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Asymptotic Solutions of Fifth Order More Critically Damped Nonlinear Systems in the Case of Four Repeated Roots

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Abstract In this article, we have modified the Krylov-Bogoliubov-Mitropolskii (KBM) method, which is one of the most widely used methods to delve into the transient behavior of oscillating systems, to find out the solutions of fifth order more critically damped nonlinear systems. In this paper, we have considered the asymptotic solutions of fifth order more critically damped nonlinear systems when the four eigenvalues are equal and another one is distinct. This article suggests that the perturbation solutions obtained by the modified KBM method for both the cases (when repeated eigenvalues are greater than the distinct eigenvalue, and when the distinct eigenvalue is greater than repeated eigenvalues) satisfactorily correspond to the numerical solutions obtained by *Mathematica 9.0*.

Keywords: *KBM, asymptotic solution, more critically damped system, nonlinearity, eigenvalues*

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1. Introduction

In order to obtain solutions of nonlinear systems, the asymptotic method of Krylov–Bogoliubov–Mitropolskii (KBM) [1,2] is regarded as convenient and one of the widely-used tools. For the systems with periodic solutions with small nonlinearities, the method was first extended by Krylov and Bogoliubov [2]. Later, it was amplified and justified by Bogoliubov and Mitropolskii [1]. For nonlinear systems affected by strong linear damping forces, Popov [3] extended the method. However, due to its physical significance, Popov's method was rediscovered by Mendelson [4]. Then, this method was extended by Murty and Deekshatulu [5] for over–damped nonlinear systems. Sattar [6] studied the second order critically-damped nonlinear systems by using of the KBM method. Murty [7] proposed a unified KBM method for second order nonlinear systems which covers the undamped, over-damped and damped oscillatory cases. After that, Osiniskii [8] first developed the KBM method to solve third-order nonlinear differential systems imposing some restrictions, which makes the solution over-simplified. Mulholland [9] removed these restrictions and found desired solutions of third order nonlinear systems. Sattar [10] examined solutions of three-dimensional over-damped nonlinear systems. Shamsul [11] propounded an asymptotic method for second order over-damped and critically damped nonlinear systems. Later, Shamsul [12] extended the method presented in [11] to third-order over-damped nonlinear systems under some special conditions. Akbar *et al.* [13] generalized the method and showed that their method is easier than the method of Murty et al. [14]. Recently Rahaman and Rahman [15] have suggested analytical approximate solutions of fifth order more critically damped systems in the case of smaller triply repeated roots. Moreover, Rahaman and Kawser [16] have also proposed asymptotic solutions of fifth order critically damped nonlinear systems with pair wise equal eigenvalues and another is distinct. Further, Rahaman *et al.* [17] suggested an asymptotic method of Krylov-Bogoliubov-Mitropolskii for fifth order critically damped nonlinear systems. Again, Rahaman and Kawser [18] expounded analytical approximate solutions of fifth order more critically damped nonlinear systems

In this paper, we seeks to investigate an asymptotic solution of fifth order more critically damped nonlinear system, based upon the KBM method. In this study, we suggest that the perturbation results obtained by the presented technique reveal good coincidence with numerical results obtained by *Mathematica 9.0*.

2. The Method

Consider a fifth order non-linear differential system of the form

$$x^{(v)} + k_1 x^{(iv)} + k_2 \dddot{x} + k_3 \ddot{x} + k_4 \dot{x} + k_5 x = -\varepsilon f(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(iv)}) \tag{1}$$

where $x^{(v)}$ and $x^{(iv)}$ stand for the fifth and fourth derivatives respectively, and over dots are used for the first, second and third derivatives of x with respect to t; k_1, k_2, k_3, k_4, k_5 are constants, ε is a sufficiently small parameter and $f(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(iv)})$ is the given nonlinear function. As the unperturbed equation (1) is of fifth order, so it has five real negative eigenvalues, where four eigenvalues are equal and the other one is distinct. Suppose the eigenvalues are $-\lambda, -\lambda, -\lambda, -\lambda$ and μ.

When $\varepsilon = 0$, the equation (1) becomes linear and the solution of the corresponding linear equation is

$$x(t,0) = (a_0 + b_0 t + c_0 t^2 + d_0 t^3)e^{-\lambda t} + h_0 e^{-\mu t} \tag{2}$$

where a_0, b_0, c_0, d_0 and h_0 are constants of integration.

When $\varepsilon \neq 0$, Murty [7] and Shamsul [19], we look for a solution of equation (1) in an asymptotic expansion of the form

$$\begin{aligned} x(t,\varepsilon) &= (a + bt + ct^2 + dt^3)e^{-\lambda t} + he^{-\mu t} \\ &+ \varepsilon u_1(a,b,c,d,h,t) + \ldots \end{aligned} \tag{3}$$

where a, b, c, d and h are the functions of t and they satisfy the first order differential equations

$$\begin{aligned} \dot{a} &= \varepsilon A_1(a,b,c,d,h,t) + \ldots \\ \dot{b} &= \varepsilon B_1(a,b,c,d,h,t) + \ldots \\ \dot{c} &= \varepsilon C_1(a,b,c,d,h,t) + \ldots \\ \dot{d} &= \varepsilon D_1(a,b,c,d,h,t) + \ldots \\ \dot{h} &= \varepsilon H_1(a,b,c,d,h,t) + \ldots \end{aligned} \tag{4}$$

Now differentiating (3) five times with respect to t, substituting the value of x and the derivatives $x^{(v)}, x^{(iv)}, \dddot{x}, \ddot{x}, \dot{x}$ in the original equation (1) utilizing the relations presented in (4) and, finally, extracting the coefficients of ε, we obtain

$$\begin{aligned} &e^{-\lambda t}(D + \mu - \lambda)\Big\{\Big(t^3D^3 + 12t^2D^2 \\ &+36tD + 24\Big)D_1 + \Big(t^2D^3 + 8tD^2 \\ &+12\,D\Big)C_1 + \Big(tD^3 + 4D^2\Big)B_1 + D^3A_1\Big\} \\ &+e^{-\mu t}(D + \lambda - \mu)^4 H_1 + (D+\lambda)^4 \\ &(D+\mu)u_1 = -f^0(a,b,c,d,h,t) \end{aligned} \tag{5}$$

where $f^{(0)}(a,b,c,d,h,t) = f(x, \dot{x}, \ddot{x}, \dddot{x}, x^{iv})$ and $x(t,0) = (a_0 + b_0 t + c_0 t^2 + d_0 t^3)e^{-\lambda t} + h_0 e^{-\mu t}$.

We have expanded the function $f^{(0)}$ in the Taylor's series (Sattar [20], Shamsul [19]) about the origin in power of t. Therefore, we obtain

$$f^{(0)} = \sum_{q=0}^{\infty}\left\{t^q \sum_{i,j=0}^{\infty} F_{q,j}(a,b,c,d,h)e^{-(i\lambda + j\mu)t}\right\} \tag{6}$$

Thus, using (6), the equation (5) becomes

$$\begin{aligned} &e^{-\lambda t}(D + \mu - \lambda)\Big\{\Big(t^3D^3 + 12t^2D^2 \\ &+36tD + 24\Big)D_1 + \Big(t^2D^3 + 8tD^2 \\ &+12\,D\Big)C_1 + \Big(tD^3 + 4D^2\Big)B_1 + D^3A_1\Big\} \\ &+e^{-\mu t}(D + \lambda - \mu)^4 H_1 + (D+\lambda)^4(D+\mu)u_1 \\ &= -\sum_{q=0}^{\infty}\left\{t^q \sum_{i,j=0}^{\infty} F_{q,j}(a,b,c,d,h)e^{-(i\lambda + j\mu)t}\right\}. \end{aligned} \tag{7}$$

