American Journal of Applied Mathematics and Statistics, 2014, Vol. 2, No. 6, 369-375
Available online at http://pubs.sciepub.com/ajams/2/6/3

DOI:10.12691/ajams-2-6-3

The Approximate Method for Solving the Boundary Integral Equations of the Problem of Wave Scattering by Superconducting Lattice

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Received October 18, 2014; Revised November 10, 2014; Accepted November 19, 2014

Abstract In this article the method for numerical solution of boundary integral equations of the original problem is proposed. This method is one of the modifications of Nystrom-type methods; particularly the method of discrete vortices. The convergence of the numerical solutions to the exact solution of the problem is guaranteed by propositions proved in this article. Also, the rate of convergence of the approximate solutions to the exact solution had been found.

Keywords: *singular integral equation, modification of method of discrete vortices, existence of approximate solution, the rate of convergence of the approximate solutions*

Cite This Article: Gandel Yu. V., and Dushkin V.D., “The Approximate Method for Solving the Boundary Integral Equations of the Problem of Wave Scattering by Superconducting Lattice.” *American Journal of Applied Mathematics and Statistics*, vol. 2, no. 6 (2014): 369-375. doi: 10.12691/ajams-2-6-3.

1. Introduction

The Nystrom-type methods are popular methods of numerical solution of boundary integral equations of mathematical diffraction theory [1,2,3,4]. In the countries of the former Soviet Union, some variants of Nystrom-type methods are called methods of discrete vortices [5,6] or methods of discrete singularities [7]. In the articles [1,2,3,4] and many others (see. [8]) the solution of mathematical electrodynamics problems is carried out in two stages.

At the first stage the initial boundary-value problem for the Helmholtz equation is reduced to an equivalent system of boundary integral equations using the method of parametric representation of integral operators [8,9,10]. On the second stage the resulting system of integral equations are solved numerically by Nystrom-type methods. Using this approach the electrodynamics’ structures containing not perfectly conducting and superconducting elements [12,13,14,15,16] can also be investigated numerically.

In particular, the system of boundary integral equations of the problem of electromagnetic waves scattering by a system of superconducting band had been taken in the article [12].

The scheme for numerical solution of this problem and the results of numerical experiments had been proposed in the article [13]. A rigorous mathematical justification for the numerical solution of various electromagnetic problems by the method of discrete singularities [6,17,18] had existed at the time of publication of the articles [12,13]. But the systems of boundary integral equations of the problem of electromagnetic waves scattering by a system of superconducting band are different from the systems of integral equations of other problems that were previously solved numerically by the method of discrete singularities.

Due to these differences rigorous mathematical schemes for the numerical solution of the problem of electromagnetic waves scattering by a system of superconducting band has not been given until now.

In the articles [1,2,16] the integral equations have the same properties as the boundary equations of the problem of wave scattering by superconducting tapes.

Therefore, the rigorous justification of schemes for the numerical solution of the problem of wave scattering by superconducting tapes is of interest.

In this article a scheme for the numerical solution of boundary integral equations of the problem of wave scattering by superconducting lattice has been justified. Also, the rate of convergence of the approximate solutions to the exact solution has been found.

2. The System of Boundary Integral Equations

The system of boundary integral equations of the problem of wave scattering by superconducting tape consists of integral equations of two different types.

The Equations of the first type are singular integral equations of the first kind:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\tau-\xi}\frac{\vartheta(\tau)d\tau}{\sqrt{1-\tau^2}}+\frac{1}{\pi}\int_{-1}^{1}Q(\xi,\tau)\frac{\vartheta(\tau)d\tau}{\sqrt{1-\tau^2}}$$
$$-\frac{C_1}{\pi}\int_{-1}^{\xi}\frac{\vartheta(\tau)d\tau}{\sqrt{1-\tau^2}}=f(\xi),\quad |\xi|<1; \tag{1}$$

with the additional condition

$$\int_{-1}^{1}\frac{\vartheta(\tau)d\tau}{\sqrt{1-\tau^2}}=0. \tag{2}$$

The equations of the second type are Fredholm equations of the second kind:

$$v(\xi)-\frac{C_2}{\pi}\sqrt{1-\xi^2}\cdot\int_{-1}^{1}\ln|\tau-\xi|\frac{v(\tau)d\tau}{\sqrt{1-\tau^2}}$$
$$+\frac{1}{\pi}\int_{-1}^{1}K\ \ (\xi,\tau)\frac{v\ \ (\tau)d\tau}{\sqrt{1-\tau^2}}=g\ \ (\xi),\quad |\xi|<1. \tag{3}$$

