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Total Domination Subdivision Number in Strong Product Graph

P. Jeyanthi[1,*], G. Hemalatha[2], B. Davvaz[3]

[1]Department of Mathematics, Govindammal Aditanar College for Women, Tiruchendur, Tamil Nadu, India
[2]Department of mathematics, Shri Andal Alagar College of Engineering, Mamandur, Kancheepuram, Tamil Nadu, India
[3]Department of Mathematics, Yazd University, Yazd, Iran
*Corresponding author: jeyajeyanthi@rediffmail.com

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Abstract A set *D* of vertices in a graph G(V,E) is called a total dominating set if every vertex $v \in V$ is adjacent to an element of *D*. The domination subdivision number of a graph *G* is the minimum number of edges that must be subdivided in order to increase the domination number of a graph. In this paper, we determine the total domination number for strong product graph and establish bounds on the total domination subdivision number for strong product graph.

Keywords: *total dominating set, strong product graph, total domination number*

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1. Introduction

Let $G=(V,E)$ be a simple graph on the vertex set *V*. In a graph *G*, a set $D \subseteq V$ is a dominating set of G if every vertex in $V-D$ is adjacent to some vertex in *D*. The domination number of a graph *G* is the minimum size of a dominating set of vertices in *G*, denoted by $\gamma(G)$. A thorough study of fundamental domination appears in [2]. The concept of total domination in graphs was introduced by Cokayne, Dawes and Hedetemini [1]. A set of vertices in a graph $G(V,E)$ is called a total dominating set if every vertex $v \epsilon V$ is adjacent to an element of S. The total domination number of some cartesian products of two paths P_n and P_m, are investigated in [8,9]. The values of $\gamma_t(P_{n,m})$ for *n* = 2, 3, 4 are determined in [8], and for *n* = 5, 6 are determined in [9].

Let *G* and *H* be the two graphs with the set of vertices $U=\{u_1,u_2,\ldots,u_n\}$ and $V=\{v_1,v_2,\ldots,v_m\}$ respectively. The strong product of G and H is the graph $G \boxtimes H$ formed by the vertices $V=\{(u_i,v_j): 1 \le i \le n, 1 \le j \le m\}$ and two vertices (u_i,v_j) and (u_s,v_t) are adjacent in $G \boxtimes H$ if and only if $(u_i = u_s \text{and} v_j adj v_t)$, $(u_i adj u_s \text{and} v_j = v_t)$ or $(u_i adj u_s \text{and} v_j adj v_t)$. Domination number is rather difficult to construct graphs with large value of $Sd\gamma(G), Sd\gamma_t(G)$ and the first conjecture on this subject was that $Sd\gamma(G) \le 3$ for every G [10]. The concept of total domination subdivision number $Sd\gamma_t(G)$ was due to Haynes et al [3]. Haynes et.al [4] studied the total domination subdivision number of graphs, for instance, they showed that $Sd\gamma_t(G) \le 3$ holds for a graph having three or more pair wise adjacent simplicial vertices. In [5] the authors proved the total domination subdivision number of trees. Constant upper bounds on the total domination number for several families of graphs were determined in [3]. Nasrin Soltankhah showed that for any $m,n \ge 3$, $Sd\gamma_t(G) \le 3$ [7]. The behaviour of several graph parameters in product graphs has become an interesting topic of research [6]. G. Yero and J. A. Rodr´ıguez-Vel´azquez [11] proved that for any $m,n \ge 2, \gamma(P_m \boxtimes P_n) = \left\{ \left\lceil \frac{m}{3} \right\rceil \left\lceil \frac{n}{3} \right\rceil \right\}$. In this paper is to establish a bound of this type on $Sd\gamma_t(P_n \boxtimes P_m)$.

