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Approximation of Conjugate Series of the Fourier Series of a Function of Class W(L^p,ξ(t)) by Product Means

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Abstract In this paper a theorem on degree of approximation of a function $f \in W\left(L^p, \xi(t)\right)$ by product summability $(E,q)\left(\overline{N}, p_n\right)$ of the conjugate series of Fourier series associated with f has been established.

Keywords: *degree of approximation,* $W\left(L^p, \xi(t)\right)$ *class of function,* (E,q) *mean,* $\left(\overline{N}, p_n\right)$ *mean,* $(E,q)\left(\overline{N}, p_n\right)$ *product mean, Fourier series, conjugate series, Lebesgue integral.*

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1. Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive real numbers such that

$$P_n = \sum_{\upsilon=0}^{n} p_\upsilon \to \infty,\ n \to \infty, P_{-i} = p_{-i} = 0, i \ge 0 \quad (1.1)$$

The sequence –to-sequence transformation

$$t_n = \frac{1}{P_n}\sum_{\upsilon=0}^{n} p_\upsilon s_\upsilon \quad (1.2)$$

defines the sequence $\{t_n\}$ of the $\left(\overline{N}, p_n\right)$ -mean of the sequence $\{s_n\}$ generated by the sequence of coefficient $\{p_n\}$. If

$$t_n \to s, as \quad n \to \infty \quad (1.3)$$

then the series $\sum a_n$ is said to be $\left(\overline{N}, p_n\right)$ summable to s.

The conditions for regularity of $\left(\overline{N}, p_n\right)$- summability are easily seen to be

$$\begin{cases} (i)\ P_n \to \infty, as\ n \to \infty, \\ (ii)\ \sum_{i=0}^{n} p_i \le C\left|P_n\right|, as\ n \to \infty. \end{cases} \quad (1.4)$$

The sequence to-sequence transformation, [1]

$$T_n = \frac{1}{(1+q)^n}\sum_{\upsilon=0}^{n}\binom{n}{\upsilon} q^{n-\upsilon} s_\upsilon \quad (1.5)$$

defines the sequence $\{T_n\}$ of the (E,q) mean of the sequence $\{s_n\}$. If

$$T_n \to s, as \quad n \to \infty \quad (1.6)$$

then the series $\sum a_n$ is said to be (E,q) summable to s.

Clearly (E,q) method is regular. Further, the (E,q) transform of the $\left(\overline{N}, p_n\right)$ transform of $\{s_n\}$ is defined by

$$\begin{aligned} \tau_n &= \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} t_k \\ &= \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_\upsilon s_\upsilon\right\} \end{aligned} \quad (1.7)$$

If

$$\tau_n \to s, as \quad n \to \infty \quad (1.8)$$

then $\sum a_n$ is said to be $(E,q)\left(\overline{N}, p_n\right)$-summable to s.

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi,\pi)$. Then the Fourier series associated with f at any point x is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \quad (1.9)$$

and the conjugate series of the Fourier series (1.9) is

$$\sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x) \quad (1.10)$$

Let $\bar{s}_n(f;x)$ be the n-th partial sum of (1.10). The L_∞-norm of a function $f: R \to R$ is defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in R\} \quad (1.11)$$

and the L_υ-norm is defined by

$$\|f\|_\upsilon = \left(\int_0^{2\pi} |f(x)|^\upsilon\right)^{\frac{1}{\upsilon}}, \upsilon \geq 1 \quad (1.12)$$

The degree of approximation of a function $f: R \to R$ by a trigonometric polynomial $P_n(x)$ of degree n under norm $\|\ .\ \|_\infty$ is defined by [6]

$$\|P_n - f\|_\infty = \sup\{|p_n(x) - f(x)| : x \in R\} \quad (1.13)$$

and the degree of approximation $E_n(f)$ of a function $f \in L_\upsilon$ is given by

$$E_n(f) = \min_{P_n} \|P_n - f\|_\upsilon \quad (1.14)$$

This method of approximation is called Trigonometric Fourier approximation.