Following the KBM method, Murty and Deekshatulu [21], Sattar [20], Shamsul [22], Shamsul and Sattar [23] imposed the condition that u_1 does not contain the fundamental terms of $f^{(0)}$. Therefore, equation (7) can be separated for unknown functions A_1, B_1, C_1, D_1, H_1 and u_1 in the following way:

$$\begin{aligned} &e^{-\lambda t}(D + \mu - \lambda)\Big\{\Big(t^3D^3 + 12t^2D^2 \\ &+36tD + 24\Big)D_1 + \Big(t^2D^3 + 8tD^2 \\ &+12\,D\Big)C_1 + \Big(tD^3 + 4D^2\Big)B_1 + D^3A_1\Big\} \\ &+e^{-\mu t}(D + \lambda - \mu)^4 H_1 = \\ &-\sum_{q=0}^{3}\left\{t^q \sum_{i,j=0}^{\infty} F_{q,j}(a,b,c,d,h)e^{-(i\lambda + j\mu)t}\right\} \end{aligned} \tag{8}$$

$$\begin{aligned} &(D+\lambda)^4(D+\mu)u_1 = \\ &-\sum_{q=4}^{\infty}\left\{t^q \sum_{i,j=0}^{\infty} F_{q,j}(a,b,c,d,h)e^{-(i\lambda + j\mu)t}\right\} \end{aligned} \tag{9}$$

Now equating the coefficients of the various power of t from equation (8), we obtain

$$\begin{aligned} &e^{-\lambda t}(D + \mu - \lambda)\{24D_1 + 12DC_1 \\ &+4D^2B_1 + D^3A_1\} + e^{-\mu t}(D + \lambda - \mu)^4 H_1 \\ &= -\sum_{i,j=0}^{\infty} F_{0,j}(a,b,c,d,h)e^{-(i\lambda + j\mu)t} \end{aligned} \tag{10}$$

$$\begin{aligned} &e^{-\lambda t}(D + \mu - \lambda)\{36D\,D_1 + 8D^2C_1 + D^3\,B_1\} \\ &= -\sum_{i,j=0}^{\infty} F_{1,j}(a,b,c,d,h)e^{-(i\lambda + j\mu)t} \end{aligned} \tag{11}$$

$$\begin{aligned} &e^{-\lambda t}(D + \mu - \lambda)\{12\,D^2D_1 + D^3\,C_1\} \\ &= -\sum_{i,j=0}^{\infty} F_{2,j}(a,b,c,d,h)e^{-(i\lambda + j\mu)t} \end{aligned} \tag{12}$$

$$\begin{aligned} &e^{-\lambda t}(D + \mu - \lambda)D^3D_1 \\ &= -\sum_{i,j=0}^{\infty} F_{3,j}(a,b,c,d,h)e^{-(i\lambda + j\mu)t} \end{aligned} \tag{13}$$

Here, we have four equations (10), (11), (12) and (13) for determining the unknown functions A_1, B_1, C_1, D_1 and H_1. Thus, to obtain the unknown functions A_1, B_1, C_1, D_1 and H_1, we need to impose some conditions (Shamsul [26, 22, 24, 25]) between the eigenvalues. Different authors have imposed different conditions according to the behavior of the systems, such as Shamsul [25] imposed the condition

$$i_1\lambda_1 + i_2\lambda_2 + ... + i_n\lambda_n \le (i_1 + i_1 + ... + i_1)(\lambda_1 + \lambda_2 + ... + \lambda_n)/n$$

In this study, we have investigated solutions for both the cases $\lambda >> \mu$ and $\lambda << \mu$. Therefore, we obtain the value of D_1 from equation (13), and substituting the value of D_1 in equation (12), we get the value of C_1, and using these values of C_1 and D_1 in equation (11), we find the value of B_1. Now we will be able to separate the equation (10) for unknown functions A_1 and H_1 for both the conditions $\lambda >> \mu$ and $\lambda << \mu$; and solving them for A_1 and H_1. Since $\dot{a}, \dot{b}, \dot{c}, \dot{d}$ and $\dot{h}$ are proportional to the small parameter, so they are slowly varying functions of time t, and for first approximate solution, we may consider them as constants which are presented in the right side. This assumption was first made by Murty and Deekshatulu [21]. Thus, the solutions of the equation (4) become

$$\begin{aligned} a &= a_0 + \varepsilon\int_0^t A_1(a,b,c,d,h,t)dt \\ b &= b_0 + \varepsilon\int_0^t B_1(a,b,c,d,h,t)dt \\ c &= c_0 + \varepsilon\int_0^t C_1(a,b,c,d,h,t)dt \\ d &= d_0 + \varepsilon\int_0^t D_1(a,b,c,d,h,t)dt \\ h &= h_0 + \varepsilon\int_0^t H_1(a,b,c,d,h,t)dt \end{aligned} \tag{14}$$

Equation (9) is a non-homogeneous linear ordinary differential equation; therefore, it can be solved by the well-known operator method. Substituting the values of a, b, c, d, h and u_1 in the equations (3), we get the complete solution of (1). Therefore, the determination of the first approximate solution is complete.

3. Example

As an example of the above technique, we have considered the Duffing type equation of fifth order nonlinear differential system:

$$x^{(v)} + k_1x^{(iv)} + k_2\dddot{x} + k_3\ddot{x} + k_4\dot{x} + k_5x = -\varepsilon x^3 \tag{15}$$

Comparing equation (13) and equation (1), we obtain

$$f(x,\dot{x},\ddot{x},\dddot{x},x^{(iv)}) = x^3$$

Therefore,

$$\begin{aligned} f^{(0)} = &\{a^3 + 3a^2bt + 3ab^2t^2 + 3a^2ct^2 + b^3t^3 + \\ &6abct^3 + 3a^2dt^3 + 3b^2ct^4 + 3ac^2t^4 + \\ &6abdt^4 + 3bc^2t^5 + 3bc^2t^5 + 3b^2dt^5 + \\ &6acdt^5 + c^3t^6 + 6bcdt^6 + 3ad^2t^6 + \\ &3c^2dt^7 + 3bd^2t^7 + 3cd^2t^8 + d^3t^9\}e^{-3\lambda t} \\ &+ \left(3ah^2 + 3bh^2t + 3ch^2t^2 + 3dh^2t^3\right)e^{-(\lambda+2\mu)t} \\ &+ \left(3a^2h + 6abht + 3b^2ht^2 + 6acht^2 + \right. \\ &+ 6bcht^3 + 6adht^3 + 3c^2ht^4 + 6bdht^4 + \\ &\left. + 6cdht^5 + 3d^2ht^6\right)e^{-(2\lambda+\mu)t} + h^3e^{-3\mu t} \end{aligned} \tag{16}$$

For equation (15), the equation (9) to equation (13) respectively become

$$\begin{aligned} (D+\lambda)^4(D+\mu)u_1 = &-\{3b^2ct^4 + 3ac^2t^4 + \\ &6abdt^4 + 3bc^2t^5 + 3b^2dt^5 + 6acdt^5 + c^3t^6 \\ &+ 6bcdt^6 + 3ad^2t^6 + 3c^2dt^7 + 3bd^2t^7 + \\ &3cd^2t^8 + d^3t^9\}e^{-3\lambda t} - \{3c^2ht^4 + 6bdht^4 \\ &+ 6cdht^5 + 3d^2ht^6\}e^{-(2\lambda+\mu)t} \end{aligned} \tag{17}$$

$$\begin{aligned} &e^{-\lambda t}(D+\mu-\lambda)\{24D_1 + 12DC_1\, 4D^2B_1 \\ &+ D^3A_1\} + e^{-\mu t}(D+\lambda-\mu)^4H_1 = \\ &-\{a^3e^{-3\lambda t} + 3a^2he^{-(2\lambda+\mu)t} + 3ah^2 \\ &e^{-(\lambda+2\mu)t} + h^3e^{-3\mu t}\} \end{aligned} \tag{18}$$

$$\begin{aligned} &e^{-\lambda t}(D+\mu-\lambda)\{36DD_1 + 8D^2C_1 + D^3B_1\} \\ &= -\{3a^2be^{-3\lambda t} + 6abhe^{-(2\lambda+\mu)t} + 3bh^2e^{-(\lambda+2\mu)t}\} \end{aligned} \tag{19}$$

$$\begin{aligned} &e^{-\lambda t}(D+\mu-\lambda)\{12D^2D_1 + D^3C_1\} = -3\{(a^2b + a^2c) \\ &e^{-3\lambda t} + 3(b^2h + 2ach)e^{-(2\lambda+\mu)t} + 3ch^2e^{-(\lambda+2\mu)t}\} \end{aligned} \tag{20}$$

$$\begin{aligned} &e^{-\lambda t}(D+\mu-\lambda)D^3D_1 = -3\{(b^3 + 6abc + 3a^2d)e^{-3\lambda t} \\ &+ 6(bch + adh)e^{-(2\lambda+\mu)t} + 3dh^2e^{-(\lambda+2\mu)t}\} \end{aligned} \tag{21}$$