2. Main Result

In this section, we first determine the value of the total domination number of $P_m \boxtimes P_n$ for $m \le 4$. Since $P_1 \boxtimes P_n \simeq P_n$, we have:

Proposition 2.1. For any $n \ge 2$, we have

$$\gamma_t(P_1 \boxtimes P_n) = \begin{cases} \frac{n}{2} & \text{if} \quad n \equiv 0 \pmod 4 \\ \frac{n}{2}+1 & \text{if} \quad n \equiv 1,2,3 \pmod 4). \end{cases}$$

Lemma 2.2. We have $\gamma_t(P_1 \boxtimes P_n) = \begin{cases} 2 & \text{for } n = 2,3,4 \\ 3 & \text{for } n = 5. \end{cases}$

Proof: To obtain totally dominate the vertices (u_2, v_1) and (u_2, v_2), we need two vertices (u_1, v_1) and (u_1, v_2). Therefore, $\gamma_t(P_2 \boxtimes P_2) = 2$. Last column of $P_2 \boxtimes P_3$ is totally dominated by $P_2 \boxtimes P_2$. Hence, $\gamma_t(P_2 \boxtimes P_3) = 2$. Let us consider $P_2 \boxtimes P_3$ as block B. The last three columns of $P_2 \boxtimes P_4$ is block B. The first column of $P_2 \boxtimes P_4$ can be totally dominated by B. Hence, $\gamma_t(P_2 \boxtimes P_4) = 2$. In $P_2 \boxtimes P_5$, to totally dominate a vertex (u_1, v_4), we need one vertex among $\{(u_2, v_3), (u_2, v_4), (u_1, v_4)\}$. Hence, $\gamma_t(P_2 \boxtimes P_5) = 3$. The first three columns of $P_2 \boxtimes P_5$ is block B and also the last column of $P_2 \boxtimes P_5$ is totally dominated by the fourth column. This completes the proof.

Proposition 2.3. For any $n \geq 6$, we have

$$\gamma_t(P_2 \boxtimes P_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 4 \\ \frac{n+1}{2} & \text{if } n \equiv 1,3 \pmod 4 \\ \frac{n+2}{2} & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof:

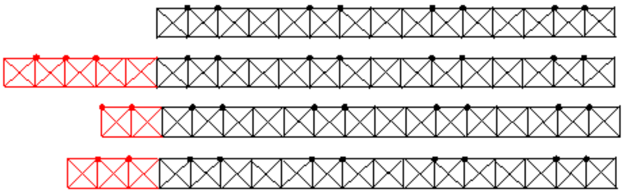

Figure 1. $P_2 \boxtimes P_n$

Let S be a total dominating set of $P_2 \boxtimes P_n$. Since $\gamma_t(P_2 \boxtimes P_4) = 2$. Suppose that C_j, C_{j+1}, C_{j+2} and C_{j+3} are four consecutive columns of $P_2 \boxtimes P_n$. To totally dominate the vertices (u_1, v_{j+1}) and (u_1, v_{j+2}), we need one vertex among $\left\{ \begin{matrix} (u_1, v_j), (u_1, v_{j+2}), (u_2, v_j), \\ (u_2, v_{j+1}), (u_2, v_{j+2}) \end{matrix} \right\}$ and one more vertex among $\left\{ \begin{matrix} (u_1, v_{j+1}), (u_1, v_{j+3}), (u_2, v_{j+1}), \\ (u_2, v_{j+2}), (u_2, v_{j+3}) \end{matrix} \right\}$.

Now, to describe the total dominating set S, we consider block $B \simeq P_2 \boxtimes P_4$ and $\cap B = \{(u_1, v_2), (u_1, v_3)\}$. If $n \equiv 0 \pmod 4$, then $P_2 \boxtimes P_n$ can be partitioned with $\frac{n}{4}$ number of blocks B. If $n \equiv 1 \pmod 4$, then $P_2 \boxtimes P_n$ can be partitioned with $\frac{n-5}{4}$ number of blocks B, plus a block $B' \simeq P_2 \boxtimes P_5$ and $S \cap B' = \{(u_1, v_2), (u_1, v_3), (u_1, v_4)\}$. If $n \equiv 2 \pmod 4$, then $P_2 \boxtimes P_n$ can be partitioned with $\frac{n-2}{4}$ number of blocks B, plus a block $B' \simeq P_2 \boxtimes P_2$ and $S \cap B' = \{(u_1, v_1), (u_1, v_2)\}$. If $n \equiv 3 \pmod 4$, then $P_2 \boxtimes P_n$ can be partitioned with $\frac{n-3}{4}$ number of blocks B, plus a block $B' \simeq P_2 \boxtimes P_3$ and $S \cap B' = \{(u_1, v_1), (u_1, v_2)\}$. This completes the proof.

