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Common Fixed Point Theorem for Weakly Contractive Mappings in Cone Metric Spaces

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Abstract In this paper, we prove a unique common fixed point theorem for weakly contractive three mappings in cone metric spaces. Our results generalize and extend some recent results in the literature.

Keywords: *Fixed point, common fixed point, regular cone, weakly compatible*

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1. Introduction

In 2007, metric space was generalized by Huang and Zhang [6] and introduced cone metric space replacing the set of real numbers by an ordered Banach space. They have proved some fixed point theorems for contractive mappings in cone metric spaces. Later on many authors have extended and generalized the fixed point theorems of Huang and Zhang [6] (see, for exampls [1,2,3,8]. In 2008, Dutta and Chudahury [4] introduced the concept of generalized weakly contractive mapping and proved some fixed point theorems. In2010 Choudhury and Meitya [5] proved some fixed point theorems for weakly contractive mapping in cone metric spaces. In 2011, Hui-ru Zhao, Byung-Soo Lee and Nan-jing Huang [7] proved some common fixed point theorems for weakly compatible mappings in cone metric space under certain contractive conditions and they have generalized the fixed point theorems of [4] and [5].In this paper, we prove a common fixed point theorem for three weakly contractive three self-mappings in cone metric spaces. Our result extends and improves the results of [7].

In this paper B stands for a real Banach space and θ is the zero element, P is a normal cone in B with $P^0 \neq \emptyset$ where ≼ is a partial order with respect to P.

We recall some of the definitions are in [6] which are useful in the sequel.

Definition 1.1. Let P be a subset of B. Then P is called a cone if the following conditions are satisfied:

(i) P is closed and $P \neq \{\theta\}$;

(ii) $a, b \in \mathbb{R}$, $a,b \geq 0$, $x, y \in P \Longrightarrow ax + by \in P$;

(iii) $x \in P \cap (-P) \Longrightarrow x = \theta$.

For a cone P, define a partial ordering ≼ with respect to P by $x \preccurlyeq y$ if and only if $y\text{-}x \in P$. We shall write $x \prec y$ to indicate that $x \preccurlyeq y$ but $x \neq y$, while $x \ll y$ will stand for $y\text{-}x \in \text{int } P$, where, int P is the interior of P.

Definition 1.2. The cone P in a real Banach space B is called norm if there is a number M > 0 such that for all

$$x, y \in B, \theta \preceq x \preceq y \Rightarrow \| x \| \leq M \| y \|.$$

The least positive number M>0 satisfying the above relation is called a normal constant.

The cone P is said to be regular if for every increasing sequence bounded above is convergent, that is, if $\{x_n\}$ is a sequence such that $x_1 \preccurlyeq x_2 \preccurlyeq x_1 \preccurlyeq \ldots \preccurlyeq x_n \preccurlyeq y$ for some $y \in B$, then there exists $x \in B$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.

Similarly, if every decreasing sequence which is bounded below is convergent, then the cone P is also regular. It is well known that every regular cone is a normal cone.

Definition 1.3. Let X be a nonempty set. Let $d: X \times X \to B$ be a mapping satisfies the following

(i) $\theta \preccurlyeq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ iff $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iii) $d(x, y) \preccurlyeq d(x, z) + d(y, z)$ for all $x, y \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.4. Let (X, d) be cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for any $c \in B$ with $\theta \ll c$, there exists a natural number N such that $d(x_n, x) \ll c$ for all $n > N$, then $\{x_n\}$ is said to be convergent in X. We denote this $x_n \to x$ as $n \to +\infty$.

Definition 1.5. Let (X, d) be cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in B$ with $\theta \ll c$, there exists a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m > N$ then $\{x_n\}$ is called a Cauchy sequence in X.

The space (X, d) is called a complete cone metric space if every Cauchy sequence is convergent.