The solution of the equation (21), therefore, is

$$\begin{aligned} D_1 = -\Bigg\{ &\frac{(b^3 + 6abc + 3a^2d)e^{-2\lambda t}}{8\lambda^3(3\lambda-\mu)} + \\ &\frac{24(bch + adh)e^{-(\lambda+\mu)t}}{8\lambda(\lambda+\mu)^3} + \frac{3dh^2e^{-2\mu t}}{8\mu^3(\lambda+\mu)}\Bigg\} \end{aligned} \tag{22}$$

Putting the value of D_1 from equation (22) into equation (20), we obtain

$$e^{-\lambda t}(D+\mu-\lambda)D^3C_1=-3\left\{\left(a^2b+a^2c\right)+\frac{6\left(b^3+6abc+3a^2d\right)}{\lambda}-\frac{3\left(b^3+6abc+3a^2d\right)}{-3\lambda+\mu}e^{-3\lambda t}\right\}-\left\{\frac{9(bch+adh)}{\lambda}+3\left(b^2h+2ach\right)+\frac{72(bc+ad)h}{\lambda+\mu}\right\}e^{-(2\lambda+\mu)t}-\left\{3ch^2+\frac{9(2\lambda+\mu)}{\mu(\lambda+\mu)}e^{-(\lambda+2\mu)t}\right\} \qquad (23)$$

Therefore, the solution of the equation (23) is

$$C_1=-\left\{\frac{(9\lambda+\mu)(bch+adh)}{2\lambda^2(\lambda+\mu)^4}+\frac{3\left(b^2h+2ach\right)}{2\lambda(\lambda+\mu)^3}\right\}e^{-(\lambda+\mu)t}-\left\{\frac{3\left(ab^2+a^2c\right)}{8\lambda^3(3\lambda-\mu)}+\frac{3(7\lambda-2\mu)\left(b^3+6abc+3a^2d\right)}{8\lambda^4(3\lambda-\mu)^2}\right\}e^{-2\lambda t}-\left\{\frac{3ch^2}{8\mu^3(\lambda+\mu)}+\frac{9(2\lambda+3\mu)dh^2}{8\mu^4(\lambda+\mu)^2}\right\}e^{-2\mu t} \qquad (24)$$

Putting the value of C_1 and D_1 from equation (24) and (22) into equation (19), we obtain

$$e^{-\lambda t}(D+\mu-\lambda)D^3B_1=-\left\{\frac{6(7\lambda-2\mu)\left(ab^2+a^2c\right)}{\lambda(3\lambda-\mu)}+3a^2b+3\frac{\left(59\lambda^2-34\lambda\mu+5\mu^2\right)\left(b^3+6abc+3a^2d\right)}{\lambda^2(3\lambda-\mu)^2}\right\}e^{-3\lambda t}-\left\{\frac{\left(b^2h+2ach\right)}{\lambda(\lambda+\mu)}3(9\lambda+\mu)+\frac{9\left(49\lambda^2+10\lambda\mu+\mu^2\right)bch}{\lambda^2(\lambda+\mu)^2}+6abh+9\mu adh\frac{\left(-204\lambda^3-2\lambda^2\mu+4\mu^2\lambda+\mu^3\right)}{\lambda^2(\lambda+\mu)^2(\mu-3\lambda)^2}\right\}e^{-(2\lambda+\mu)t}-\left\{\frac{6(2\lambda+3\mu)ch^2}{\mu(\lambda+\mu)}+3bh^2+\frac{9\left(5\lambda^2+14\lambda\mu+11\mu^2\right)dh^2}{\mu^2(\lambda+\mu)^2}\right\}e^{-(\lambda+2\mu)t} \qquad (25)$$

Thus, the solution of the equation (25) is

$$B_1=-\left\{\frac{3(7\lambda-2\mu)\left(ab^2+a^2c\right)}{4\lambda^4(3\lambda-\mu)^2}+\frac{3a^2b}{8\lambda^3(3\lambda-\mu)}+3(59\lambda^2-34\lambda\mu+5\mu^2)\frac{\left(b^3+6abc+3a^2d\right)}{8\lambda^5(3\lambda-\mu)^3}\right\}e^{-2\lambda t}-\left\{\frac{3abh}{\lambda(\lambda+\mu)^3}+\frac{3(9\lambda+\mu)}{2\lambda^2(\lambda+\mu)^4}\left(b^2h+2ach\right)+\frac{3\left(49\lambda^2+10\lambda\mu+\mu^2\right)(bch+adh)}{2\lambda^3(\lambda+\mu)^5}\right\}e^{-(\lambda+\mu)t}-\left\{\frac{9dh^2\left(5\lambda^2+14\lambda\mu+11\mu^2\right)}{8\mu^5(\lambda+\mu)^3}+\frac{3bh^2}{8\mu^3(\lambda+\mu)}+\frac{3(2\lambda+3\mu)ch^2}{4\mu^4(\lambda+\mu)^2}\right\}e^{-2\mu t} \qquad (26)$$

Now applying the conditions $\lambda >> \mu$ in equation (18), we obtain the following equations for unknown functions A_1 and H_1:

$$(D+\mu-\lambda)A_1=-\left\{\frac{15(b^3+6abc+3a^2d)}{\lambda^3}+\frac{15\left(ab^2+a^2c\right)}{\lambda^2}+\frac{6(b^3+6abc+3a^2d)}{(3\lambda-\mu)^3}+\frac{12(b^3+6abc+3a^2d)}{\lambda(3\lambda-\mu)^2}+a^3+\frac{15(b^3+6abc+3a^2d)}{\lambda^2(3\lambda-\mu)}+\frac{6a^2b}{\lambda}+\frac{12\left(ab^2+a^2c\right)}{\lambda(3\lambda-\mu)}+\frac{3a^2b}{3\lambda-\mu}+\frac{6ab^2+a^2c}{(3\lambda-\mu)^2}\right\}e^{-2\lambda t}-\left\{\frac{720(bch+adh)}{(\lambda+\mu)^3}+\frac{3abh}{\lambda}+3\lambda^3a^2h+\frac{3(b^2h+2ach)}{2\lambda^2}+\frac{180(bch+adh)}{\lambda(\lambda+\mu)^2}+\frac{9(bch+adh)}{2\lambda^3}+\frac{36(bch+adh)}{\lambda^2(\lambda+\mu)}+\frac{60(b^2h+2ach)}{(\lambda+\mu)^2}+\frac{12(b^2h+2ach)}{\lambda(\lambda+\mu)}+\frac{24abh}{(\lambda+\mu)}\right\}e^{-(\lambda+\mu)t}-\left\{\frac{3(\lambda-\mu)^3(2\lambda+3\mu)bh^2}{\mu(\lambda+\mu)}+3ah^2(\lambda-\mu)^3-\frac{3(\lambda-\mu)^3\left(5\lambda^2+14\lambda\mu+11\mu^2\right)ch^2}{\mu^2(\lambda+\mu)^2}+\left(\begin{array}{l}5\lambda^3+20\lambda^2\mu\\+29\lambda\mu^2+16\mu^3\end{array}\right)\frac{9(\lambda-\mu)^3dh^2}{\mu^3(\lambda+\mu)^3}\right\}e^{-2\mu t} \qquad (27)$$

And

$$(D+\lambda-\mu)H_1=-h^3e^{-2\mu t}\frac{\begin{pmatrix}27\lambda^6+54\lambda^5\mu-28\lambda^3\mu^3\\-3\lambda^2\mu^4+6\lambda\mu^5-\mu^6\end{pmatrix}}{(3\lambda-\mu)^3(\lambda+\mu)^3} \quad (28)$$