Proposition 2.4. For $n \geq 3$, the total domination number of $P_2 \boxtimes P_n$ and $P_3 \boxtimes P_n$ are same.

Proof: Last two rows of $P_3 \boxtimes P_n$ is considered as blocks $B \simeq P_2 \boxtimes P_n$ and the first row of $P_3 \boxtimes P_n$ is totally dominated by B, which completes the proof.

Observation 2.5. For $n \geq 1$, we have $P_k \boxtimes P_n \simeq P_n \boxtimes P_k$.

Proposition 2.6. For any $n \geq 4$, we have

Proof:

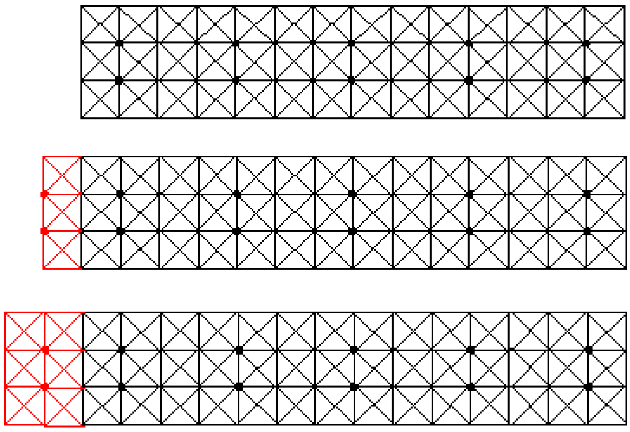

Figure 2. $P_4 \boxtimes P_n$

Suppose that S is a total dominating set of $P_4 \boxtimes P_n$. Let us consider $P_4 \boxtimes P_3$ as block. Since the total domination number of $P_4 \boxtimes P_3$ is 2. We have $S \cap B = \{(u_2, v_2), (u_3, v_2)\}$. Let C_j, C_{j+1} and C_{j+2} be three consecutive columns of $P_4 \boxtimes P_n$. To totally dominate the vertices (u_2, v_{j+1}) and (u_3, v_{j+1}), we need one vertex among $\left\{ \begin{matrix} (u_1, v_j), (u_1, v_{j+1}), (u_1, v_{j+2}), \\ (u_2, v_j), (u_2, v_{j+2}), (u_3, v_j), \\ (u_3, v_{j+1}), (u_3, v_{j+2}) \end{matrix} \right\}$ and one more vertex among

$$\left\{ \begin{matrix} (u_2, v_j), (u_2, v_{j+1}), (u_2, v_{j+2}), (u_3, v_j), \\ (u_3, v_{j+2}), (u_4, v_j), (u_4, v_{j+1}), (u_4, v_{j+2}) \end{matrix} \right\}.$$

Now, to describe our total dominating set S, we consider block $B \simeq P_4 \boxtimes P_3$. If $n \equiv 0 \pmod 3$, then $P_4 \boxtimes P_n$ can be partitioned with $\frac{n}{3}$ number of blocks B. If $n \equiv 1 \pmod 3$, then $P_4 \boxtimes P_n$ can be partitioned with $\frac{n-1}{3}$

number of blocks B, plus a block $B' \simeq P_4 \boxtimes P_1$ and $S \cap B' = \{(u_2,v_1),(u_3,v_1)\}$. If $n \equiv 2 \pmod 3$, then $P_4 \boxtimes P_n$ can be partitioned with $\frac{n-2}{3}$ number of blocks *B*, plus a block $B' \simeq P_4 \boxtimes P_2$ and $S \cap B' = \{(u_2,v_1),(u_3,v_1)\}$. This completes the proof.