A function $f(x) \in Lip\,\alpha$ if

$$|f(x+t) - f(x)| = O(|t|^\alpha), 0 < \alpha \leq 1, t > 0 \quad (1.15)$$

and $f(x) \in Lip(\alpha, r)$, for $0 \leq x \leq 2\pi$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O(|t|^\alpha), \quad (1.16)$$

$$0 < \alpha \leq 1, r \geq 1, t > 0$$

For a given positive increasing function $\xi(t)$, the function $f(x) \in Lip(\xi(t), r)$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O(\xi(t)), \quad (1.17)$$

$$r \geq 1, t > 0.$$

For a given positive increasing function $\xi(t)$ and an integer $p > 1$ the function $f(x) \in W(L^p, \xi(t))$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^p (\sin x)^{p\beta} dx\right)^{\frac{1}{p}} = O(\xi(t)), \quad (1.18)$$

$$\beta \geq 0.$$

We use the following notation throughout this paper:

$$\psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}, \quad (1.19)$$

and

$$\bar{K}_n(t) = \frac{1}{\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{\frac{1}{P_k} \sum_{\upsilon=0}^{k} p_\upsilon \frac{\cos\frac{t}{2} - \cos\left(\upsilon + \frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\} \quad (1.20)$$

Further, the method $(E,q)(\bar{N}, p_n)$ is assumed to be regular throughout the paper.

2. Known Theorems

Dealing with the degree of approximation by the product, Misra et al [2] proved the following theorem using $(E,q)(\bar{N}, p_n)$ mean of conjugate series of Fourier series:

2.1. Theorem

If f is a $2\pi-$ periodic function of class $Lip\,\alpha$, then degree of approximation by the product $(E,q)(\bar{N}, p_n)$ summability means of the conjugate series (1.10) of the Fourier series (1.9) is given by $\|\tau_n - f\|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right), 0 < \alpha < 1$ where τ_n is as defined in (1.7).

Recently *Misra* et al [3] established a theorem on degree of approximation by the product mean $(E,q)(\bar{N}, p_n)$ of the conjugate series of Fourier series of a function of class $Lip(\alpha, r)$. They prove:

2.2. Theorem

If f is a $2\pi-$ periodic function of class $Lip(\alpha, r)$, then degree of approximation by the product $(E,q)(\bar{N}, p_n)$ means of the conjugate series (1.10) of the Fourier series (1.9) is given by $\|\tau_n - f\|_\infty = O\left(\frac{1}{(n+1)^{\alpha+\frac{1}{r}}}\right), 0 < \alpha < 1, r \geq 1$, where τ_n is as defined in (1.7).

Extending to the function of the class $Lip(\xi(t), r)$, very recently *Misra* et al [4] have proved a theorem on degree of approximation by the product mean $(E,q)(\bar{N}, p_n)$ of the conjugate series of the Fourier series of a function of class $Lip(\xi(t), r)$. They prove:

2.3. Theorem

Let $\xi(t)$ be a positive increasing function and f a $2\pi-$ Periodic function of the class $Lip(\xi(t),r)$, $r\geq 1, t>0$. Then degree of approximation by the product $(E,q)(\overline{N},p_n)$ summability means on the conjugate series (1.10) of the Fourier series (1.9) is given by $\|\tau_n - f\|_\infty = O\left((n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right)$, $r\geq 1.$, where τ_n is as defined in (1.7).

Further extending to the class of functions $W(L^p,\xi(t))$, $p>1$, in the present paper, we establish the following theorem:

3. Main result

3.1. Theorem

Let $\xi(t)$ be a positive increasing function and f a $2\pi-$ Periodic function of the class $W(L^p,\xi(t))$, $p>1, t>0$. Then degree of approximation by the product $(E,q)(\overline{N},p_n)$ summability means on the conjugate series (1.10) of the Fourier series (1.9) is given by

$$\|\tau_n - f\|_r = O\left((n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right), r\geq 1 \quad (3.1.1),$$

provided

$$\left(\int_0^{\frac{1}{n+1}}\left(\frac{t\psi(t)\sin^\beta t}{\xi(t)}\right)^r dt\right)^{\frac{1}{r}} = O\left(\frac{1}{n+1}\right) \quad (3.1.2)$$

and

$$\left(\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^r dt\right)^{\frac{1}{r}} = O\left((n+1)^\delta\right) \quad (3.1.3)$$

hold uniformly in x with $\frac{1}{r}+\frac{1}{s}=1$, where δ is an arbitrary number such that $s(1-\delta)-1>0$ and τ_n is as defined in (1.7).

