1.636	1.531	1.545	1.576	2.992	2.995	3.027	3.050
		MSE	0.061	0.034	0.041	0.024	0.001	0.001	0.003	0.009
		VAR	0.006	0.009	0.005	0.005	0.005	0.003	0.001	0.000
		BIAS	0.061	0.034	0.041	0.024	0.001	0.001	0.003	0.009
		LSCI	0.185	0.255	0.225	0.214	0.131	0.137	0.140	0.163

Under a type II progressive interval censored, using the different simulation methods, on the hand, were obtained the unknown parameters dependent on Monte Carlo simulation and Bootstrap method using maximum likelihood estimation and from other hand were used MCMC to obtain the unknown parameters using Bayes estimation.

In general, tables from (1) to (9) show that variance is usually smaller and bias is usually larger in both the estimation methods by using different simulations. The mean-squared error (MSE) associated with both MLE and Bayes estimates of the parameters decrease with increasing the sample size n. Also, it decreases when m is large.

With increasing n, there is an improvement in the value of estimators regardless of the type of estimation and the method of simulation. At each table with increasing m, there is an improvement in the value of the estimators, and also estimators obtained from progressive schemes p_3 and p_4 are the best forever. By comparison with the different methods of simulation the worst methods was bootstrap and which fail to give good estimators. Also, the estimators which were obtained from maximum likelihood estimation and Bayes estimation approximately one. In other words, the difference between them was trivial.

Under a type II progressive interval censored inspection scheme, that trial is terminated after the k^{th} inspection if the total number of failed units is equal to or exceeds m. In the fact, the total number of failure units greater than the value of m and know $\tilde{m}$ which refer to estimated value to estimate value which obtained in accordance with the conditions of the experiment.

Table 10. The average values (AVE), mean square error (MSE), variance (VAR), bias and 95% CI of thetotal number of failed unitswith different simulation methods

Simulation	m	$\tilde{m}$				95% Confidence Intervals	
		AVE	MSE	VAR	BIAS		
Monte Carlo	5	9	0.005	0.002	0.056	8.990	9.010
	10	14	0.005	0.004	0.022	13.991	14.009
	15	18	0.008	0.007	0.039	17.984	18.016
Bootstrap	5	11	0.012	0.009	0.056	10.976	11.024
	10	16	0.214	0.205	0.097	15.581	16.419
	15	19	0.017	0.014	0.051	18.967	19.033
MCMC	5	7	0.002	0.000	0.038	6.996	7.004
	10	12	0.001	0.001	0.001	11.998	12.002
	15	16	0.001	0.006	0.082	8.990	9.010

We considered the following values: $n = 20$ and $m = 5,10$ and 15,we computed $\tilde{m}$ using Monte Carlo simulation and Bootstrap method using maximum likelihood estimation and MCMC using Bayes estimation to study of 10000 samples. The results are displayed in Table 10, By compared between simulation methods and both of estimations, which referred to MCMC gave the smallest $\tilde{m}$, so, MCMC is the better in economic terms, where decrease of failure units (unobserved) which mean ending the experiment early and therefore is the best estimate. Bootstrap method is the worst, where the experiment ends in late unlike other methods.

References

[1] Ashour and Afify (2008), Estimations of the Parameters of Exponentiated Weibull Family with Type II Progressive Interval Censoring with Random Removals. *Journal of Applied Sciences Research, 4(11): 1428-1442.*

[2] Bebbington, M., Lai, C. D. and Zitikis, R. (2007), A flexible Weibull extension. *Reliability Engineering and System Safety, 92, 719-726.*

[3] El-Gohary A,El-Bassiouny, A. H. and El-Morshedy, M. (2015), Inverse Flexible Weibull Extension Distribution. *International Journal of Computer Applications (0975-8887), 115 (2) 46-51.*

[4] Gilks W. R., Richardson S. and Spiegelhalter D. J. (1996), *Markov chain Monte Carlo in Practices*, Chapman and Hall, London.

[5] Green E. J., Roesch F. A., Smith .A .F .M, and Strawderman W. E., (1994), Bayesian estimation for the three-parameter Weibull distribution with tree diameter data. *Biometrica (50) 254-269.*

[6] Gupta A. , Mukherjee B. and Upadhyay S. K., (2008), A Bayes study using Markov Chain Monte Carlo simulation, *Reliability Engineering & System Safety*, (93) 1434-1443.

[7] Hall, P. (1988), Theoretical comparison of Bootstrap confidence intervals, *Annals of Statistics 16, 927-953.*

[8] Hastings, W. K. (1970), Monte Carlo sampling methods using markov chains and their applications, *Biometrika*, 57(1):97-109.

[9] Metropolis N., Rosenbluth A.W., Rosenbluth M. N., Teller A. H. and Teller E., (1953), Equations of state calculations by fast computing machine, *Journal of Chemical Physics,* (21) 1087-1091.