Theorem 2.7. We have

$$\gamma_t(P_m \boxtimes P_n) = \begin{cases} \left(\frac{m}{2}\right)\frac{n}{3} & \text{if } \ m \equiv 0 \pmod 4 \\ \left(\frac{m}{2}+1\right)\frac{n}{3} & \text{if } \ m \equiv 1,2,3 \pmod 4. \end{cases}$$

Proof:

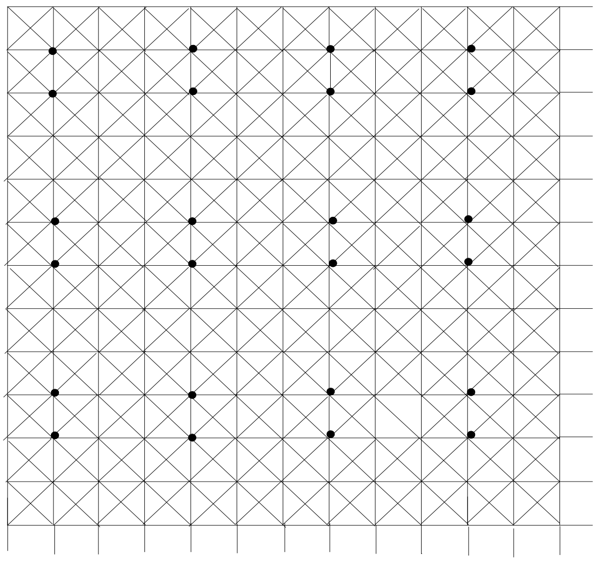

Figure 3. $P_m \boxtimes P_n$

Let *S* be a total dominating set of $P_m \boxtimes P_n$. Since each column of $P_m \boxtimes P_n$ is isomorphic to $P_m \boxtimes P_1$. By Proposition 2.1 and Observation 2.5, we have

$$\gamma_t(P_m \boxtimes P_1) = \begin{cases} \frac{m}{2} & \text{if } \ m \equiv 0 \pmod 4 \\ \frac{m}{2}+1 & \text{if } \ m \equiv 1,2,3 \pmod 4. \end{cases}$$

Let us consider $P_m \boxtimes P_1$ as block. Now to describe our total dominating set *S*, we consider block $B \simeq P_m \boxtimes P_1$. If $m \equiv 0 \pmod 4$, then $P_m \boxtimes P_n$ can be partitioned with $\left\lceil \frac{n}{3} \right\rceil$ number of blocks *B*. By Proposition 2.1 and Observation 2.5, we obtain $\gamma_t(P_m \boxtimes P_n) = \left(\frac{m}{2}\right)\left\lceil \frac{n}{3} \right\rceil$. If $m \equiv 1,2,3, \pmod 4$, then $P_m \boxtimes P_n$ can be partitioned with $\left\lceil \frac{n}{3} \right\rceil$ number of blocks *B*. By Proposition 2.1 and Observation 2.5, we obtain $\gamma_t(P_m \boxtimes P_n) = \left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right)\left\lceil \frac{n}{3} \right\rceil$. This completes the proof.

3. Subdivision Number for the Strong Product Graph

Proposition 2.8. For $P_2 \boxtimes P_2$, we have $Sd\gamma_t(P_2 \boxtimes P_2) = 2$.

Proof: Let *S* be a total dominating set of $P_2 \boxtimes P_2$ and $S = \{(u_1,v_1),(u_1,v_2)\}$. Let $(P_2 \boxtimes P_2)'$ be obtain from $P_2 \boxtimes P_2$ by subdividing an edge $(u_1,v_1)(u_2,v_1)$ and adding new vertex called *x*. Now, there is no change in total domination number, i.e, $\gamma_t(P_2 \boxtimes P_2)' = \gamma_{tr}(P_2 \boxtimes P_2)$.

Let $(P_2 \boxtimes P_2)''$ be obtain from $P_2 \boxtimes P$ by subdividing the edges $(u_1,v_1)(u_2,v_1)$, $(u_2,v_1)(u_1,v_2)$ and adding new vertices respectively called *x* and *y*. So, we need three vertices for totally domination. Therefore, $S'' = \{(u_1,v_1),(u_1,v_2),(u_2,v_2)\}$.

Thus, $\gamma_t(P_2 \boxtimes P_2)'' = 3$. By Lemma.2.2, we obtain that the total domination number of $(P_2 \boxtimes P_2)''$ is greater than the total domination number of $P_2 \boxtimes P_2$. This completes the proof.

Proposition 2.9. For $B \simeq P_2 \boxtimes P_2$, we have $S \cap B = \{(u_1,v_1),(u_1,v_2)\}$.

