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Inference on P(X<Y) for Extreme Values

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Abstract The article considers the problem of R=P(X<Y), when X and Y belong to independently distributed two extreme value distributions. Maximum likelihood estimate of R has been found out and the estimates assuming different distributions have been compared for complete samples. Lower confidence limits of R have been found out by Delta method and bootstrap method. The Bayes estimate of R has also been calculated using MCMC approach.

Keywords: *Bayes estimate, delta method, Lower Confidence Limit, Metropolis-Hastings algorithm*

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1. Introduction

Inference of $R = P(X < Y)$ is used in various applications e.g. stress-strength reliability, statistical tolerancing, measuring demand-supply system performance, measuring heritability of a genetic trait, bio-equivalence study etc. It is observed especially in military and medical sciences that the system designers, reliability practitioners and experts in medical field seek to assign high probability to the event that the system/unit remains operable at its minimum strength encountering maximum stress at that time epoch. To meet this objective, it seems reasonable to define $R = P(X < Y)$ with $X = \max(X_1, X_2, ...X_k)$ and $Y = \min(Y_1, Y_2, ...Y_k)$.

Now the cumulative distribution function of X is given by

$$\begin{aligned} F_X(x) &= P(X = \max(X_1, X_2, ...X_k) < x) \\ &= P(X_1 < x, X_2 < x, ...X_k < x) \\ &= \prod_{i=1}^{k} P(X_i < x) \text{ if } X_i^{'}s \text{ are independent} \\ &= \left[P\left(X_1 < x\right)\right]^k \end{aligned}$$

if $X_i^{'}s$ are independent and identical, and the cumulative distribution function of Y is given by

$$\begin{aligned} F_Y(y) &= P(Y = \min(Y_1, Y_2, ...Y_k) < y) \\ &= 1 - P(Y = \min(Y_1, Y_2, ...Y_k) > y) \\ &= 1 - P(Y_1 > y, Y_2 > y, ...Y_k > y) \\ &= 1 - \prod_{i=1}^{k} P(Y_i > y) \text{ if } Y_i^{'}s \text{ are independent} \\ &= 1 - \left[P\left(Y_1 > y\right)\right]^k \end{aligned}$$

if $Y_i^{'}s$ are independent and identical.

Here we assume that X_i and $Y_i, i = 1(1)k$ follow independent Weibull distributions with common shape parameter and the probability density functions are given by

$$f_{X_i}(x) = \alpha\lambda_1^{\alpha} e^{-(\lambda_1 x)^{\alpha}}, x > 0, \lambda_1, \alpha > 0$$

and

$$g_{Y_i}(y) = \alpha\lambda_2^{\alpha} e^{-(\lambda_2 y)^{\alpha}}, y > 0, \lambda_2, \alpha > 0$$

respectively.

Then

$$F_X(x) = \left[1 - e^{-(\lambda_1 x)^{\alpha}}\right]^k$$

and the probability density function is

$$f_X(x) = k\alpha\lambda_1^{\alpha}\left[1 - e^{-(\lambda_1 x)^{\alpha}}\right]^{k-1} x^{\alpha-1} e^{-(\lambda x_1)^{\alpha}}$$

and

$$G_Y(y) = 1 - \left[1 - e^{-k(\lambda_2 y)^{\alpha}}\right]$$

and the probability density function is

$$g_Y(y) = k\alpha\lambda_2^{\alpha} y^{\alpha-1} e^{-k(\lambda_2 y)^{\alpha}}$$

Hence

$$\begin{aligned} R &= P(X < Y) \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{y} f_X(x)dx\right] dG_Y(y) \end{aligned}$$

$$=\int_{-\infty}^{\infty}F_X(y)dG_Y(y)$$

$$=\int_0^{\infty}\left[1-e^{-(\lambda_1 y)^{\alpha}}\right]^k k\alpha\lambda_2^{\alpha}y^{\alpha-1}e^{-k(\lambda_2 y)^{\alpha}}dy \qquad (1.1)$$

$$=\int_0^1\left[1-w^{\frac{1}{k}\left(\frac{\lambda_1}{\lambda_2}\right)^{\alpha}}\right]^k dw=\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{1}{1+\frac{j}{k}\left(\frac{\lambda_1}{\lambda_2}\right)^{\alpha}}$$