4. Required Lemmas

We require the following Lemmas to prove the theorem.

LEMMA 4.1:

$$|\overline{K}_n(t)| = O(n) \quad , 0\leq t\leq \frac{1}{n+1}.$$

Proof:

For $0\leq t\leq \frac{1}{n+1}$, we have sin nt $\leq$ n sin t then

$$|\overline{K}_n(t)| = \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\cos\upsilon t.\cos\frac{t}{2}+\sin\upsilon t.\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\left(\frac{\cos\frac{t}{2}\left(2\sin^2\upsilon\frac{t}{2}\right)}{\sin\frac{t}{2}}+\sin\upsilon t\right)\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\left(O\left(2\sin\upsilon\frac{t}{2}\sin\upsilon\frac{t}{2}\right)+\upsilon\sin t\right)\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\left(O(\upsilon)+O(\upsilon)\right)\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{O(k)}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\right|$$

$$= O(n)$$

This proves the lemma.

LEMMA 4.2:

$$|\overline{K}_n(t)| = O\left(\frac{1}{t}\right), \ for \ \frac{1}{n+1}\leq t\leq\pi.$$

Proof:

For $\frac{1}{n+1}\leq t\leq\pi$, by Jordan's lemma, we have $\sin\left(\frac{t}{2}\right)\geq\frac{t}{\pi}$.

Then

$$|\overline{K}_n(t)| = \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$= \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon \frac{\cos\frac{t}{2}-\cos\upsilon t.\cos\frac{t}{2}+\sin\upsilon t.\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}\frac{\pi}{2t}p_\upsilon\left(\cos\frac{t}{2}\left(2\sin^2\upsilon\frac{t}{2}\right)+\sin\upsilon\frac{t}{2}.\sin\frac{t}{2}\right)\right\}\right|$$

$$\leq \frac{\pi}{2\pi(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\right\}\right|$$

$$= \frac{1}{2(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\right\}\right|$$

$$= \frac{1}{2(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\right|$$

$$= O\left(\frac{1}{t}\right)$$

This proves the lemma.

5. Proof of Main Theorem

Using Riemann–Lebesgue theorem, for the n-th partial sum $\overline{s}_n(f;x)$ of the conjugate Fourier series (1.10) of $f(x)$ and following *Titchmarch* [5], we have

$$\overline{s_n}(f;x) - f(x) = \frac{2}{\pi}\int_0^\pi \psi(t)\frac{\cos\frac{t}{2}-\sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}dt,$$

Using (1.2), the $(\overline{N},p_n)$ transform of $\overline{s_n}(f;x)$ is given by

$$t_n - f(x) = \frac{2}{\pi P_n}\int_0^\pi \psi(t)\sum_{k=0}^{n}p_k\frac{\cos\frac{t}{2}-\sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}dt\ ,$$

Denoting the $(E,q)(\overline{N},p_n)$ transform of $\overline{s_n}(f;x)$ by τ_n, we have

$$\|\tau_n - f\| = \frac{2}{\pi(1+q)^n}\int_0^\pi \psi(t)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\sin\left(\upsilon+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\right\}dt \tag{5.1}$$

$$= \int_0^\pi \psi(t)\ \overline{K_n}(t)\,dt = \left\{\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right\}\psi(t)\ \overline{K_n}(t)\ dt$$

$$= I_1 + I_2,\ say$$

Now

$$|I_1| = \left|\frac{2}{\pi(1+q)^n}\int_0^{1/n+1}\psi(t)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\right\}dt\right|$$

$$\leq \left|\int_0^{\frac{1}{n+1}}\psi(t)\ \overline{K_n}(t)\,dt\right|$$

$$\leq \left(\int_0^{\frac{1}{n+1}}\left|\frac{t\,\psi(t)\sin^\beta t}{\xi(t)}\right|^r dt\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{n+1}}\left|\frac{\xi(t)\overline{K}_n(t)}{t\sin^\beta t}\right|^s dt\right)^{\frac{1}{s}}$$

