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Common Fixed Point Theorem in Cone Metric Spaces

K. Prudhvi*

Department of Mathematics, University College of Science, Saifabad, Osmania University, Hyderabad, Telangana State, India
*Corresponding author: prudhvikasani@rocketmail.com

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Abstract In this paper, we study the existence of coincidence points and a unique common fixed point theorem for three self-mappings in cone metric spaces, where the cone is not necessarily normal. This result extends and improves recent related results in the literature.

Keywords: *coincidence point, cone metric space, common fixed point*

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1. Introduction

Huang and Zhang [5] generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space. They proved some fixed point theorems of contractive type mappings over cone metric spaces. Subsequently, many authors generalized their fixed point theorems to different types using with and without normality (see, e.g., [1,2,3,6,7,8]). Recently, Guangxing Song et.al. [4] obtained a new common fixed point theorems for two maps in cone metric spaces, and omitting the assumption of normality. In this paper, we proved a fixed point theorem for three maps in cone metric spaces. Our result extends and improves the results of [4].

We recall some definitions and properties of cone metric spaces due to Huang and Zhang [5].

Definition 1.1. Let E be a real Banach space and P a subset of E .The set P is called a cone if and only if:

(a). P is closed, non –empty and $P \neq \{0\}$;

b). a,b∈ℝ, $a,b \geq 0$, x,y∈P implies ax+by∈P;

(c). x∈P and -x∈p implies x = 0.

Definition 1.2. Let P be a cone in a Banach space E, define partial ordering '$\leq$' with respect to P by $x \leq y$ if and only if y-x∈P. We shall write x<y to indicate x≤y but $x \neq y$ while x<<y will stand for y-x∈Int P, where Int P denotes the interior of the set P. This cone P is called an order cone.

Definition 1.3. Let E be a Banach space and $P \subset E$ be an order cone .The order cone P is called normal if there exists L>0 such that for all x, y∈E,

$$0 \leq x \leq y \text{ implies } \|x\| \leq L\|y\|.$$

The least positive number L satisfying the above inequality is called the normal constant of P.

Definition 1.4. Let X be a nonempty set of E. Suppose that the map d: $X \times X \to E$ satisfies:

(d1). $0 \leq d(x, y)$ for all x, y∈X and
d(x,y) = 0 if and only if x = y;

(d2). d(x, y) = d(y, x) for all x, y∈X;

(d3). $d(x, y) \leq d(x, z)+d(z, y)$ for all x, y, z∈X.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

It is clear that the concept of a cone metric space is more general than that of a metric space.

Example 1.5. [5] Let E =R^2, P = $\{(x, y) \in E$ such that : x, $y \geq 0\} \subset R^2$, X = R and

$$d : X \times X \to E \text{ such that } d(x, y) = (|x-y|, \alpha|x-y|),$$

where $\alpha \geq 0$ is a constant . Then (X, d) is a cone metric space.

Definition 1.6. Let (X, d) be a cone metric space .We say that $\{x_n\}$ is

(i) a Cauchy sequence if for every c in E with c>>0, there is N such that for all

$$n, m > N, d(x_n, x_m) << c;$$

(ii) convergent sequence if for any c>>0, there is an positive integer N such that for all n>N, $d(x_n, x) <<c$, for some fixed x in X .We denote this $x_n \to x$ (as $n \to \infty$) .

The space (X, d) is called a complete cone metric space if every Cauchy sequence is convergent ([5]).

Definition 1.8. **[1]** For the mapping f, g: X →X. If w = fz = gz for some z in X, then z is called a coincidence point of f and g and w is called a point of coincidence of f and g.

Definition 1.7. Let f, g: X→ X. Then the pair (f, g) is said to be (IT)-commuting at z∈X if f(g(z))=g(f(z)) with f(z)=g(z).

2. Main Result

In this section, we proved a fixed point theorem for three self- mappings in cone metric spaces and without assuming the normality.

Theorem 2.1. Let (X, d) be a cone metric space and P an order cone and f, g,h: X→X be self-maps satisfying the following condition

$$d(fx,\ gy) \le a_1 d(hx,\ hy) + a_2(fx,\ hx) + a_3 d(gy,\ hy) + a_4 d(hx, gy) + a_5 d(fx, hy) \quad (1)$$

for all x, y∈X, where $a_i \ge 0$ (i= 1,2,3,4,5) be constants ($a_1+a_2+a_3+2a_4+a_5 < 1$) .

If $f(X) \cup g(X) \subseteq h(X)$ and h(X) is a complete subspace of X. Then the maps f,g and h have a coincidence point p in X. Moreover if (f, h) and (g, h) are (IT)-Commuting at p, then f, g and h have a unique common fixed point.