Definition 1.6. [7] Let I, J,T :X→X be three mappings with $T(X) \subseteq I(X)$ and $T(X) \subseteq J(X)$ and x_0 be a given point. Choose $x_1 \in X$ such that $Tx_{2n} = Ix_{2n+1}$ and $Tx_{2n+1} = Jx_{2n+2}$, n = 0,1,2,… then the sequence $\{Tx_n\}$ is called a T-I-J-sequence with initial point x_0.

Definition 1.7. [9] Let X be a non-empty set and I,T:X→X be two mappings. The pair of mappings I and T is said to be weakly compatible if for every $x\in X, T(Ix) = I(Tx)$ whenever $Tx = Ix$.

2. Main Results

In this section we obtain a common fixed point theorem for three maps in complete cone metric space.

The following theorem is extended improved the Theorem 2.1 of [7].

Theorem 2.1. Let (X, d) be a complete cone metric space, P be a regular cone, and $d(x, y)\in P^0$ for all $x, y\in X$ with $x \neq y$. Let I, J, T:X→X be mappings with $T(X) \subseteq I(X)$ and $T(X)\subseteq J(X)$ satisfying the following condition:

$$\psi\big(d(Tx,Ty)\big)\le\psi\big(d(Ix,\ Iy)\big)-\varphi\big(d(Jx,Jy)\big) \qquad (1)$$

for all $x, y\in X$, where $\psi:P^0 \cup \{\theta\} \to P^0 \cup \{\theta\}$ is continuous and monotone, $\varphi: P^0 \cup \{\theta\} \to P^0 \cup \{\theta\}$ is continuous, $\psi(t) = \theta$ if and only if $t = \theta$ and $\varphi(t) = \theta$ if and only if $t = \theta$. Suppose that

(i) I(X) or J(X) or T(X) is a complete subspace of X.

(ii){T, I} and {T, J} are weakly compatible.

(iii) either $d(x,y)\ll c$ or $d(x,y)\gg c$ for all $x, y\in X$ and $c\in P^0$.

Then I, J and T have a unique common fixed point in X. Moreover, for any $x_0\in X$ every T-I-J sequence with initial point x_0 converges to the common fixed point.

Proof. First we shall prove that $\{Tx_n\}$ is a Cauchy sequence. If $Tx_{2n} = Tx_{2n-1}$ for some n, then from (1), it follows that

$$\begin{aligned}\psi\big(d(Tx_{2n},Tx_{2n+1})\big)&\le\psi\big(d(Ix_{2n},Ix_{2n+1})\big)-\varphi\big(d(Jx_{2n},Jx_{2n+1})\big)\\&=\psi\big(d(Tx_{2n-1},Tx_{2n})\big)-\varphi\big(d(Tx_{2n-1},Tx_{2n})\big)\\&=\theta.\end{aligned}$$

$$\Rightarrow d\big(Tx_{2n},\ Tx_{2n+1}\big)=\theta.\text{ So, }Tx_{2n}=Tx_{2n+1}.$$

By the induction, we know that $Tx_m = Tx_{n-1}$ for all $m\in\mathbb{N}$ with $m \ge n$. Therefore, $\{Tx_n\}$ is a Cauchy sequence.

Now assume $Tx_{2n}\neq Tx_{2n-1}$ for all $n\in\mathbb{N}$.

From (1), we have

$$\begin{aligned}\psi\big(d(Tx_{2n},Tx_{2n+1})\big)&\le\psi\big(d(Ix_{2n},\ Ix_{2n+1})\big)-\varphi\big(d(Jx_{2n},Jx_{2n+1})\big)\\&=\psi\big(d(Tx_{2n-1},Tx_{2n})\big)-\varphi\big(d(Tx_{2n-1},Tx_{2n})\big).\end{aligned}$$

Since, ψ, φ are continuous,

$$\psi(y)\le\psi(y)-\varphi(y),\text{ that is }\varphi(y)\le\theta.$$

This shows that $y = \theta$, and so

$$d\big(Tx_{2n},Tx_{2n+1}\big)\to\theta\text{ as }n\to\infty.$$

Suppose that $\{Tx_n\}$ is not a Cauchy sequence, then there exists a $c\in B$ with $c\gg\theta$ and two subsequences $\{T x_{n_k}\}$ and $\{T x_{m_k}\}$ of $\{Tx_n\}$ with $n_k > m_k$ such that $d(Tx_{n_k},Tx_{m_k})\ge c$.