In the equations (1), (3) it is assumed that

$$f(\xi)\in C^{\mu,\gamma}([-1,1]),g(\xi)\in C^{0,\psi}([-1,1]), \tag{4}$$

$$Q(\xi,\tau)\in C^{\mu,\gamma}([-1,1]\times[-1,1]), \tag{5}$$

$$\mathrm{K}(\xi,\tau)\in C^{0,\psi}([-1,1]\times[-1,1]), \tag{6}$$

$$\gamma>0,\quad 0<\psi\le\frac{1}{2}.$$

Let $L_{2,\alpha}$, $\left(\alpha=\pm\frac{1}{2}\right)$, are the Hilbert spaces of measurable functions with the inner product

$$(u,v)_\alpha=\int_{-1}^{1}u(\tau)\cdot\overline{v}(\tau)\cdot\left(1-\tau^2\right)^\alpha d\tau \tag{7}$$

and norm $\|v\|_\alpha=\sqrt{(v,v)_\alpha}$. Also we define the subspaces

$$L_{2,\alpha}^0=\left\{u\in L_{2,\alpha}\middle|(u,1)_\alpha=0\right\}.$$

We introduce the operators (see [8,19]):

$$\Lambda:L_{2,-\frac{1}{2}}\to L_{2,-\frac{1}{2}},$$
$$(\Lambda\ u)(\xi)=\frac{1}{\pi}\int_{-1}^{1}\ln|\tau-\xi|\frac{u(\tau)d\tau}{\sqrt{1-\tau^2}},\quad |\xi|<1; \tag{8}$$

$$K:L_{2,-\frac{1}{2}}\to L_{2,-\frac{1}{2}},$$
$$(Ku)(\xi)==\frac{1}{\pi}\int_{-1}^{1}K(\xi,\tau)\frac{u(\tau)d\tau}{\sqrt{1-\tau^2}},\quad |\xi|<1; \tag{9}$$

$$Q:L_{2,-\frac{1}{2}}\to L_{2,\frac{1}{2}},$$
$$(Qu)(\xi)=\frac{1}{\pi}\int_{-1}^{1}Q(\xi,\tau)\frac{u(\tau)d\tau}{\sqrt{1-\tau^2}},\quad |\xi|<1; \tag{10}$$

$$J:L_{2,-\frac{1}{2}}\to L_{2,\frac{1}{2}},\quad (Ju)(\xi)=\frac{1}{\pi}\int_{-1}^{\xi}\frac{V(\tau)d\tau}{\sqrt{1-\tau^2}},\ |\xi|<1; \tag{11}$$

$$P:L_{2,-\frac{1}{2}}\to L_{2,-\frac{1}{2}},(\mathrm{P}u)(\xi)=\sqrt{1-\xi^2}\cdot u(\xi),|\xi|<1; \tag{12}$$

$$\Gamma:L_{2,-\frac{1}{2}}^0\to L_{2,\frac{1}{2}},(\Gamma u)(\xi)=\frac{1}{\pi}\int_{-1}^{1}\frac{u(t)}{t-\xi}\frac{dt}{\sqrt{1-t^2}},|\xi|<1 \tag{13}$$

and mappings

$$A:L_{2,-\frac{1}{2}}^0\to L_{2,\frac{1}{2}},\quad A=\Gamma-c_1J+Q; \tag{14}$$

$$B:L_{2,-\frac{1}{2}}\to L_{2,-\frac{1}{2}},\quad B=I-c_2(P\Lambda)+K. \tag{15}$$

With the preceding notation (8)-(15), equation (1) with the additional condition (2) and equation (3) can be written as

$$A\vartheta=f, \tag{16}$$

$$Bv=g. \tag{17}$$

Proposition 1

The operator $A:L_{2,-\frac{1}{2}}^0\to L_{2,\frac{1}{2}}$ is invertible and operator A^{-1} is bounded.

Proof of Proposition 1

The operator A – is the sum of invertible operator $\Gamma:L_{2,-\frac{1}{2}}^0\to L_{2,\frac{1}{2}}$ (see [8,19]) and a compact operator $-c_1J+Q$. Hence, by virtue of Nikolsky criterion (see [20], p.150),

$$ind\left(A\Big|_{L_{1,-\frac{1}{2}}^0\to L_{1,\frac{1}{2}}}\right)=0.$$

From the uniqueness of the problem solutions (1)-(2) follows that

$$\dim\ker\left(A\Big|_{L_{2,-\frac{1}{2}}^0\to L_{2,\frac{1}{2}}}\right)=0.$$

Hence,

$$A\left(L_{1,-\frac{1}{2}}^0\right)=L_{1,\frac{1}{2}}.$$

Therefore operator A is bijective and bounded. So, by the Banach Isomorphism Theorem (see in [20], p.113) the operator A has the bounded inverse.