Solution of the equations (27) and (28) are

$$A_1=-\left\{\frac{a^3}{8\lambda^3(3\lambda-\mu)}+\frac{3(59\lambda^2-34\lambda\mu+5\mu^2)}{8\lambda^3(3\lambda-\mu)^3}\right.$$
$$(ab^2+a^2c)+\frac{3\left(194\lambda^3-169\lambda^2\mu+50\lambda\mu^2-5\mu^3\right)}{8\lambda^6(3\lambda-\mu)^4}$$
$$\left.(b^3+6abc+3a^2d)+\frac{3(7\lambda-2\mu)a^2b}{8\lambda^4(3\lambda-\mu)^2}\right\}e^{-2\lambda t}-$$
$$\left\{\frac{3\left(49\lambda^2+10\lambda\mu+\mu^2\right)(b^2h+2ach)}{4\lambda^3(\lambda+\mu)^5}\right.$$
$$+\frac{3a^2h}{2\lambda(\lambda+\mu)^3}+\frac{3(9\lambda+\mu)abh}{2\lambda^2(\lambda+\mu)^4}$$
$$\left.+9\begin{pmatrix}209\lambda^3+59\lambda^2\mu\\+11\lambda\mu^2+\mu^3\end{pmatrix}\frac{(bch+adh)}{4\lambda^4(\lambda+\mu)^6}\right\}e^{-(\lambda+\mu)t}$$
$$-\left\{\frac{3\left(5\lambda^2+14\lambda\mu+11\mu^2\right)ch^2}{8\mu^5(\lambda+\mu)^3}\right.$$
$$+\frac{9\left(5\lambda^3+20\lambda^2\mu+29\lambda\mu^2+16\mu^3\right)dh^2}{8\mu^6(\lambda+\mu)^4} \quad (29)$$
$$\left.+\frac{3ah^2}{8\mu^3(\lambda+\mu)}+\frac{3(2\lambda+3\mu)bh^2}{8\mu^4(\lambda+\mu)^2}\right\}e^{-2\mu t}$$

$$H_1=-h^3e^{-2\mu t}$$
$$\frac{\left(27\lambda^6+54\lambda^5\mu-28\lambda^3\mu^3-3\lambda^2\mu^4+6\lambda\mu^5-\mu^6\right)}{(3\lambda-\mu)^3(\lambda+\mu)^3} \quad (30)$$

Now applying the condition $\lambda << \mu$ in equation (18), we obtain the following equations for unknown functions A_1 and H_1:

$$(D+\mu-\lambda)A_1=-\left\{\frac{3\left(59\lambda^2-34\lambda\mu+5\mu^2\right)}{\lambda^2(\mu-3\lambda)^2}\right.$$
$$(ab^2+a^2c)+\frac{3\left(194\lambda^3-169\lambda^2\mu+50\lambda\mu^2-5\mu^3\right)}{\lambda^3(3\lambda-\mu)^3} \quad (31)$$
$$\left.(b^3+6abc+3a^2d)+a^3+\frac{3(7\lambda-2\mu)a^2b}{\lambda(3\lambda-\mu)}\right\}e^{-2\lambda t}$$

And

$$(D+\lambda-\mu)H_1=-\left\{\frac{3(49\lambda^2+10\lambda\mu+\mu^2)(b^2h+ach)}{2\lambda^2(\lambda+\mu)^2}\right.$$
$$+\frac{9(bch+adh)\left(209\lambda^3+59\lambda^2\mu+11\lambda\mu^2+\mu^3\right)}{2\lambda^3(\lambda+\mu)^3}$$
$$\left.+3a^2h+\frac{3(9\lambda+\mu)abh}{\lambda(\lambda+\mu)}\right\}e^{-2\lambda t}-\left\{\frac{3(2\lambda+3\mu)b^2h}{\mu(\lambda+\mu)}\right.$$
$$+\frac{9\left(5\lambda^3+20\lambda^2\mu+29\lambda\mu^2+16\mu^3\right)d^2h}{\mu^3(\lambda+\mu)^3}+3a^2h \quad (32)$$
$$\left.+\frac{3\left(5\lambda^2+14\lambda\mu+11\mu^2\right)ch^2}{\mu^2(\lambda+\mu)^2}\right\}e^{-(\lambda+\mu)t}$$
$$-h^3e^{-2\mu t}\frac{\begin{pmatrix}27\lambda^6+54\lambda^5\mu-28\lambda^3\mu^3\\-3\lambda^2\mu^4+6\lambda\mu^5-\mu^6\end{pmatrix}}{(3\lambda-\mu)^3(\lambda+\mu)^3}$$

Thus, the solution of the equations (31) and (32) are

$$A_1=-\left\{\frac{a^3}{8\lambda^3(3\lambda-\mu)}+\frac{3(59\lambda^2-34\lambda\mu+5\mu^2)}{8\lambda^3(3\lambda-\mu)^3}\right.$$
$$(ab^2+a^2c)+\frac{3\left(194\lambda^3-169\lambda^2\mu+50\lambda\mu^2-5\mu^3\right)}{8\lambda^6(3\lambda-\mu)^4} \quad (33)$$
$$\left.(b^3+6abc+3a^2d)+\frac{3(7\lambda-2\mu)a^2b}{8\lambda^4(3\lambda-\mu)^2}\right\}e^{-2\lambda t}$$

$$H_1=-\frac{3}{2}\left\{\frac{(49\lambda^2+10\lambda\mu+\mu^2)(b^2h+2ach)}{\lambda^2(\lambda+\mu)^6}\right.$$
$$+\frac{3(bch+adh)\left(209\lambda^3+59\lambda^2\mu+11\lambda\mu^2+\mu^3\right)}{\lambda^3(\lambda+\mu)^7}$$
$$\left.+\frac{2a^2h}{(\lambda+\mu)^4}+\frac{2(9\lambda+\mu)abh}{\lambda(\lambda+\mu)^5}\right\}e^{-2\lambda t}$$
$$-\frac{1}{16}\left\{\frac{(2\lambda+3\mu)}{\mu^5(\lambda+\mu)}bh^2+\frac{3\begin{pmatrix}5\lambda^3+20\lambda^2\mu\\+29\lambda\mu^2+16\mu^3\end{pmatrix}dh^2}{\mu^7(\lambda+\mu)^3}\right.$$
$$\left.+\frac{ah^2}{\mu^4}+\frac{\left(5\lambda^2+14\lambda\mu+11\mu^2\right)ch^2}{\mu^6(\lambda+\mu)^2}\right\}e^{-(\lambda+\mu)t}$$
$$-h^3e^{-2\mu t}\frac{\begin{pmatrix}27\lambda^6+545\lambda^5\mu-28\lambda^3\mu^3\\-3\lambda^2\mu^4+6\lambda\mu^5-\mu^6\end{pmatrix}}{(3\lambda-\mu)^3(\lambda-3\mu)^4(\lambda+\mu)^3} \quad (34)$$

Finally, the solution of the equation (17) for u_1 is

$$
\begin{aligned}
u_1 = \{ & 3(b^2c + ac^2 + 2abd)(r_1 + r_2t + r_3t^2 + r_4t^3 + r_5t^4) \\
& + 3(bc^2 + b^2d + 2acd)\begin{pmatrix} r_6 + r_7t + r_8t^2 \\ +r_9t^3 + r_{10}t^4 + r_{11}t^5 \end{pmatrix} \\
& + (c^3 + 6bcd + 3ad^2)\begin{pmatrix} r_{12} + r_{13}t + r_{14}t^2 + r_{15}t^3 \\ +r_{16}t^4 + r_{17}t^5 + r_{18}t^6 \end{pmatrix} \\
& + 3(c^2d + bd^2)\begin{pmatrix} r_{19} + r_{20}t + r_{21}t^2 + r_{22}t^3 \\ +r_{23}t^4 + r_{24}t^5 + r_{25}t^6 + r_{26}t^7 \end{pmatrix} \\
& + 3cd^2\begin{pmatrix} r_{27} + r_{28}t + r_{29}t^2 + r_{30}t^3 + r_{31}t^4 \\ +r_{32}t^5 + r_{33}t^6 + r_{34}t^7 + r_{35}t^8 \end{pmatrix} \\
& + d^3\begin{pmatrix} r_{36} + r_{37}t + r_{38}t^2 + r_{39}t^3 + r_{40}t^4 \\ +r_{41}t^5 + r_{42}t^6 + r_{43}t^7 + r_{44}t^8 + r_{45}t^9 \end{pmatrix}\Bigg\} e^{-3\lambda t} \\
& + \{3(c^2h + 2bdh)\left(r_{46} + r_{47}t + r_{48}t^2 + r_{49}t^3 + r_{50}t^4\right) \\
& + 6cdh\left(r_{51} + r_{52}t + r_{53}t^2 + r_{54}t^3 + r_{55}t^4 + r_{56}t^5\right) \\
& + 3d^2h\begin{pmatrix} r_{57} + r_{58}t + r_{59}t^2 + r_{60}t^3 \\ +r_{61}t^4 + r_{62}t^5 + r_{63}t^6 \end{pmatrix} e^{-(2\lambda+\mu)t}
\end{aligned}
\tag{35}
$$