Moreover, corresponding to m_k we can choose n_k as the smallest integer satisfying

$$d(Tx_{n_k},Tx_{m_k})\ge c\text{ and }n_k > m_k.$$

Thus, $d(Tx_{n_k},Tx_{m_k})\ll\varepsilon$,

And

$$\begin{aligned}c&\le d(Tx_{n_k},Tx_{m_k})\\&\le d(Tx_{n_k},Tx_{n_{k-1}})+d(Tx_{n_{k-1}},Tx_{m_k})\\&\le d(Tx_{n_k},Tx_{n_{k-1}})+c.\end{aligned}$$

Letting $k\to\infty$, we get that

$$d(Tx_{n_k},Tx_{m_k})\to c\ \ as\ k\to\infty.$$

By the triangle inequality

$$\begin{aligned}d(Tx_{n_k},Tx_{m_k})\le d(Tx_{n_k},Tx_{n_{k-1}})+d(Tx_{n_{k-1}},Tx_{m_{k-1}})\\+d(Tx_{m_{k-1}},Tx_{m_k})\end{aligned}$$

and

$$\begin{aligned}d(Tx_{n_{k-1}},Tx_{m_{k-1}})\le d(Tx_{n_{k-1}},Tx_{n_k})+d(Tx_{n_k},Tx_{m_k})\\+d(Tx_{m_k},Tx_{m_{k-1}})\end{aligned}$$

it is easy to see that

$$d(Tx_{n_{k-1}},Tx_{m_{k-1}})\to c\big(k\to\infty\big).$$

Now by (1)

$$\begin{aligned}\psi\Big(d(Tx_{n_k},Tx_{m_k})\Big)&\le\psi\Big(d(Ix_{n_k},Ix_{m_k})\Big)-\varphi\Big(d(Jx_{n_k},Jx_{m_k})\Big)\\&=\psi\Big(d(Tx_{n_{k-1}},Tx_{m_{k-1}})\Big)-\varphi\Big(d(Tx_{n_{k-1}},Tx_{m_{k-1}})\Big).\end{aligned}$$

And so $\psi(c)\le\psi(c) - \varphi(c)$, which is a contradiction to $c\gg\theta$.

Therefore, $\{Tx_n\}$ is a Cauchy sequence.

Since T(X) or I(X) or J(X) is complete and $T(X) \subseteq I(X)$ there exists $u\in I(X)$ such that $Tx_n \to u$ (as $n\to\infty$), $Ix_n \to u$, $Jx_n \to u$(as $n\to\infty$).

Choose $z\in u$ such that Iz = u such that Iz = u and Jz = u.

To prove Tz = u.

$$\begin{aligned}\psi\big(d(u,Tz)\big)&=\lim_{n\to\infty}\varphi\big(d(Tx_n,Tz)\big)\\&\le\lim_{n\to\infty}\psi\big(d(Ix_n,Iz)\big)-\lim_{n\to\infty}\psi\big(d(Jx_n,Jz)\big)\\&=\theta.\end{aligned}$$

$\Rightarrow u = Tz$ and so $Iz = Jz = u$,

we have TIz = IIz and TJz = JJz.

Also Tu = Iu and Tu = Ju.

Since,

$$\begin{aligned}\psi\big(d(u,Tu)\big)&=\psi\big(d(Tz,Tu)\big)\\&\le\psi\big(d(Iz,Iu)\big)-\varphi\big(d(Jz,Ju)\big)\\&=\psi\big(d(u,Tu)\big)-\varphi\big(d(u,Tu)\big).\end{aligned}$$

We have $\varphi(d(u,Tu))\le\theta$.

$\Rightarrow d(u,Tu) = \theta$.

That is, u = Tu.

Therefore, Iu = Ju = Tu = u, u is a common fixed point of I, J and T.