If k=1 i.e. in case of stress-strength reliability for component only, $R=\frac{\lambda_1^{\alpha}}{\lambda_1^{\alpha}+\lambda_2^{\alpha}}$ and its inferential aspects have been studied in McCool (1991) and Mukherjee and Maiti (1998), if $\alpha=1$ also, then $R=\frac{\lambda_1}{\lambda_1+\lambda_2}$. It is just exponential case and the case is studied by a host of authors. If $\alpha=1$ only, then the situation reduces to exponential case of a system with k identical components and then

$$R=\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{1}{1+\frac{j}{k}\frac{\lambda_1}{\lambda_2}}.$$

In this article, we have attempted estimation problem of R for Weibull family of distributions. We have found out maximum likelihood estimate (MLE) of R for complete samples. An emphasis has been given for finding out lower confidence limits (lcls) as this is the one of practical importance-practioners want to assert that the system is at least attained this limit. We use delta method and bootstrap method to find out lcls. We also derive Bayes estimate of R using MCMC approach.

The paper is organized as follows. Section 2 is devoted for finding out MLE and lcls of R. Bayes estimation of R has been discussed in section 3. Simulation results have been discussed in section 4. Data analysis has been presented in Section 5 and section 6 concludes.

2. Inference about R

2.1. Maximum Likelihood Estimation of R

To compute the MLE of R, we have to obtain the MLEs of λ_1 and λ_2. Suppose $(X_1, X_2, ..., X_m)$ is a random sample from $f_X(\lambda_1,\alpha)$ and $(Y_1, Y_2, ..., Y_n)$ is a random sample from $g_X(\lambda_2,\alpha)$. Hence, the underlying log-likelihood function is

$$l(\lambda_1,\lambda_2,\alpha)=(m+n)\ln k+(m+n)\ln\alpha+m\alpha\ln\lambda_1$$
$$+n\alpha\lambda_2+(\alpha-1)\left[\sum_{i=1}^{m}\ln X_i+\sum_{j=1}^{n}\ln Y_j\right]-k\lambda_2^{\alpha}\sum_{j=1}^{n}Y_j^{\alpha}$$
$$-\lambda_1^{\alpha}\sum_{i=1}^{m}X_i^{\alpha}+(k-1)\sum_{i=1}^{m}\ln\left[1-e^{-(\lambda_1 X_i)^{\alpha}}\right]$$

Then the MLE of λ_1 is to be obtained from the relation

$$\hat{\lambda}_1(\alpha)=\left[\frac{m}{\sum_{i=1}^{m}X_i^{\alpha}-(k-1)\sum_{i=1}^{m}\frac{X_i e^{-(\lambda_1 X_i)^{\alpha}}}{1-e^{-(\lambda_1 X_i)^{\alpha}}}}\right]^{\frac{1}{\alpha}}$$

and that of λ_2 is from

$$\hat{\lambda}_2(\alpha)=\left[\frac{n}{k\sum_{j=1}^{n}Y_i^{\alpha}}\right]^{\frac{1}{\alpha}}$$

and the MLE of α is to be obtained by solving the equation

$$\frac{m+n}{\alpha}+m\ln\lambda_1+n\ln\lambda_2+\left[\sum_{i=1}^{m}\ln X_i+\sum_{j=1}^{n}\ln Y_j\right]$$
$$-k\left[\lambda_2^{\alpha}\ln\lambda_2\sum_{j=1}^{n}Y_j^{\alpha}+\alpha\lambda_2^{\alpha}\sum_{j=1}^{n}Y_j^{\alpha-1}\right]$$
$$-\left[\lambda_1^{\alpha}\ln\lambda_1\sum_{i=1}^{m}X_i^{\alpha}+\alpha\lambda_1^{\alpha}\sum_{i=1}^{m}X_i^{\alpha-1}\right]+(k-1)$$
$$\sum_{i=1}^{m}\frac{e^{-(\lambda_1 X_i)^{\alpha}}(\lambda_1 X_i)^{\alpha}\ln(\lambda_1 X_i)}{1-e^{-(\lambda_1 X_i)^{\alpha}}}=0$$

An estimate $\hat{R}$ of R is to be obtained from (1.1) replacing λ_1,λ_2 and α by $\hat{\lambda}_1(\hat{\alpha}),\hat{\lambda}_2(\hat{\alpha})$ and $\hat{\alpha}$ respectively.