where

$$r_1 = \frac{3}{32\lambda^8(3\lambda-\mu)^5}\begin{pmatrix} 4387\lambda^4 - 5132\lambda^3\mu + 2290\lambda^2\mu^2 \\ -460\lambda\mu^3 + 35\mu^4 \end{pmatrix},$$

$$r_2 = \frac{3}{4\lambda^7(\mu-3\lambda)^4}\begin{pmatrix} 194\lambda^3 - 169\lambda^2\mu \\ +50\lambda\mu^2 - 5\mu^3 \end{pmatrix},$$

$$r_3 = \frac{3}{8\lambda^6(3\lambda-\mu)^3}(59\lambda^2 - 34\lambda\mu + 5\mu^2),$$

$$r_4 = \frac{1}{4\lambda^5(\mu-3\lambda)^2}(7\lambda + 2\mu),$$

$$r_5 = \frac{1}{48\lambda^5 - 16\lambda^4\mu},$$

$$r_6 = \frac{15}{32\lambda^9(\mu-3\lambda)^6}\begin{pmatrix} 11191\lambda^5 - 16472\lambda^4\mu - 9850\lambda^3\mu^2 \\ -2980\lambda^2\mu^3 + 455\lambda\mu^4 - 28\mu^5 \end{pmatrix},$$

$$r_7 = 5r_1, r_8 = \frac{5}{2}r_2, r_9 = 5r_3, r_{10} = \frac{5}{4}r_4, r_{11} = r_5,$$

$$r_{12} = \frac{45}{16\lambda^{10}(3\lambda-\mu)^7}\begin{pmatrix} 26500\lambda^6 - 47090\lambda^5\mu + \\ 35365\lambda^4\mu^2 - 14320\lambda^3\mu^3 + \\ 3290\lambda^2\mu^4 - 406\lambda\mu^5 + 21\mu^6 \end{pmatrix},$$

$$r_{13} = \frac{3}{2}r_6, r_{14} = 3r_7, r_{15} = \frac{1}{2}r_8, r_{16} = \frac{3}{2}r_9,$$

$$r_{17} = \frac{6}{5}r_{10}, r_{18} = r_{11},$$

$$r_{19} = \frac{315}{16\lambda^{11}(\mu-3\lambda)^8}\begin{pmatrix} 59305\lambda^7 - 123635\lambda^6\mu + \\ 111910\lambda^5\mu^2 - 56845\lambda^4\mu^3 \\ +17465\lambda^3\mu^4 - 3241\lambda^2\mu^5 \\ +336\lambda\mu^6 - 15\mu^7 \end{pmatrix},$$

$$r_{20} = 7r_{12}, r_{21} = \frac{7}{2}r_{13}, r_{22} = \frac{7}{3}r_{14}, r_{23} = \frac{7}{4}r_{15},$$

$$r_{24} = \frac{7}{5}r_{16}, r_{25} = \frac{7}{6}r_{17}, r_{26} = r_{18},$$

$$r_{27} = \frac{315}{32\lambda^{12}(3\lambda-\mu)^9}\begin{pmatrix} (2031445\lambda^8 - 4865000\lambda^7\mu \\ +5158540\lambda^6\mu^2 - 3154840\lambda^5\mu^3 \\ +1214990\lambda^4\mu^4 - 301336\lambda^3\mu^5 \\ +46956\lambda^2\mu^6 - 4200\lambda\mu^7 + 165\mu^8 \end{pmatrix},$$

$$r_{28} = \frac{1}{8}r_{19}, r_{29} = \frac{1}{4}r_{20}, r_{30} = \frac{4}{3}r_{21}, r_{31} = 2r_{22},$$

$$r_{32} = \frac{8}{5}r_{23}, r_{33} = \frac{4}{3}r_{25}, r_{34} = \frac{8}{7}r_{25}, r_{35} = r_{26},$$

$$r_{36} = \frac{2835}{32\lambda^{13}(\mu-3\lambda)^{10}}\begin{pmatrix} 4196575\lambda^9 - 11360390\lambda^8\mu \\ +13819060\lambda^7\mu^2 - 9890800\lambda^6\mu^3 \\ +4582970\lambda^5\mu^4 - 1423996\lambda^4\mu^5 \\ +296436\lambda^3\mu^6 - 39840\lambda^2\mu^7 \\ +3135\lambda\mu^8 - 110\mu^9 \end{pmatrix},$$

$$r_{37} = 9r_{27}, r_{38} = \frac{9}{2}r_{28}, r_{39} = 3r_{29}, r_{40} = \frac{3}{4}r_{30}, r_{41} = \frac{189}{105}r_{31},$$

$$r_{42} = \frac{3}{2}r_{32}, r_{43} = \frac{9}{7}r_{33}, r_{44} = \frac{9}{8}r_{34}, r_{45} = r_{35},$$

$$r_{46} = \frac{3}{4\lambda^5(\lambda+\mu)^8}\begin{pmatrix} 769\lambda^4 + 268\lambda^3\mu + 70\lambda^2\mu^2 \\ +12\lambda\mu^3 + \mu^4 \end{pmatrix},$$

$$r_{47} = \frac{3}{2\lambda^4(\lambda+\mu)^7}(209\lambda^3 + 11\lambda\mu^2 + \mu^3),$$

$$r_{48} = \frac{3}{2\lambda^3(\lambda+\mu)^6}(49\lambda^2 + 10\lambda\mu + \mu^2),$$

$$r_{49} = \frac{1}{\lambda^2(\lambda+\mu)^5}(9\lambda + \mu), \quad r_{50} = \frac{1}{2\lambda(\lambda+\mu)^4},$$

$$r_{51} = \frac{15}{8\lambda^6(\mu+\lambda)^9}\begin{pmatrix} 2561\lambda^5 + 1037\lambda^4\mu + 338\lambda^3\mu^2 \\ +82\lambda^2\mu^3 + 13\lambda\mu^4 + \mu^5 \end{pmatrix},$$

$$r_{52} = 5r_{46}, r_{53} = \frac{5}{2}r_{47}, r_{54} = \frac{5}{3}r_{48}, r_{55} = \frac{5}{4}r_{49}, r_{56} = r_{50},$$

$$r_{57} = \frac{45}{8\lambda^7(\lambda+\mu)^{10}}\begin{pmatrix} 7937\lambda^6 + 3598\lambda^5\mu \\ +1375\lambda^4\mu^2 + 420\lambda^3\mu^3 \\ +95\lambda^2\mu^4 + 14\lambda\mu^5 + \mu^6 \end{pmatrix},$$

$$r_{58}=3r_{51}, r_{59}=3r_{52}, r_{60}=\frac{1}{2}r_{53},$$
$$r_{61}=\frac{3}{2}r_{54}, r_{62}=\frac{6}{5}r_{55}, r_{63}=r_{56}$$