Proposition 2

The operator $B:L_{2,-\frac{1}{2}}^0\to L_{2,\frac{1}{2}}$ is invertible and the operator B^{-1} is bounded.

Proof of Proposition 2

The operator $P\Lambda$ is compact as the composition of the bounded operator P and compact operator Λ. This follows from the compactness of the operator $-c_2(P\Lambda)+K$ and the Fredholm Theorem (see. [20], p.146), that

$$ind\left(B\big|_{L_{2,-\frac{1}{2}} \to L_{2,-\frac{1}{2}}} \right) = 0.$$

From the uniqueness of boundary integral equation solution (3) follows that

$$\dim\ker\left(B\big|_{L_{2,-\frac{1}{2}} \to L_{2,-\frac{1}{2}}} \right) = 0.$$

Hence

$$B\left(L_{2,-\frac{1}{2}} \right) = L_{2,-\frac{1}{2}}.$$

Therefore operator B is bijective and bounded. So by the Banach Isomorphism Theorem (see in [20], p.113) the operator B has the bounded inverse.

3. The Approximate System of Integral Equations and Its Properties

Let us consider sets of points

$$t_k^n = \cos\left(\frac{2k-1}{2n}\pi\right), \quad k=1,\dots,n; \tag{18}$$

$$t_{0,j}^n = \cos\left(\frac{j}{n}\pi\right), \quad j=1,\dots,n-1. \tag{19}$$

Here the points $\left\{t_k^n\right\}_{k=1}^n$ are the zeros of Chebychev polynomials of the first kind $T_n(\tau)$ and the points $\left\{t_{0,j}^n\right\}_{j=1}^{n-1}$ are the zeros of Chebychev polynomials of the second kind $U_{n-1}(\xi)$.

Let us consider the basis polynomials

$$l_{1,n-1,k}(\tau) = \left[1 + 2\sum_{p=1}^{n-1} T_p(\tau)\cdot T_p\left(t_k^n\right)\right]\cdot\frac{1}{n}, \quad k=1,\dots,n; \tag{20}$$

$$l_{2,n-2,j}(\xi) = \frac{U_{n-1}(\xi)}{U'_{n-1}\left(t_{0,j}^n\right)\left(\xi - t_{0,j}^n\right)}, \quad j=1,\dots,n-1; \tag{21}$$

which has the properties

$$l_{1,n-1,k}(t_i^n) = \delta_{i,k}, \qquad i=1,\dots,n;$$

$$l_{2,n-2,j}(t_{0,l}^n) = \delta_{jl}, \quad l=1,\dots,n-1;$$

where $\delta_{i,k}$ is the Kronecker delta.

We introduce functions:

$$K_n(\xi,\tau) = \sum_{j=1}^{n}\sum_{k=1}^{n} K\left(t_j^n, t_k^n\right)\cdot l_{1,n-1,j}(\xi)\cdot l_{1,n-1,k}(\tau); \tag{22}$$

$$Q_n(\xi,\tau) = \sum_{j=1}^{n-1}\cdot\sum_{k=1}^{n} Q\left(t_{0,j}^n, t_k^n\right)\cdot l_{2,n-2,j}(\xi)\cdot l_{1,n-1,k}(\tau); \tag{23}$$

$$f_n(\tau) = \sum_{k=1}^{n} f\left(t_k^n\right)\cdot l_{1,n-1,k}(\tau); \tag{24}$$

$$g_n(\xi) = \sum_{j=1}^{n-1} g\left(t_{0,j}^n\right)\cdot l_{2,n-2,j}(\xi). \tag{25}$$

The functions $f_n(\tau)$ are the Lagrange interpolation polynomials of the function $f(\tau)$ with the nodal points $\left\{t_k^n\right\}_{k=1}^n$ and the functions $g_n(\xi)$ are the interpolation polynomials of the function $g(\xi)$ with the nodal points $\left\{t_{0,j}^n\right\}_{j=1}^{n-1}$. The functions $K_n(\xi,\tau)$ are the Lagrange interpolation polynomials of the function $K(\xi,\tau)$ with the nodal points $\left\{t_k^n\right\}_{k=1}^n$, with respect to both variables. The functions $Q_n(\xi,\tau)$ are the Lagrange interpolation polynomials of the function $Q(\xi,\tau)$ with the nodal points $\left\{t_{0,j}^n\right\}_{j=1}^{n-1}$ with respect to variable ξ and with the nodal points $\left\{t_k^n\right\}_{k=1}^n$ with respect to variable τ.