Proof. Suppose x_0 is an arbitrary point of X, and define the sequence $\{y_n\}$ in X
such that $y_{2n} = fx_{2n} = hx_{2n+1}$,
and $y_{2n+1} = gx_{2n+1} = hx_{2n+2}$, for all n = 0,1,2,3,…..

By (1), we have

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(fx_{2n,} gx_{2n+1}) \\
&\le a_1 d(hx_{2n}, hx_{2n+1}) + a_2 d(fx_{2n}, hx_{2n}) + \\
&\quad a_3 d(gx_{2n+1,} hx_{2n+1}) + a_4 d(hx_{2n}, gx_{2n+1}) \\
&\quad + a_5 d(fx_{2n}, hx_{2n+1}), \\
&\le a_1 d(y_{2n-1}, y_{2n}) + a_2 d(y_{2n}, y_{2n-1}) + \\
&\quad a_3 d(y_{2n+1}, y_{2n}) + a_4 d(y_{2n-1}, y_{2n+1}) \\
&\quad + a_5 d(y_{2n}, y_{2n}), \\
&\le a_1 d(y_{2n-1}, y_{2n}) + a_2 d(y_{2n}, y_{2n-1}) \\
&\quad + a_3 d(y_{2n+1}, y_{2n}) + a_4[d(y_{2n-1}, y_{2n+1}) \\
&\quad + d(y_{2n}, y_{2n+1})], \\
&\le (a_1 + a_2 + a_4) d(y_{2n-1}, y_{2n}) \\
&\quad + (a_3 + a_4) d(y_{2n,} y_{2n+1}),
\end{aligned}$$

$$\begin{aligned}
1-(a_3+a_4) d(y_{2n}, y_{2n+1}) &\le (a_1 + a_2 + a_4)(y_{2n-1,} y_{2n}) \\
&\le \frac{(a_1+a_2+a_4)}{1-(a_3+a_4)} d(y_{2n-1,} y_{2n}).
\end{aligned}$$

Put, $k = \dfrac{(a_1+a_2+a_4)}{1-(a_3+a_4)} < 1.$

$$d(y_{2n}, y_{2n+1}) \le k d(y_{2n-1,} y_{2n}). \quad (2)$$

Similarly it can be shown that

$$d(y_{2n+1,} y_{2n+2}) \le k d(y_{2n,} y_{2n+1}).$$

Therefore, for all n,

$$d(y_{n+1,} y_{n+2}) \le k d(y_{n,} y_{n+1}) \le \ldots \le k^{n+1} d(y_0, y_1)$$

Now, for any m>n,

$$\begin{aligned}
d(y_{n,} y_m) &\le d(y_{n,} y_{n+1}) + d(y_{n+1,} y_{n+2}) \\
&\quad + \ldots + d(y_{m-1}, y_m), \\
&\le (k^n + k^{n+1} + \ldots + k^{m-1}) d(y_1, y_0), \\
&\le \frac{k^n}{1-k} d(y_1, y_0).
\end{aligned}$$

Let 0<<c be given. Choose δ>0 such that c+ $N_\delta(0) \subseteq P$, where $N_\delta(0)=\{x \in E: \|x\| < \delta\}$.

Also choose a natural number N_1 such that

$\dfrac{k^n}{1-k}$ $d(y_1, y_0) \in N_\delta(0)$, for all $n \ge N_1$.

Then $\dfrac{k^n}{1-k}$ $d(y_1, y_0) << c$, for all $n \ge N_1$.

Thus, $d(y_n, y_m) \le \dfrac{k^n}{1-k} d(y_1, y_0) << c$, for all m>n.

Therefore, $\{y_n\}$ is a Cauchy sequence. Since h(X) is complete, there exists q in h(X) such that h(p)=q. We shall show that hp = fp = gp. Note that d(hp,fp) = d(q,fp).

Let us estimate d(hp, fp).We have by (1) and the triangle inequality

$$\begin{aligned}
d(hp,\ fp) &\le d(hp,\ hx_{2n}) + d(hx_{2n}, fp) \\
&= d(hp, hx_{2n}) + d(fp, gx_{2n-1}) \\
&\le d(q, hx_{2n}) + a_1 d(hp, hx_{2n-1}) + a_2 d(fp, hp) \\
&\quad + a_3 d(gx_{2n-1,} hx_{2n-1}) + a_4 d(hp, gx_{2n-1}) \\
&\quad + a_5 d(fp, hx_{2n-1}), \\
&\le d(q, hx_{2n}) + a_1 d(q, hx_{2n-1}) + a_2 d(fp, hp) \\
&\quad + a_3 d(hx_{2n}, hx_{2n-1}) + a_4 d(q, hx_{2n}) \\
&\quad + a_5 d(fp, hx_{2n-1}), \\
&\le d(q,\ hx_{2n}) + a_1 d(q, hx_{2n-1})\ a_2 d(fp, hp) \\
&\quad + a_3 \left[d(hx_{2n,} q) + d(q, hx_{2n-1})\right] \\
&\quad + a_4 d(q, hx_{2n}) + a_5 \left[d(fp, hp) + d(hp, hx_{2n-1})\right], \\
&\le (1 + a_3 + a_4) d(q, hx_{2n}) + (a_2 + a_5) d(fp, hp) \\
&\quad + (a_1 + a_3 + a_5) d(q, hx_{2n-1}).
\end{aligned}$$