We have already mentioned that when $\alpha=1$, it reduces to exponential case. We will concentrate further inference in this situation only. Under such situation, the estimates of λ_1 and λ_2 are of the form

$$\hat{\lambda}_1=\left[\frac{m}{\sum_{i=1}^{m}X_i-(k-1)\sum_{i=1}^{m}\frac{X_i e^{-(\lambda_1 X_i)}}{1-e^{-(\lambda_1 X_i)}}}\right]$$

$$\hat{\lambda}_2=\left[\frac{n}{k\sum_{j=1}^{n}Y_j}\right]$$

respectively.

Let us write

$$W=\begin{pmatrix}W_{11} & 0\\ 0 & W_{22}\end{pmatrix}$$

Where

$$W_{11}=-\frac{\partial^2 l(\lambda_1,\lambda_2)}{\partial\lambda_1^2}=\frac{m}{\lambda_1^2}+(k-1)\sum_{i=1}^{m}\frac{X_i^2 e^{-\lambda_1 X_i}}{\left(1-e^{-\lambda_1 X_i}\right)^2},$$

$$W_{22}=-\frac{\partial^2 l(\lambda_1,\lambda_2)}{\partial\lambda_2^2}=\frac{n}{\lambda_2^2}$$

Now, the asymptotic variance-covariance matrix of $(\hat{\lambda}_1,\hat{\lambda}_2)$ is given by

$$V=W^{-1}=\begin{pmatrix} W_{11}^{-1} & 0 \\ 0 & W_{22}^{-1}\end{pmatrix}$$

Let $G=(G_1,G_2)'$, with $G_1=\frac{\partial R}{\partial\lambda_1}, G_2=\frac{\partial R}{\partial\lambda_2}$ yield the asymptotic variance of $\hat{R}$ as $S_\Delta^2(\hat{R})=G'VG=\frac{G_1^2}{W_{11}}+\frac{G_2^2}{W_{22}}$.

Here $\frac{\partial R}{\partial\lambda_1}=-\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{j}{k\lambda_2\left(1+\frac{j}{k}\frac{\lambda_1}{\lambda_2}\right)^2}$

and

$$\frac{\partial R}{\partial\lambda_2}=\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{j\lambda_1}{k\lambda_2^2\left(1+\frac{j}{k}\frac{\lambda_1}{\lambda_2}\right)^2}.$$

Assuming $\frac{R-\hat{R}}{S_\Delta(\hat{R})}$ as a standard normal variate a lower confidence bound to R can be constructed.

Remark 2.1 $S_\Delta(\hat{R})$ is to be obtained by replacing the parameters by their ML estimators.

2.2. Bootstrap Lower Confidence Limits

In this subsection, we propose to use two lower confidence limits based on the parametric bootstrap methods; (i) percentile bootstrap method (we call it from now on as Boot-p) based on the idea of Efron (1982, 1988), (ii) bootstrap-t method (we refer it as Boot-t from now on) based on the idea of Hall (1988). We illustrate briefly how to estimate lower confidence limits of R using both methods.

Boot-p Methods:

Step 1: From the sample $\{x_1,.......,x_m\}$ and $\{y_1,......y_n\}$, compute $\frac{R-\hat{R}}{S_\Delta(\hat{R})}$ and $\hat{\lambda}_2$.

Step 2: Using $\hat{\lambda}_1$ generate a bootstrap sample $\{x_1^*,.........,x_m^*\}$ and similarly using $\hat{\lambda}_2$ generate a bootstrap sample $\{y_1^*,.......,y_n^*\}$. Based on $\{x_1^*,.........,x_m^*\}$ and $\{y_1^*,.......,y_n^*\}$ compute the bootstrap estimate of R using (1), say $\hat{R}^*$.