Let us consider the following system of integral equations:

$$\begin{aligned}&\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\tau-\xi}\frac{\vartheta_n(\tau)d\tau}{\sqrt{1-\tau^2}} + \frac{1}{\pi}\int_{-1}^{1} Q_n(\xi,\tau)\frac{\vartheta_n(\tau)d\tau}{\sqrt{1-\tau^2}}\\ &-\frac{C_1}{\pi}\int_{-1}^{\xi}\frac{\vartheta_n(\tau)d\tau}{\sqrt{1-\tau^2}} = f_n(\xi), \quad |\xi|<1;\end{aligned} \tag{26}$$

$$\int_{-1}^{1}\frac{\vartheta_n(\tau)d\tau}{\sqrt{1-\tau^2}} = 0; \tag{27}$$

$$\begin{aligned}&v_n(\xi) + \frac{1}{\pi}\int_{-1}^{1} K_n(\xi,\tau)\frac{v_n(\tau)d\tau}{\sqrt{1-\tau^2}} -\\ &-\frac{C_2}{\pi}\sum_{k=1}^{n}\sqrt{1-\left(t_k^n\right)^2}\cdot l_{1,n-1,k}(\xi)\cdot\int_{-1}^{1}\ln\left|\tau - t_k^n\right|\frac{v_n(\tau)d\tau}{\sqrt{1-\tau^2}}\\ &= g_n(\xi), \quad |\xi|<1\end{aligned} \tag{28}$$

The functions $\vartheta_n(\tau)$ and $v_n(\tau)$ are sought in the class of polynomials of degree n-1. Subsequently, the reasoning shows the existence of such solutions.

We define subspaces $\Pi_{n,\alpha}$ of spaces $L_{2,\alpha}$. Elements of these subspaces are polynomials of degree n. Also we are taking under considerations the spaces

$$\Pi_{n,\alpha}^0 = \left\{ u \in \Pi_{n,\alpha} \,\middle|\, (u,1)_\alpha = 0 \right\}.$$

We introduce the operators:

$$P_n : \Pi_{n-1,-\frac{1}{2}} \to \Pi_{n-1,-\frac{1}{2}},$$

$$(\mathrm{P}_n u)(\xi) = \sum_{k=1}^{n} \sqrt{1-\left(t_k^n\right)^2}\, u\left(t_k^n\right) \cdot l_{1,n-1,k}(\xi); \tag{29}$$

$$K_n : \Pi_{n-1,-\frac{1}{2}} \to \Pi_{n-1,-\frac{1}{2}},$$

$$(K_n u)(\xi) = \frac{1}{\pi} \int_{-1}^{1} K_n(\xi,\tau) \frac{u(\tau)\,d\tau}{\sqrt{1-\tau^2}}, \quad |\xi| < 1; \tag{30}$$

$$Q_n : \Pi^0_{n-1,-\frac{1}{2}} \to \Pi_{n-2,\frac{1}{2}},$$

$$(Q_n u)(\xi) = \frac{1}{\pi} \int_{-1}^{1} Q_n(\xi,\tau) \frac{u(\tau)\,d\tau}{\sqrt{1-\tau^2}}, \quad |\xi| < 1; \tag{31}$$

$$A_n : \Pi^0_{n-1,-\frac{1}{2}} \to \Pi_{n-2,\frac{1}{2}},$$

$$A_n = \Gamma - c_1 J + Q_n; \tag{32}$$

$$B_n : \Pi_{n,-\frac{1}{2}} \to \Pi_{n,-\frac{1}{2}},$$

$$B_n = I - c_2 (P_n \Lambda_n) + K_n. \tag{33}$$

In notation (29)-(33) the problems (26)-(27) and (28) have the form:

$$A_n \vartheta_n = f_n, \tag{34}$$

$$B_n v_n = g_n. \tag{35}$$

The following estimates hold true [8,20]:

$$\left\| f - f_n \right\|_{L_{2,\frac{1}{2}}} \le \frac{M_1}{n^{\mu+\gamma}}, \tag{36}$$

$$\left\| g - g_n \right\|_{L_{2,-\frac{1}{2}}} \le \frac{M_2}{n^{\psi}}, \tag{37}$$

$$\left\| Q - Q_n \right\|_{L_{2,-\frac{1}{2}} \to L_{2,-\frac{1}{2}}} \le \frac{M_3}{n^{\mu+\gamma}}, \tag{38}$$

$$\left\| K - K_n \right\|_{L_{2,-\frac{1}{2}} \to L_{2,-\frac{1}{2}}} \le \frac{M_4}{n^{\psi}}. \tag{39}$$

These estimates are the consequences of Jackson's Theorems (see Corollary 1 of Th. 2 in [21], p.128). The constants M_1 and M_3 depend only upon μ and γ. Also the constants M_2 and M_4 depend only upon ψ.