where $\frac{1}{r}+\frac{1}{s}=1$, using Hölder's inequality

$$= O(1)\left(\int_0^{\frac{1}{n+1}}\left(\frac{\xi(t)}{t^{1+\beta}}\right)^s dt\right)^{\frac{1}{s}},$$

$$using\ Lemma 4.1\ and\ (3.1.2)$$

$$= O\left(\xi\left(\frac{1}{n+1}\right)\right)\left(\int_0^{\frac{1}{n+1}}\frac{dt}{t^{(1+\beta)s}}\right)^{\frac{1}{s}} \tag{5.2}$$

$$= O\left(\xi\left(\frac{1}{n+1}\right)\right)O\left((n+1)^{-\frac{1}{s}+1+\beta}\right)$$

$$= O\left(\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+\frac{1}{r}}\right)$$

Next

$$|I_2| \le \left(\int_{\frac{1}{n+1}}^{\pi} \left| \frac{t^{-\delta}\psi(t)\sin^{\beta} t}{\xi(t)} \right|^r dt \right)^{\frac{1}{r}} \left(\int_{\frac{1}{n+1}}^{\pi} \left| \frac{\xi(t)\bar{K}_n(t)}{t^{-\delta}\sin^{\beta} t} \right|^s dt \right)^{\frac{1}{s}}$$

where $\frac{1}{r}+\frac{1}{s}=1$, using Hölder's inequality

$$= O((n+1)^{\delta}) \left(\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\xi(t)}{t^{\beta+1-\delta}} \right)^s dt \right)^{\frac{1}{s}}$$, using Lemma 4.2 and

$$(3.1.3)\ = O((n+1)^{\delta}) \left(\int_{\frac{1}{\pi}}^{n+1} \left(\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-\beta-1}} \right)^s \frac{dy}{y^2} \right)^{\frac{1}{s}}$$

since $\xi(t)$ is a positive increasing function, so is $\xi(1/y)/(1/y)$. Using second mean value theorem we get

$$= O((n+1)^{1+\delta}\, \xi\left(\frac{1}{n+1}\right)) \left(\int_{\varepsilon}^{n+1} \frac{dy}{y^{s(\delta-\beta-1)+2}} \right)^{\frac{1}{s}},$$

$$\text{for some } \frac{1}{\pi} \le \varepsilon \le n+1$$

$$O\left((n+1)^{1+\delta} \xi\left(\frac{1}{n+1}\right) \right) O\left((n+1)^{\beta+1-\delta-\frac{1}{s}} \right) \qquad (5.3)$$

$$= O\left((n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right) \right)$$

Then from (5.2) and (5.3), we have

$$|\tau_n - f(x)| = O\left((n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right) \right), \text{ for } r \ge 1$$

$$\|\tau_n - f(x)\|_r = \left(\int_0^{2\pi} O\left((n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right) \right)^r dx \right)^{\frac{1}{r}}, r \ge 1.$$

$$= O\left((n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right) \right) \left(\int_0^{2\pi} dx \right)^{\frac{1}{r}}$$

$$= O\left((n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right) \right)$$

This completes the proof of the theorem.

6. Corollaries

Following corollaries can be derived from the main theorem.

Corollary 6.1: The degree of approximation of a function f belonging to the class $Lip(\alpha, r), 0 < \alpha \le 1, r \ge 1$ is given by

$$\|\tau_n - f\|_r = O\left((n+1)^{-\alpha+\frac{1}{r}} \right) .$$

Proof: The corollary follows by putting $\beta = 0$ and $\xi(t) = t^{\alpha}$ in the main theorem.

Corollary 6.2: The degree of approximation of a function f belonging to the class $Lip(\alpha), 0 < \alpha \le 1$ is given by

$$\|\tau_n - f\|_{\infty} = O\left((n+1)^{-\alpha} \right) .$$

Proof: The corollary follows by letting $r \to \infty$ in corollary 6.1.

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