Substituting the values of A_1, B_1, C_1, D_1 and H_1 from equations (29), (33), (26), (24), (22), (32) and (34) into equation (4), we obtain, when $\lambda >> \mu$ then $\dot{a}$ becomes

$$\begin{aligned}
\dot{a} = -\varepsilon\Bigg[&\Bigg\{\frac{a^3}{8\lambda^3(3\lambda-\mu)}+\frac{3(59\lambda^2-34\lambda\mu+5\mu^2)}{8\lambda^3(3\lambda-\mu)^3} \\
&(ab^2+a^2c)+\frac{3(194\lambda^3-169\lambda^2\mu+50\lambda\mu^2-5\mu^3)}{8\lambda^6(3\lambda-\mu)^4} \\
&(b^3+6abc+3a^2d)+\frac{3(7\lambda-2\mu)a^2b}{8\lambda^4(3\lambda-\mu)^2}\Bigg\}e^{-2\lambda t}+ \\
&\Bigg\{3(49\lambda^2+10\lambda\mu+\mu^2)\frac{(b^2h+2ach)}{4\lambda^3(\lambda+\mu)^5}+3a^2h \\
&\frac{1}{2\lambda(\lambda+\mu)^3}+3abh\frac{3(9\lambda+\mu)}{2\lambda^2(\lambda+\mu)^4}+9(bch+adh) \\
&\frac{(209\lambda^3+59\lambda^2\mu+11\lambda\mu^2+\mu^3)}{4\lambda^4(\lambda+\mu)^6}\Bigg\}e^{-(\lambda+\mu)t}+ \\
&\Bigg\{\frac{3(5\lambda^2+14\lambda\mu+11\mu^2)ch^2}{8\mu^5(\lambda+\mu)^3}+\frac{3ah^2}{8\mu^3}\frac{1}{(\lambda+\mu)} \\
&+\frac{9(5\lambda^3+20\lambda^2\mu+29\lambda\mu^2+16\mu^3)dh^2}{8\mu^6(\lambda+\mu)^4}+ \\
&+\frac{3(2\lambda+3\mu)bh^2}{8\mu^4(\lambda+\mu)^2}\Bigg\}e^{-2\mu t}\Bigg]
\end{aligned}$$

And when $\lambda << \mu$ then $\dot{a}$ becomes

$$\begin{aligned}
\dot{a} = -\varepsilon\Bigg[&\Bigg\{\frac{a^3}{8\lambda^3(3\lambda-\mu)}+\frac{3(59\lambda^2-34\lambda\mu+5\mu^2)}{8\lambda^3(3\lambda-\mu)^3} \\
&(ab^2+a^2c)+\frac{3\left(194\lambda^3-169\lambda^2\mu+50\lambda\mu^2-5\mu^3\right)}{8\lambda^6(3\lambda-\mu)^4} \\
&(b^3+6abc+3a^2d)+\frac{3(7\lambda-2\mu)a^2b}{8\lambda^4(3\lambda-\mu)^2}\Bigg\}e^{-2\lambda t}\Bigg]
\end{aligned}$$

$$\begin{aligned}
\dot{b} = -\varepsilon\Bigg[&\Bigg\{\frac{3(7\lambda-2\mu)\left(ab^2+a^2c\right)}{4\lambda^4(3\lambda-\mu)^2}+\frac{3a^2b}{8\lambda^3(3\lambda-\mu)} \\
&+\frac{3\left(59\lambda^2-34\lambda\mu+5\mu^2\right)\left(b^2h+2ach\right)}{2\lambda^2(\lambda+\mu)^4}\Bigg\}e^{-2\lambda t} \\
&+\Bigg\{\frac{3abh}{\lambda(\lambda+\mu)^3}+\frac{3(9\lambda+\mu)\left(b^2h+2ach\right)}{2\lambda^2(\lambda+\mu)^4} \\
&+\left(49\lambda^2+10\lambda\mu+\mu^2\right)\frac{3(bch+adh)}{2\lambda^3(\lambda+\mu)^5}\Bigg\}e^{-(\lambda+\mu)t} \\
&+\Bigg\{\frac{3bh^2}{8\mu^3(\lambda+\mu)}+dh^2\frac{9\left(5\lambda^2+14\lambda\mu+11\mu^2\right)}{8\mu^5(\lambda+\mu)^3} \\
&+\frac{3(2\lambda+3\mu)ch^2}{4\mu^4(\lambda+\mu)^2}\Bigg\}e^{-2\mu t}\Bigg]
\end{aligned}$$

$$\begin{aligned}
\dot{c} = -\varepsilon\Bigg[&\Bigg\{\frac{(9\lambda+\mu)(bch+adh)}{2\lambda^2(\lambda+\mu)^4}+\frac{3\left(b^2h+2ach\right)}{2\lambda(\lambda+\mu)^3}\Bigg\}e^{-(\lambda+\mu)t} \\
&+\Bigg\{\frac{3\left(ab^2+a^2c\right)}{8\lambda^3(3\lambda-\mu)}+\frac{3(7\lambda-2\mu)\left(b^3+6abc+3a^2d\right)}{8\lambda^4(3\lambda-\mu)^2}\Bigg\} \\
&e^{-2\lambda t}+\Bigg\{\frac{3ch^2}{8\mu^3(\lambda+\mu)}+\frac{9(2\lambda+3\mu)dh^2}{8\mu^4(\lambda+\mu)^2}\Bigg\}e^{-2\mu t}\Bigg]
\end{aligned}$$

$$\begin{aligned}
\dot{d} = -\varepsilon\Bigg\{&\frac{\left(b^3+6abc+a^2c\right)e^{-2\lambda t}}{8\lambda^3(3\lambda-\mu)}+ \\
&\frac{24(bch+adh)e^{-(\lambda+\mu)t}}{8\lambda^3(\lambda+\mu)^3}+\frac{3dh^2e^{-2\mu t}}{8\mu^3(\lambda+\mu)}\Bigg\}
\end{aligned} \qquad (36)$$

Again, when $\lambda >> \mu$ then $\dot{h}$ becomes

$$\dot{h} = -\varepsilon h^3 e^{-2\mu t}\frac{\left(27\lambda^6+54\lambda^5\mu-28\lambda^3\mu^3-3\lambda^2\mu^4+6\lambda\mu^5-\mu^6\right)}{(3\lambda-\mu)^3(\lambda+\mu)^3}$$

Further, when $\lambda << \mu$ then $\dot{h}$ becomes

$$\begin{aligned}
\dot{h} = -\varepsilon\Bigg[&\frac{3}{2}\Bigg\{\frac{3(49\lambda^2+10\lambda\mu+\mu^2)(b^2h+2ach)}{\lambda^2(\lambda+\mu)^6} \\
&+\frac{2a^2h}{(\lambda+\mu)^4}+\frac{3(bch+adh)\begin{pmatrix}209\lambda^3+59\lambda^2\mu \\ +11\lambda\mu^2+\mu^3\end{pmatrix}}{\lambda^3(\lambda+\mu)^7} \\
&+abh\frac{2(9\lambda+\mu)}{\lambda(\lambda+\mu)^5}\Bigg\}e^{-2\lambda t}+\frac{1}{16}\Bigg\{\frac{(2\lambda+3\mu)bh^2}{\mu^5(\lambda+\mu)} \\
&+\frac{3dh^2}{\mu^7(\lambda+\mu)^3}\left(5\lambda^3+20\lambda^2\mu+29\lambda\mu^2+16\mu^3\right) \\
&+\left(5\lambda^2+14\lambda\mu+11\mu^2\right)\frac{ch^2}{\mu^6(\lambda+\mu)^2}+\frac{a^2h}{\mu^4}\Bigg\}e^{-(\lambda+\mu)t} \\
&+\begin{pmatrix}27\lambda^6+545\lambda^5\mu \\ -28\lambda^3\mu^3-3\lambda^2\mu^4 \\ +6\lambda\mu^5-\mu^6\end{pmatrix}\frac{h^3e^{-2\mu t}}{(\lambda-3\mu)^4(\lambda+\mu)^3(3\lambda-\mu)^3}\Bigg]
\end{aligned}$$

Here, all of the equations (36) have no exact solutions, but since $\dot{a}, \dot{b}, \dot{c}, \dot{d}$ and $\dot{h}$ are proportional to the small parameter ε, they are slowly varying functions of time *t*. Therefore, it is possible to replace a, b, c, d and h by their respective values obtained in linear case (i.e., the values of a, b, c, d and h obtained when $\varepsilon = 0$) in the right hand side of equations (36). This type of replacement was first introduced by Murty and Deekshatulu [5] and Mutry *et.al.* [14] in order to solve similar types of nonlinear equations.