Proposition 3

For all natural n the following inequality holds true

$$\left\| A - A_n \right\|_{\Pi^0_{n-1,-\frac{1}{2}} \to L_{2,\frac{1}{2}}} \le \frac{M_3}{n^{\mu+\gamma}}. \tag{40}$$

In (40) constant M_3 depends only upon μ and γ. Besides,

$$\left\| A - A_n \right\|_{\Pi^0_{n-1,-\frac{1}{2}} \to L_{2,\frac{1}{2}}} \to 0, \quad n \to \infty.$$

Proof of Proposition 3

From (38) and equality

$$\left\| A - A_n \right\|_{\Pi^0_{n-1,-\frac{1}{2}} \to L_{2,\frac{1}{2}}} = \left\| Q - Q_n \right\|_{\Pi^0_{n-1,-\frac{1}{2}} \to L_{2,\frac{1}{2}}} \tag{41}$$

it follow that proposition 3 is valid.

Proposition 4

For all natural n the following inequality holds true

$$\left\| B - B_n \right\|_{\Pi_{n-1,-\frac{1}{2}} \to L_{2,\frac{1}{2}}} \le \frac{M^*}{n^{\psi}}. \tag{42}$$

In (42) M^* depends only upon ψ. Besides,

$$\left\| B - B_n \right\|_{\Pi_{n-1,-\frac{1}{2}} \to L_{2,-\frac{1}{2}}} \to 0, \quad n \to \infty. \tag{1}$$

Proof of Proposition 4

If the function $s(\xi)$ is the polynomial of degree n then

$$\sqrt{1-\xi^2} \cdot s(\xi) \in C^{0,\frac{1}{2}}([-1,1]).$$

From that fact and by corollary of Jackson's Theorems [21], p. 128 follow:

$$\left\| Pu - P_n u \right\|_{L_{2,-\frac{1}{2}}} \le \frac{M_5}{\sqrt{n}}, \quad u \in \Pi_{n-1,-\frac{1}{2}}. \tag{43}$$

Hence, the estimation (43) shows that the operator

$$W_n : \Pi_{n-1,-\frac{1}{2}} \to L_{2,\frac{1}{2}}, \quad W_n = P - P_n \tag{44}$$

and operator

$$P_n : \Pi_{n-1,-\frac{1}{2}} \to L_{2,\frac{1}{2}}$$

are bounded. Furthermore

$$\left\| P - P_n \right\|_{\Pi_{n-1,-\frac{1}{2}} \to L_{2,\frac{1}{2}}} \le \frac{M_5}{\sqrt{n}}, \quad \forall n \in N. \tag{45}$$

The following inequality clearly holds

$$\begin{aligned} &\left\| B - B_n \right\|_{\Pi_{n-1,-\frac{1}{2}} \to L_{2,-\frac{1}{2}}} \\ &\le |c_2| \cdot \left\| P - P_n \right\|_{\Pi_{n-1,-\frac{1}{2}} \to L_{2,-\frac{1}{2}}} \cdot \left\| \Lambda \right\|_{\Pi_{n-1,-\frac{1}{2}} \to L_{2,-\frac{1}{2}}} \\ &+ \left\| K - K_n \right\|_{\Pi_{n-1,-\frac{1}{2}} \to L_{2,-\frac{1}{2}}}. \end{aligned} \tag{46}$$