$$\begin{aligned}
1-(a_2+a_5) d(hp,\ fp) &\le (1 + a_3 + a_4) d(q, hx_{2n}) \\
&\quad + (a_1 + a_3 + a_5) d(q, hx_{2n-1})
\end{aligned}$$

$$\begin{aligned}
d(hp, fp) &\le \frac{a_1 + a_3 + a_4}{a_1 - a_2 - a_5} d(q, hx_{2n}) \\
&\quad + \frac{a_1 + a_3 + a_5}{a_1 - a_2 - a_3} d(q, hx_{2n-1}).
\end{aligned} \quad (3)$$

Suppose 0<<c and there exists n_0∈N such that

$$d(q, hx_{2n}) << \frac{c}{2} \frac{a_1 - a_2 - a_3}{a_1 + a_3 + a_5}. \quad (4)$$

and

$$d(q, hx_{2n-1}) << \frac{c}{2} \frac{a_1 - a_2 - a_3}{a_1 + a_3 + a_5} \text{for all } n \ge n_0. \quad (5)$$

From (4), (5) and (3) it follows that

$$d(hp,\ fp) << \frac{c}{2} + \frac{c}{2} = c.$$

And hence, d(hp, fp)<< $\frac{c}{r}$ for every $r \in N$.

Since, $\frac{c}{r}$ - d(hp, fp) $\in$ int P, and P is closed, then as $r \to \infty$ we have that - d(hp, fp) $\in$ P. Since d(hp, fp)>0, therefore d(hp, fp) $\in$ P and so d(hp, fp) $\in P \cap (-P) = \{o\}$.

$\Longrightarrow$ d(hp, fp) = 0.

Hence, d(hp, fp) = 0.

Similarly, we can show that hp = gp.

$$\text{Thus, } q = hp = fp = gp \text{ and hence p is a point of coincidence point of } f, g \text{ and } h. \quad (6)$$

Since, (f, h),(g, h) are (IT)-Commuting at p. We get by (6) and (1)

$$\begin{aligned} d(ffp, fp) = d(ffp, gp) &\le a_1 d(hfp, hp) + a_2 d(ffp, hfp) \\ &\quad + a_3 d(gp, hp) + a_4 d(hfp, gp) \\ &\quad + a_5 d(ffp, hp), \\ &\le a_1 d(ffp, fp) + a_2 d(ffp, ffp) \\ &\quad + a_3 d(hp, hp) + a_4 d(ffp, fp) + a_5 d(ffp, fp), \\ &\le (a_1 + a_4 + a_5) d(ffp, fp), \\ &< d(ffp, fp), \end{aligned}$$

since $a_1 + a_4 + a_5 < 1$, a contradiction.

Therefore, ffp = fp. fp = ffp = fhp = hfp.

$\Rightarrow$ ffp = hfp = fp(= q).

Therefore, fp(= q) is a common fixed point of f and h. (7)

Similarly , we can get that

gp = ggp = ghp = hgp.

$\Rightarrow$ ggp = hgp = gp = q.

Therefore, gp (= q) is a common fixed point of g and h. (8)

Since, fp = gp = q.

Therefore from , (7) and (8) it follows that f, g, and h have a common fixed point namely q.

Uniqueness: Let q_1 is another fixed point of f, g and h, then

$$\begin{aligned} d(q, q_1) = d(fq, gq_1) &\le a_1 d(hq, hq_1) + a_2 d(fq, hq) \\ &\quad + a_3 d(gq_1, hq_1) + a_4 d(hq, gq_1) \\ &\quad + a_5 d(fq, hq_1), \\ &\le a_1 d(q, q_1) + a_2 d(q, q) + a_3 d(q_1, q_1) \\ &\quad + a_4 d(q, q_1) + a_5 d(q, q_1), \\ &\le (a_1 + a_4 + a_5) d(q, q_1). \end{aligned}$$

Hence, $d(q, q_1) = 0$ and so, $q = q_1$.

Therefore, f, g, and h have a unique common fixed point.

Remark 2.2. If we choose h = g and g = f in the above Theorem 2.1, then we obtain the Theorem 2.1 of [4].

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