Therefore, the solutions to the equations (36) are, when $\lambda >> \mu$ then a becomes

$$
\begin{aligned}
a = a_0 - \varepsilon \Bigg[\Bigg\{ & \frac{a_0^3}{8\lambda^3(3\lambda-\mu)} + \frac{3(59\lambda^2 - 34\lambda\mu + 5\mu^2)}{8\lambda^3(3\lambda-\mu)^3}(a_0 b_0^2 + a_0^2 c_0) \\
& + \frac{3\begin{pmatrix}194\lambda^3 - 169\lambda^2\mu \\ +50\lambda\mu^2 - 5\mu^3\end{pmatrix}}{8\lambda^6(3\lambda-\mu)^4}(b_0^3 + 6a_0 b_0 c_0 + 3a_0^2 d_0) + \frac{3(7\lambda-2\mu)a^2 b}{8\lambda^4(3\lambda-\mu)^2} \Bigg\} \frac{(1-e^{-2\lambda t})}{2\lambda} \\
& + \Bigg\{ \frac{3\left(49\lambda^2 + 10\lambda\mu + \mu^2\right)(b_0^2 h_0 + 2a_0 c_0 h_0)}{4\lambda^3(\lambda+\mu)^5} + \frac{3a_0^2 h_0}{2\lambda(\lambda+\mu)^3} + \frac{3(9\lambda+\mu)a_0 b_0 h_0}{2\lambda^2(\lambda+\mu)^4} \\
& + \begin{pmatrix}209\lambda^3 + 59\lambda^2\mu \\ +11\lambda\mu^2 + \mu^3\end{pmatrix} \frac{9(b_0 c_0 h_0 + a_0 d_0 h_0)}{4\lambda^4(\lambda+\mu)^6} \Bigg\} \frac{(1-e^{-(\lambda+\mu)t})}{(\lambda+\mu)} \\
& + \Bigg\{ \frac{\left(5\lambda^2 + 14\lambda\mu + 11\mu^2\right)}{8\mu^5(\lambda+\mu)^3} 3c_0 h_0^2 + \frac{9\left(5\lambda^3 + 20\lambda^2\mu + 29\lambda\mu^2 + 16\mu^3\right)d_0 h_0^2}{8\mu^6(\lambda+\mu)^4} \\
& + \frac{3a_0 h_0^2}{8\mu^3(\lambda+\mu)} + \frac{3(2\lambda+3\mu)b_0 h_0^2}{8\mu^4(\lambda+\mu)^2} \Bigg\} \frac{(1-e^{-2\mu t})}{2\mu} \Bigg]
\end{aligned}
$$

Again, when $\lambda << \mu$ then a becomes

$$
\begin{aligned}
a = a_0 - \varepsilon \Bigg[\Bigg\{ & \frac{a_0^3}{8\lambda^3(3\lambda-\mu)} + \frac{3(59\lambda^2 - 34\lambda\mu + 5\mu^2)}{8\lambda^3(3\lambda-\mu)^3}(a_0 b_0^2 + a_0^2 c_0) \\
& + \frac{3\begin{pmatrix}194\lambda^3 - 169\lambda^2\mu \\ +50\lambda\mu^2 - 5\mu^3\end{pmatrix}}{8\lambda^6(3\lambda-\mu)^4}(b_0^3 + 6a_0 b_0 c_0 + 3a_0^2 d_0) \\
& + \frac{3(7\lambda-2\mu)a_0^2 b_0}{8\lambda^4(3\lambda-\mu)^2} \Bigg\} \frac{(1-e^{-2\lambda t})}{2\lambda} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
b = b_0 - \varepsilon \Bigg[\Bigg\{ & \frac{3(7\lambda-2\mu)\left(a_0 b_0^2 + a_0^2 c_0\right)}{4\lambda^4(3\lambda-\mu)^2} + \frac{3a_0^2 b_0}{8\lambda^3(3\lambda-\mu)} \\
& + \frac{3\left(59\lambda^2 - 34\lambda\mu + 5\mu^2\right)\begin{pmatrix}b_0^3 + 6a_0 b_0 c_0 \\ +3a_0^2 d_0\end{pmatrix}}{2\lambda^2(\lambda+\mu)^4} \Bigg\} \frac{(1-e^{-2\lambda t})}{2\lambda} \\
& + \Bigg\{ \frac{3a_0 b_0 h_0}{\lambda(\lambda+\mu)^3} + \frac{\left(b_0^2 h_0 + 2a_0 c_0 h_0\right)}{2\lambda^2(\lambda+\mu)^4} 3(9\lambda+\mu) \\
& + 3\left(49\lambda^2 + 10\lambda\mu + \mu^2\right)(b_0 c_0 h_0 + a_0 d_0 h_0)\frac{1}{2\lambda^3(\lambda+\mu)^5} \Bigg\} \\
& \frac{(1-e^{-(\lambda+\mu)t})}{(\lambda+\mu)} + \Bigg\{ \frac{9\left(5\lambda^2 + 14\lambda\mu + 11\mu^2\right)}{8\mu^5(\lambda+\mu)^3} d_0 h_0^2 \\
& + \frac{3b_0 h_0^2}{8\mu^3(\lambda+\mu)} + \frac{3(2\lambda+3\mu)c_0 h_0^2}{4\mu^4(\lambda+\mu)^2} \Bigg\} \frac{(1-e^{-2\mu t})}{2\mu} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
c = c_0 - \varepsilon \Bigg[\Bigg\{ & \frac{(9\lambda+\mu)(b_0 c_0 h_0 + a_0 d_0 h_0)}{2\lambda^2(\lambda+\mu)^4} + \frac{\left(b_0^2 h_0 + 2a_0 c_0 h_0\right)}{(\lambda+\mu)^3}\frac{3}{2\lambda} \Bigg\} \frac{(1-e^{-(\lambda+\mu)t})}{(\lambda+\mu)} \\
& + \Bigg\{ \frac{3\begin{pmatrix}a_0 b_0^2 \\ +a_0^2 c_0\end{pmatrix}}{8\lambda^3(3\lambda-\mu)} + 3(7\lambda-2\mu)\frac{\begin{pmatrix}b_0^3 + 6a_0 b_0 c_0 \\ +3a_0^2 d_0\end{pmatrix}}{8\lambda^4(3\lambda-\mu)^2} \Bigg\} \frac{(1-e^{-2\lambda t})}{2\lambda} \\
& + \Bigg\{ \frac{3c_0 h_0^2}{8\mu^3(\lambda+\mu)} + \frac{9(2\lambda+3\mu)d_0 h_0^2}{8\mu^4(\lambda+\mu)^2} \Bigg\} \frac{(1-e^{-2\mu t})}{2\mu} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
d = d_0 - \varepsilon \Bigg\{ & \frac{\left(b_0^3 + 6a_0 b_0 c_0 + a_0^2 c_0\right)(1-e^{-2\lambda t})}{16\lambda^4(3\lambda-\mu)} \\
& + \frac{24(b_0 c_0 h_0 + a_0 d_0 h_0)(1-e^{-(\lambda+\mu)t})}{8\lambda^3(\lambda+\mu)^4} \\
& + \frac{3d_0 h_0^2(1-e^{-2\mu t})}{16\mu^4(\lambda+\mu)} \Bigg\}
\end{aligned}
\qquad (37)
$$