It is well-known [8], [19], that

$$\Lambda : T_0(\tau) \mapsto (-\ln 2) \cdot T_0(\xi), \tag{47}$$

$$T_n(\tau) \mapsto -\frac{T_n(\xi)}{n} \quad , \ n \in N. \tag{48}$$

It follows from (47)-(48) that

$$\|\Lambda\|_{L_{2,-\frac{1}{2}} \to L_{2,-\frac{1}{2}}} = 1. \tag{49}$$

From (39), (45)-(46) and (45) follow that Proposition 4 is valid for

$$M^* = |c_2| \cdot M_5 + M_3.$$

We introduce the variables:

$$M_6 = \left(M_1 \cdot \left\| A^{-1} \right\|^{-1}_{L_{2,-\frac{1}{2}} \to L^0_{2,\frac{1}{2}}} \right)^{\frac{1}{\mu+\gamma}}; \tag{50}$$

$$M_7 = \left(M^* \cdot \left\| B^{-1} \right\|^{-1}_{L_{2,-\frac{1}{2}} \to L_{2,-\frac{1}{2}}} \right)^{\frac{1}{\psi}}; \tag{51}$$

$$M = \max\left(M_6, M_7\right). \tag{52}$$

In the monograph [22] you can see the following theorem.

Theorem 1

Let X and Y be normed linear spaces and let $\tilde{X} \subset X$ and $\tilde{Y} \subset Y$ be finite-dimensional subspaces of the same dimension. We consider two equations.

The equation for exact solution of the problem

$$Au = f \quad u \in X, \quad f \in Y \tag{53}$$

and the equation for the approximate solution of the problem

$$A\tilde{u} = \tilde{f} \quad \tilde{u} \in \tilde{X}, \quad \tilde{f} \in \tilde{Y} \tag{54}$$

where A and $\tilde{A}$ are the linear operators

$$A : X \to Y, \quad \tilde{A} : \tilde{X} \to \tilde{Y}. \tag{55}$$

Assume that:

1) the operator A is invertible and the operator

$$A^{-1} : Y \to X, \tag{56}$$

is bounded,

2) the inequality holds

$$p = \left\|A^{-1}\right\|_{Y \to X} \cdot \left\|A - \tilde{A}\right\|_{\tilde{X} \to Y} < 1. \tag{57}$$

Then

1) for any function $\tilde{f} \in \tilde{Y}$ the equation (54) has the unique solution $\tilde{u}^* \in \tilde{X}$;

2) let $\tilde{u}^* \in X$ be the solution of equation (53) and let

$$\delta \equiv \left\| f - \tilde{f} \right\|_Y,$$

then

$$\left\| u - \tilde{u}^* \right\|_X \le \left\| A^{-1} \right\|_{Y \to X} \cdot (1-p)^{-1} \cdot \left(\delta + p \| f \|_Y \right). \tag{58}$$

The Propositions 1-4 and Theorem 1 lead us to the following result.

Theorem 2

For all natural $n>M$ the following statements hold true.

1) The problems (34) and (35) have a unique solution.

2) $\vartheta_n \in \Pi^0_{n-1,-\frac{1}{2}}, \quad v_n \in \Pi_{n-1,-\frac{1}{2}}$.

3) The sequence $\left\{\vartheta_n\right\}_{n=M}^{\infty}$ converged to the exact solution of the problem (16) in the norm of space $L_{2,-\frac{1}{2}}$. Moreover,

$$\left\|\vartheta_n - \vartheta\right\|_{L_{2,\frac{1}{2}}} \le \frac{M_8}{n^{\mu+\gamma}}, \quad \forall n \in N, \ n > M. \tag{59}$$

4) The sequence $\left\{v_n\right\}_{n=M}^{\infty}$ converged to the exact solution of the problem (17) in the norm of space $L_{2,-\frac{1}{2}}$. Moreover,

$$\left\|v_n - v\right\|_{L_{2,-\frac{1}{2}}} \le \frac{M_9}{n^{\psi}}, \quad \forall n \in N, \ n > M. \tag{60}$$

Proof of Theorem 2

Let's define numbers:

$$p_{1,n} = \left\|A^{-1}\right\|_{L_{2,-\frac{1}{2}} \to L^0_{2,-\frac{1}{2}}} \cdot \left\|A - A_n\right\|_{\Pi^0_{n-1,-\frac{1}{2}} \to L_{2,\frac{1}{2}}}, \tag{61}$$

$$p_{2,n} \equiv \left\|B^{-1}\right\|_{L_{2,-\frac{1}{2}} \to L_{2,-\frac{1}{2}}} \cdot \left\|B - B_n\right\|_{\Pi_{n-1,-\frac{1}{2}} \to L_{2,-\frac{1}{2}}}. \tag{62}$$

From the Propositions 3-4 and the existence of bounded operators A^{-1} and B^{-1} it follows that

$$\left(p_{1,n} \le 1\right) \wedge \left(p_{2,n} \le 1\right) \quad \forall n \in N, n > M. \tag{63}$$

Appealing to the Theorem 1 and (63) it concludes the uniqueness and the existence of the solution of the problems (16)-(17).