Proof: To describe our total dominating set *S*, we consider block $B \simeq P_2 \boxtimes P_2$ and $S \cap B = \{(u_1,v_1),(u_1,v_2)\}$. Since $Sd\gamma_t(P_2 \boxtimes P_2) = 2$. Thus, we have $Sd\gamma_t(P_2 \boxtimes P_3) = 2$.

Proposition 2.10. For $P_2 \boxtimes P_4$, we have $Sd\gamma_t(P_2 \boxtimes P_4) = 1$.

Proof: Let *S* be a total dominating set of $P_2 \boxtimes P_4$ and $S = \{(u_1,v_2),(u_1,v_3)\}$. Let $(P_2 \boxtimes P_4)'$ be obtain from $P_2 \boxtimes P_4$ by subdividing an edge $(u_2,v_1)(u_1,v_2)$ and adding new vertex called *x*. To totally dominate (u_2,v_1), we need one vertex among $\{x,(u_1,v_1),(u_2,v_2)\}$. Therefore, $S' = \{(u_1,v_1),(u_1,v_2),(u_1,v_3)\}$. Thus, $\gamma_t(P_2 \boxtimes P_4)' = 3$.

By Lemma 2.2, we obtain that the total domination number of $(P_2 \boxtimes P_4)'$ is greater than the total domination number of $P_2 \boxtimes P_4$. This completes the proof.

Proposition 2.11. For $P_2 \boxtimes P_5$, we have $Sd\gamma_t(P_2 \boxtimes P_5) = 1$.

Proof: Let *S* be a total dominating set of $P_2 \boxtimes P_5$ and $S = \{(u_1,v_2),(u_1,v_3),(u_1,v_4)\}$. Let $(P_2 \boxtimes P_5)'$ be obtain from $P_2 \boxtimes P_5$ by subdividing an edge $(u_2,v_1)(u_1,v_2)$ and adding new vertex called *x*. To totally dominate (u_2,v_1), we need one vertex among $\{x,(u_1,v_1),(u_2,v_2)\}$. Therefore, $S' = \{(u_1,v_1),(u_1,v_2),(u_1,v_3),(u_1,v_4)\}$. Thus, $\gamma_t(P_2 \boxtimes P_5)' = 4$. By Lemma 2.2, we obtain that the total domination number of $(P_2 \boxtimes P_5)'$ is greater than the total domination number of $P_2 \boxtimes P_5$. This completes the proof.

Theorem 2.8. For $n \geq 4$, we have $Sd\ \gamma_t(P_2 \boxtimes P_n) = 1$.

Proof: To describe our total dominating set S, we consider block $B \simeq P_2 \boxtimes P_4$ and $S \cap B = \{(u_1, v_2), (u_1, v_3)\}$. Since $Sd\gamma_t(P_2 \boxtimes P_4) = 1$ and by Proposition 2.3, we have $Sd\gamma_t(P_2 \boxtimes P_n) = 1$.

Theorem 2.9. For $n \geq 3$, subdivision number of $P_2 \boxtimes P_n$ and $P_3 \boxtimes P_n$ are same.

Proof: Last two rows of $P_3 \boxtimes P_n$ is considered as blocks $B \simeq P_2 \boxtimes P_n$ and the first row of $P_3 \boxtimes P_n$ is totally dominated by B, which completes the proof.

Theorem 2.10. For $n \geq 4$, we have $Sd\ \gamma_t(P_4 \boxtimes P_n) = 1$.

Proof: To describe our total dominating set S, we consider block $B \simeq P_4 \boxtimes P_3$ and $S \cap B = \{(u_2, v_2), (u_3, v_2)\}$. By Theorem 2.9, we have $Sd\gamma_t(P_4 \boxtimes P_3) = 1$. Thus, $Sd\gamma_t(P_4 \boxtimes P_n) = 1$.

Theorem 2.11. For $n \geq 4$, we have $Sd\ \gamma_t(P_m \boxtimes P_n) = 1$.

Proof: To describe our total dominating set S, we consider block $B \simeq P_4 \boxtimes P_n$. By Theorem 2.10, we have $Sd\gamma_t(P_4 \boxtimes P_n) = 1$. Thus, $Sd\gamma_t(P_m \boxtimes P_n) = 1$.

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