Moreover, when $\lambda >> \mu$ then h becomes

$$
\begin{aligned}
h = h_0 - \varepsilon h_0^3 & \frac{(1-e^{-2\mu t})}{2\mu} \\
& \frac{\left(27\lambda^6 + 54\lambda^5\mu - 28\lambda^3\mu^3 - 3\lambda^2\mu^4 + 6\lambda\mu^5 - \mu^6\right)}{(3\lambda-\mu)^3(\lambda+\mu)^3}
\end{aligned}
$$

when $\lambda << \mu$ then h becomes

$$h = h_0 - \varepsilon\left[\frac{3}{2}\left\{\frac{(49\lambda^2 + 10\lambda\mu + \mu^2)(b_0^2 h_0 + 2a_0 c_0 h_0)}{8\lambda^2(\lambda+\mu)^6}\right.\right.$$

$$+ 2{a_0}^2 h_0 \frac{1}{(\lambda+\mu)^4} + \frac{\left(209\lambda^3 + 59\lambda^2\mu + 11\lambda\mu^2 + \mu^3\right)}{\lambda^3(\lambda+\mu)^7}$$

$$\left. 3(b_0 c_0 h_0 + a_0 d_0 h_0) + 2a_0 b_0 h_0 \frac{(9\lambda+\mu)}{\lambda(\lambda+\mu)^5}\right\}$$

$$\frac{(1-e^{-2\lambda t})}{2\lambda} + \frac{1}{16}\left\{\frac{(2\lambda+3\mu)b_0{h_0}^2}{\mu^5(\lambda+\mu)}\right.$$

$$+ \frac{c_0{h_0}^2}{\mu^6(\lambda+\mu)^2}\left(5\lambda^2 + 14\lambda\mu + 11\mu^2\right)$$

$$\left. + \frac{\begin{pmatrix} 5\lambda^3 + 20\lambda^2\mu \\ +29\lambda\mu^2 + 16\mu^3 \end{pmatrix}}{\mu^7(\lambda+\mu)^3} 3d_0{h_0}^2 + \frac{a_0{h_0}^2}{\mu^4}\right\}\frac{(1-e^{-(\lambda+\mu)t})}{(\lambda+\mu)}$$

$$\left. + \frac{(1-e^{-2\mu t})}{2\mu(\lambda+\mu)^3(3\lambda-\mu)^3}\frac{{h_0}^3}{(\lambda-3\mu)^4}\begin{pmatrix} 27\lambda^6 + 545\lambda^5\mu \\ -28\lambda^3\mu^3 - 3\lambda^2\mu^4 \\ +6\lambda\mu^5 - \mu^6 \end{pmatrix}\right]$$

Hence, we obtain the first approximate solution of the equation (13) as:

$$x(t,\varepsilon) = (a + bt + ct^2 + dt^3)e^{-\lambda t} + he^{-\mu t} + \varepsilon u_1 \quad (38)$$

where a, b, c, d and h are given by the equations (37) and u_1 is given by (35).

4. Results and Discussion

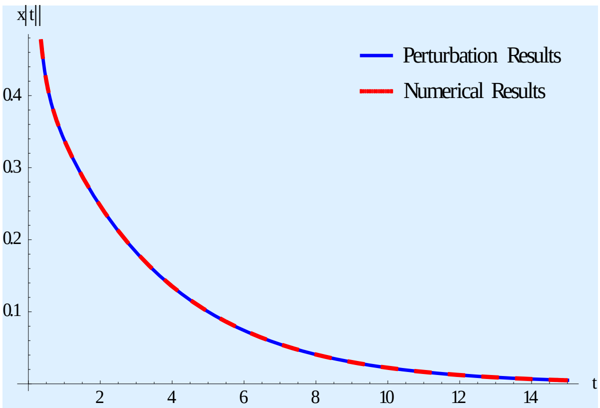

Figure 1. Comparison between perturbation and numerical results for $\varepsilon = 0.01$, $\lambda = 4.50$, $\mu = 0.30$, with the initial condition $a_0 = 0.50$, $b_0 = 0.40$, $c_0 = 0.30$, $d_0 = 0.25$ and $h_0 = 0.45$

In order to bring more efficiency to our results, the numerical results obtained by *Mathematica 9.0* are compared with the perturbation results obtained by the same program for the different sets of initial conditions. Here, we have computed $x(t,\varepsilon)$ from (38) by considering different values of λ and μ in which a, b, c, d and h are obtained from (37) and u_1 is calculated from equation (35) together with four sets of initial conditions. The corresponding numerical solutions have been computed by the *Mathematica 9.0* program for various values of t and all the perturbation solutions have been developed by a code in *Mathematica 9.0* program. All the results presented by the Figure 1 and Figure 2 for the case $\lambda >> \mu$; and Figure 3 and Figure 4 for the case $\lambda << \mu$, show the perturbation results, which are plotted by a blue line and the corresponding numerical results, which are plotted by the red line respectively.

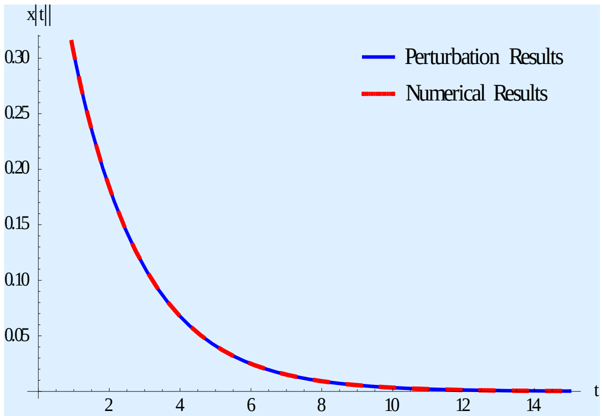

Figure 2. *Comparison between perturbation and numerical results for* $\varepsilon = 0.01$, $\lambda = 7.0$, $\mu = 0.50$, *with the initial condition* $a_0 = 0.60$, $b_0 = 0.25$, $c_0 = 0.35$, $d_0 = 0.20$ *and* $h_0 = 0.5$.

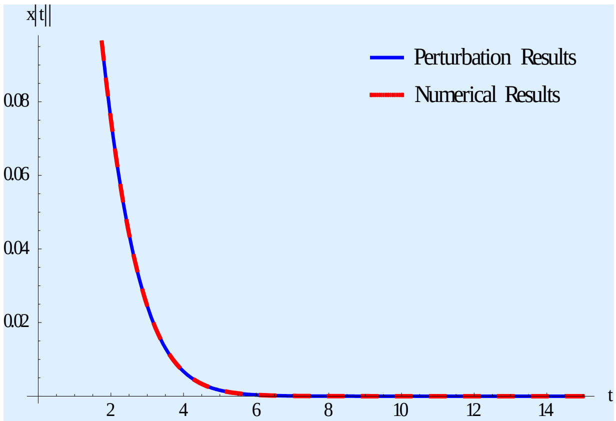

Figure 3. *Comparison between perturbation and numerical results for* $\varepsilon = 0.01$, $\lambda = 2.00$, $\mu = 9.00$, with the initial ondition $a_0 = 0.60$, $b_0 = 0.25$, $c_0 = 0.35$, $d_0 = 0.20$ and $h_0 = 0.35$

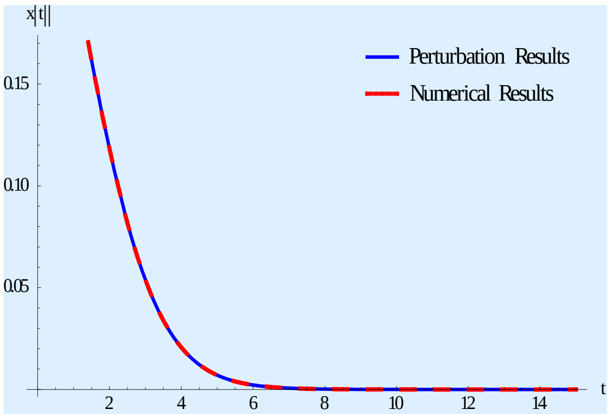

Figure 4. Comparison between perturbation and numerical results for $\varepsilon = 0.01$, $\lambda = 1.65$, $\mu = 7.00$, with the initial condition $a_0 = 0.40$, $b_0 = 0.30$, $c_0 = 0.25$, $d_0 = 0.15$ and $h_0 = 0.35$

5. Conclusion

In conclusion, it can be said that, in this article, we have successfully modified the KBM method and applied it to the fifth order more critically damped nonlinear systems. In relation to the fifth order more critically damped nonlinear systems, the solutions are obtained in such circumstances where the four eigenvalues are equal. Ordinarily, it is seen that, in the KBM method, much error occurs in the case of rapid changes of *x* with respect to time *t*. However, it has been observed in this study that, with respect to the different sets of initial conditions of the modified KBM method, the results obtained for both the cases (when $\lambda >> \mu$ and $\lambda << \mu$) correspond accurately to the numerical solutions obtained by Mathematica 9.0. We, therefore, come to the conclusion that the modified KBM method gives highly accurate results, which can be applied for different kinds of nonlinear differential systems.

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