[10] Xiang. L; Tse. S.K (2005), Maximum Likelihood Estimation In Survival Studies Under Progressive Interval Censoring With Random Removals. *Journal of Biopharmaceutical Statistics.15, 981-991.*

Appendix

The asymptotic variance-covariance matrix of the maximum likelihood estimators for parameters λ and β are given by elements of the inverse of the Fisher information matrix. Unfortunately, the exact mathematical expressions for the above expectations are very difficult to obtain. Therefore, we give the approximate (observed) asymptotic varaince-covariance matrix for the maximum likelihood estimators, which is obtained by dropping the expectation operator E, where

$$I_1^{-1}(\hat{\lambda},\hat{\beta}) = \begin{bmatrix} \left(-\frac{\partial^2 \ln L(\pi)}{\partial\lambda^2}\right) & \left(-\frac{\partial^2 \ln L(\pi)}{\partial\lambda\partial\beta}\right) \\ \left(-\frac{\partial^2 \ln L(\pi)}{\partial\lambda\partial\beta}\right) & \left(-\frac{\partial^2 \ln L(\pi)}{\partial\beta^2}\right) \end{bmatrix}_{\lambda=\hat{\lambda},\beta=\hat{\beta}} = \begin{bmatrix} V(\hat{\lambda}) & Cov(\hat{\lambda},\hat{\beta}) \\ Cov(\hat{\lambda},\hat{\beta}) & V(\hat{\beta}) \end{bmatrix}$$

Fisher information matrix and the variance-covariance matrix will be obtained by numerical technique.

From equations (5) and (6), we will determine the second partials by differentiating the first partials as following

$$\frac{\partial^2 logL}{\partial\lambda^2} = \sum_{i=1}^{k} d_i \left\{ \frac{\frac{\partial^2 F(L_i)}{\partial\lambda^2} - \frac{\partial^2 F(L_{i-1})}{\partial\lambda^2}}{F(L_i)-F(L_{i-1})} - \frac{\left[\frac{\partial F(L_i)}{\partial\lambda} - \frac{\partial F(L_{i-1})}{\partial\lambda}\right]^2}{\left[F(L_i)-F(L_{i-1})\right]^2} \right\} + \sum_{i=1}^{k} r_i \left\{ \frac{\frac{\partial^2}{\partial\lambda^2}\left[1-F(L_i)\right]}{\left[1-F(L_i)\right]} - \frac{\left[\frac{\partial}{\partial\lambda}\left[1-F(L_i)\right]\right]^2}{\left[1-F(L_i)\right]^2} \right\}$$

$$\frac{\partial^2 logL}{\partial\lambda\partial\beta} = \sum_{i=1}^{k} d_i \left\{ \frac{\frac{\partial^2 F(L_i)}{\partial\lambda\partial\beta} - \frac{\partial^2 F(L_{i-1})}{\partial\lambda\partial\beta}}{F(L_i)-F(L_{i-1})} - \frac{\left[\frac{\partial F(L_i)}{\partial\lambda} - \frac{\partial F(L_{i-1})}{\partial\lambda}\right]\left[\frac{\partial F(L_i)}{\partial\beta} - \frac{\partial F(L_{i-1})}{\partial\beta}\right]}{\left[F(L_i)-F(L_{i-1})\right]^2} \right\}$$

$$+ \sum_{i=1}^{k} r_i \left\{ \frac{\frac{\partial^2}{\partial\lambda\partial\beta}\left[1-F(L_i)\right]}{\left[1-F(L_i)\right]} - \frac{\left[\frac{\partial}{\partial\lambda}\left[1-F(L_i)\right]\right]\left[\frac{\partial}{\partial\beta}\left[1-F(L_i)\right]\right]}{\left[1-F(L_i)\right]^2} \right\}$$

$$\frac{\partial^2 logL}{\partial\beta^2} = \sum_{i=1}^{k} d_i \left\{ \frac{\frac{\partial^2 F(L_i)}{\partial\beta^2} - \frac{\partial^2 F(L_{i-1})}{\partial\beta^2}}{F(L_i)-F(L_{i-1})} - \frac{\left[\frac{\partial F(L_i)}{\partial\beta} - \frac{\partial F(L_{i-1})}{\partial\beta}\right]^2}{\left[F(L_i)-F(L_{i-1})\right]^2} \right\}$$

$$+ \sum_{i=1}^{k} r_i \left\{ \frac{\frac{\partial^2}{\partial\beta^2}\left[1-F(L_i)\right]}{\left[1-F(L_i)\right]} - \frac{\left[\frac{\partial}{\partial\beta}\left[1-F(L_i)\right]\right]^2}{\left[1-F(L_i)\right]^2} \right\}$$

$$\frac{\partial F(L_i)}{\partial\lambda} = \frac{-1}{L_i} e^{\lambda/L_i - \beta L_i} F(L_i)$$

$$\frac{\partial F(L_i)}{\partial\beta} = L_i e^{\lambda/L_i - \beta L_i} F(L_i)$$

$$\frac{\partial^2 F(L_i)}{\partial\lambda^2} = \frac{-1}{L_i}\frac{\partial F(L_i)}{\partial\lambda}\left\{1 - e^{\lambda/L_i - \beta L_i}\right\}$$

$$\frac{\partial^2 F(L_i)}{\partial \beta^2} = -L_i \frac{\partial F(L_i)}{\partial \beta}\left\{1 - e^{\lambda/L_i - \beta L_i}\right\}$$

$$\frac{\partial^2 F(L_i)}{\partial \lambda \partial \beta} = -L_i \frac{\partial F(L_i)}{\partial \lambda}\left\{1 - e^{\lambda/L_i - \beta L_i}\right\}$$

Note that: $L_i \frac{\partial F(L_i)}{\partial \lambda} = -\frac{1}{L_i}\frac{\partial F(L_i)}{\partial \beta}$