Uniqueness, let v be another common fixed point of I, J, and T such that Iv = Tv = Jv = v. Then

$$\begin{aligned}\psi\big(d(u,v)\big)&=\varphi\big(d(Tu,Tv)\big)\\&\le\psi\big(d(Iu,\ Iv)\big)-\varphi\big(d(Ju,Jv)\big)\\&=\psi\big(d(u,v)\big)-\varphi\big(d(u,v)\big).\end{aligned}$$

$$\Rightarrow\varphi\big(d(u,v)\big)\le\theta.$$

$$\Rightarrow u = v.$$

Therefore, u is a unique common fixed point of I, J and T in X.

Remark 2.2. Let ψ be an identity mapping in the Theorem 3.1, then we get the following theorem.

Theorem 2.3. Let (X, d) be a complete cone metric space, P be a regular cone, and $d(x, y)\in P^0$ for all $x, y\in X$ with $x \neq y$. Let $I, J, T:X\to X$ be mappings with $T(X) \subseteq I(X)$ and $T(X)\subseteq J(X)$ satisfying the following condition:

$$d(Tx,Ty)) \leq d(Ix,Iy)) - \varphi(d(Jx,Jy)) \quad (2)$$

for all $x, y\in X$, where, $\varphi: P^0 \cup \{\theta\} \to P^0 \cup \{\theta\}$is continuous, and $\varphi(t) = \theta$ if and only if $t = \theta$. Suppose that

(i) I(X) or J(X) or T(X) is a complete subspace of X.

(ii){T, I} and {T, J} are weakly compatible.

(iii) either $d(x,y)\ll c$ or $d(x,y)\gg c$ for all $x, y\in X$ and $c\in P^0$.

Then I, J and T have a unique common fixed point in X. Moreover, for any $x_0\in X$ every T-I-J sequence with initial point x_0 converges to the common fixed point.

Remark 2.4. Let ψ, I and J be identity mappings in the Theorem 3.1 , we get the following theorem.

Theorem 2.5. Let (X, d) be a complete cone metric space, P be a regular cone, and $d(x, y)\in P^0$ for all $x, y\in X$ with $x \neq y$. Let $T:X\to X$ be mappings such that

$$d(Tx,Ty)) \leq d(x,y) - \varphi(d(x,y)) \quad (3)$$

for all $x, y\in X$, where, $\varphi: P^0 \cup \{\theta\} \to P^0 \cup \{\theta\}$is continuous, and $\varphi(t) = \theta$ if and only if $t = \theta$. Suppose that

(i) I(X) or J(X) or T(X) is a complete subspace of X.

(ii){T, I} and {T, J} are weakly compatible.

(iii) either $d(x,y)\ll c$ or $d(x,y)\gg c$ for all $x, y\in X$ and $c\in P^0$.

Then,T has a unique fixed point in X.

Remark 2.6. Let I, J be identity mappings in the Theorem 3.1 , we get the following theorem

Theorem 2.7. Let (X, d) be a complete cone metric space, P be a regular cone, and $d(x, y)\in P^0$ for all $x, y\in X$ with $x \neq y$. Let $T:X\to X$ be mapping such that

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \varphi(d(x,y)) \quad (4)$$

for all $x, y\in X$, where $\psi:P^0 \cup \{\theta\} \to P^0 \cup \{\theta\}$ is continuous and monotone, $\varphi: P^0 \cup \{\theta\} \to P^0 \cup \{\theta\}$is continuous, $\psi(t) = \theta$ if and only if $t = \theta$ and $\varphi(t) = \theta$ if and only if $t = \theta$. Suppose that

(i) I(X) or J(X) or T(X) is a complete subspace of X.

(ii){T, I} and {T, J} are weakly compatible.

(iii) either $d(x,y)\ll c$ or $d(x,y)\gg c$ for all $x, y\in X$ and $c\in P^0$.

Then T has a unique common fixed point in X.

Remark 2.8. If we choose J = I in the above theorem then we get the Theorem 3.1 of [7].

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