Also the estimations

$$\left\|\vartheta_n - \vartheta\right\|_{L_{2,\frac{1}{2}}} \le \left\|A^{-1}\right\|_{L_{2,-\frac{1}{2}} \to L^0_{2,-\frac{1}{2}}} \cdot \left(1 - p_{1,n}\right)^{-1} \cdot \left(\left\| f_n - f \right\|_{L_{2,\frac{1}{2}}} + p_{1,n} \left\| f \right\|_{L_{2,\frac{1}{2}}} \right) \tag{64}$$

$$\left\|v_n - v\right\|_{L_{2,-\frac{1}{2}}} \le \left\|B^{-1}\right\|_{L_{2,-\frac{1}{2}} \to L_{2,-\frac{1}{2}}} \cdot \left(1 - p_{2,n}\right)^{-1} \cdot \left(\left\| g_n - g \right\|_{L_{2,-\frac{1}{2}}} + p_{2,n} \cdot \left\| g \right\|_{L_{2,-\frac{1}{2}}} \right). \tag{65}$$

follow from the statements of Theorem 1 and inequalities (63). By using (36), (37), (64), (65) and Propositions 3-4 we complete the proof of Theorem 2.

4. The Discretization of Approximate System of Integral Equations

As a result of substitution of functions ϑ_n and v_n in the left-hand side of equations (26) and (28) we obtain polynomials of degree n-2 and, n-1 respectively. The right-hand side of equations (26) and (28) are polynomials of the same degrees.

It is well-known that the interpolating polynomial of the degree n for a given set of distinct n+1 nods is unique. From that follows the statement of Proposition 5.

Proposition 5

Let the statements of Theorem 2 hold true. In order that the integral equations (26) - (28) hold true on the set of the continuum |ξ|<1 is sufficient that:

1. the integral equation (26) is carried out on a discrete set of points $\left\{t_{0,j}^n\right\}_{j=1}^{n-1}$;
2. the integral equation (28) is carried out on a discrete set of points $\left\{t_k^n\right\}_{k=1}^{n}$;
3. the additional condition (27) is fulfilled.

By the Proposition 5, the problem (34) is equivalent to the system of equation:

$$\begin{gathered}\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\tau-t_{0,j}^n}\frac{\vartheta_n(\tau)d\tau}{\sqrt{1-\tau^2}}+\frac{1}{\pi}\int_{-1}^{1}Q_n(t_{0,j}^n,\tau)\frac{\vartheta_n(\tau)d\tau}{\sqrt{1-\tau^2}}\\ -\frac{C_1}{\pi}\int_{-1}^{t_{0,j}^n}\frac{\vartheta_n(\tau)d\tau}{\sqrt{1-\tau^2}}=f_n\left(t_{0,j}^n\right),\quad j=1,\dots,n-1;\end{gathered}\tag{66}$$

$$\int_{-1}^{1}\frac{\vartheta_n(\tau)d\tau}{\sqrt{1-\tau^2}}=0\tag{67}$$

and the problem (35) is equivalent to the equation

$$\begin{gathered}v_n\left(t_l^n\right)-\frac{C_2}{\pi}\sqrt{1-\left(t_l^n\right)^2}\cdot\int_{-1}^{1}\ln\left|\tau-t_l^n\right|\frac{v_n(\tau)d\tau}{\sqrt{1-\tau^2}}\\ +\frac{1}{\pi}\int_{-1}^{1}K_n(t_l^n,\tau)\frac{v_n(\tau)d\tau}{\sqrt{1-\tau^2}}=g_n\left(t_l^n\right),\quad l=1,\dots,n.\end{gathered}\tag{68}$$

We take discretization of equations (66)-(68) using the interpolation-type quadrature formulas [8,19]:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\vartheta_n(t)}{t-t_{0j}^n}\frac{dt}{\sqrt{1-t^2}}=\frac{1}{n}\sum_{k=1}^{n}\frac{\vartheta_n(t_k^n)}{t_k^n-t_{0j}^n}\quad, j=1,\dots,n-1;\tag{69}$$

$$\frac{1}{\pi}\int_{-1}^{1}Q_n(\xi,\tau)\frac{\vartheta_n(\tau)d\tau}{\sqrt{1-\tau^2}}=\frac{1}{n}\sum_{k=1}^{n}Q_n\left(\xi,t_k^n\right)\vartheta_n(t_k^n),|\xi|<1;\tag{70}$$

$$\frac{1}{\pi}\int_{-1}^{\xi}\frac{\vartheta_n(\tau)d\tau}{\sqrt{1-\tau^2}}=\frac{1}{\pi n}\sum_{k=1}^{n}\vartheta_n(t_k^n)\cdot\rho_k^n(\xi),\quad |\xi|<1,$$

$$\rho_k^n(\xi)=\left[\pi-\arccos\xi-2\sum_{p=1}^{n-1}\frac{T_p(t_k^n)}{p}U_{p-1}(\xi)\sqrt{1-\xi^2}\right];\tag{71}$$

$$\frac{1}{\pi}\int_{-1}^{1}K_n(\xi,\tau)\frac{v_n(\tau)d\tau}{\sqrt{1-\tau^2}}=\frac{1}{n}\sum_{k=1}^{n}K_n\left(\xi,t_k^n\right)v_n(t_k^n),\ |\xi|<1;\tag{72}$$

$$\begin{gathered}\frac{1}{\pi}\int_{-1}^{1}\ln|\tau-\xi|\frac{v_n(\tau)d\tau}{\sqrt{1-\tau^2}}\\ =-\sum_{k=1}^{n}v_n(t_k^n)\left[\ln 2+2\sum_{p=1}^{n-1}T_p(\xi)\cdot\frac{T_p(t_k^n)}{p}\right]\frac{1}{n},\quad |\xi|<1.\end{gathered}\tag{73}$$

These quadrature formulas are exact for polynomials of degree n-1.

As a result of discretization the system of linear algebraic equations in the unknowns $\vartheta_n(t_k^n)$, $(k=1,\dots,n)$ and $v_n\left(t_l^n\right)$, $(l=1,\dots,n)$ had been obtained. This system has the form

$$\frac{1}{n}\sum_{k=1}^{n}a_{jk}\cdot\vartheta_n(t_k^n)=f_n\left(t_{0,j}^n\right),\quad (j=1,\dots,n-1);\tag{74}$$

$$\frac{1}{n}\sum_{k=1}^{n}\vartheta_n(t_k^n)=0;\tag{75}$$

$$v_n\left(t_l^n\right)+\frac{1}{n}\sum_{k=1}^{n}b_{lk}\cdot v_n(t_k^n)=g_n\left(t_l^n\right),\quad l=1,\dots,n;\tag{76}$$

where

$$a_{jk}=\left(t_k^n-t_{0j}^n\right)^{-1}+Q_n\left(t_{0j}^n,t_k^n\right)-C_1\left[\begin{array}{l}1-\dfrac{\arccos\left(t_{0j}^n\right)}{\pi}\\ -2\displaystyle\sum_{p=1}^{n-1}\frac{T_p(t_k^n)}{\pi p}U_{p-1}(t_{0j}^n)\sqrt{1-\left(t_{0j}^n\right)^2}\end{array}\right];\tag{77}$$

$$\begin{gathered}b_{lk}=K\left(t_l^n,t_k^n\right)+\\ +\frac{C_2}{\pi}\sqrt{1-\left(t_l^n\right)^2}\cdot\left[\ln 2+2\sum_{p=1}^{n-1}T_p(t_l^n)\cdot\frac{T_p(t_k^n)}{p}\right].\end{gathered}\tag{78}$$

The existence and the uniqueness of the solution of this linear algebraic equation system is a consequence of the existence and the uniqueness of the solution of system (66)-(68) and its equivalence to the problems (34)-(35).

After solving the system (66)-(68) of linear algebraic equations we obtain the solutions of the problems (34)-(35) by the formulas:

$$v_n(\tau)=\sum_{k=1}^{n}v_n\left(t_k^n\right)\cdot l_{1,n-1,k}(\tau),\tag{79}$$

$$\vartheta_n(\tau)=\sum_{k=1}^{n}\vartheta_n\left(t_k^n\right)\cdot l_{1,n-1,k}(\tau).\tag{80}$$

5. Conclusions

In this article the justification of the method of numerical solution of the boundary integral equation system (1)-(3) of the problem of wave scattering by superconducting tape had been done. This method is one of the modifications of method of discrete singularities. The convergence of the approximations to the exact solution is guaranteed by propositions proved in this article. Also, the rate of convergence of the approximate solutions to the exact solution had been found.

6. List of Abbreviations

$C^{\mu,\gamma}([-1,1])$ - the Hölder space. It consists of functions defined on [-1,1], which have the properties:

1. the functions have continuous derivatives up to order μ;
2. the μ-th derivatives of the functions are Hölder continuous with exponent γ, $(0<\gamma\leq 1)$.

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