Step 3 : Repeat step 2, NBOOT times.

Step 4: Let $G(x)=P(\hat{R}^*\le x)$, be the cumulative distribution function of $\hat{R}^*$.

Define $\hat{R}_{Boot-p}(x)=G^{-1}(x)$ for a given x. The approximate $100(1-\eta)\%$ lower confidence limits of R is given by $\hat{R}_{Boot-p}(1-\eta)$.

Bootstrap-t Methods:

Step 1: From the sample $\{x_1,.......,x_m\}$ and $\{y_1,......y_n\}$, compute $\frac{R-\hat{R}}{S_\Delta(\hat{R})}$ and $\hat{\lambda}_2$.

Step 2: Using $\hat{\lambda}_1$ generate a bootstrap sample $\{x_1^*,.........,x_m^*\}$ and similarly using $\hat{\lambda}_2$ generate a bootstrap sample $\{y_1^*,.......,y_n^*\}$. Based on $\{x_1^*,.........,x_m^*\}$ and $\{y_1^*,.......,y_n^*\}$ compute the bootstrap estimate of R using (1), say $\hat{R}^*$ and the following statistic:

$$T^*=\frac{\sqrt{m}\left(\hat{R}^*-\hat{R}\right)}{\sqrt{V\left(\hat{R}^*\right)}}$$

Compute $V(\hat{R}^*)$ using Remark 2.1.

Step 3 : Repeat step 2, NBOOT times.

Step 4: From the NBOOT T^* values obtained, determine the lower bound of the $100(1-\eta)\%$ confidence limits of R as follows: Let $H(x)=P(\hat{T}^*\le x)$, be the cumulative distribution function of $\hat{T}^*$. For a given x, define

$$\hat{R}_{Boot-t}=\hat{R}+m^{-\frac{1}{2}}\sqrt{V(\hat{R})}H^{-1}x.$$

Here also, $V(\hat{R})$ can be computed as mentioned in Remark 2.1. The approximate $100(1-\eta)\%$ lower confidence limit of R is given by $\hat{R}_{Boot-t}(1-\eta)$.

3. Bayes Estimation of R

In this section, we obtain the Bayes estimation of R under assumption that the shape parameters λ_1,λ_2 and α are random variables. We mainly obtain the Bayes estimate of R under the squared error loss by Gibbs sampling technique. It is assumed that λ_1,λ_2 and α have independent gamma priors with the parameter λ_1 ~ $Gamma(a_1,a_1)$, λ_2 ~ $Gamma(a_2,a_2)$ and α ~ $Gamma(a_3,a_3)$. Based on the above assumptions, we have the likelihood function of the observed data as

$$L(data\mid\lambda_1,\lambda_2,\alpha)=k^{m+n}\alpha^{m+n}\lambda_1^{m\alpha}\times$$
$$\prod_{i=1}^{m}\left[1-e^{-(\lambda_1 x_i)^\alpha}\right]^{k-1}\left[\prod_{i=1}^{m}x_i\right]^{\alpha-1}e^{-\lambda_1^\alpha\sum_{i=1}^{m}x_i^\alpha}\times$$
$$\lambda_2^{n\alpha}\left[\prod_{j=1}^{n}y_j\right]^{\alpha-1}e^{-k\lambda_2^\alpha\sum_{j=1}^{n}y_j^\varepsilon}$$

Therefore, the joint density of the data, λ_1, λ_2 and α can be obtained as

$$L(data, \lambda_1, \lambda_2, \alpha) = L(data \mid \lambda_1, \lambda_2, \alpha) \times \pi(\lambda_1)\pi(\lambda_2)\pi(\alpha)$$

where $\pi(.)$ is the prior distribution. Therefore, the joint posterior density of λ_1, λ_2 and α given data is

$$L(\lambda_1, \lambda_2, \alpha \mid data) = \frac{L(data, \lambda_1 \lambda_2 \alpha)}{\int_0^\infty \int_0^\infty \int_0^\infty L(data, \lambda_1, \lambda_2, \alpha)\, d\lambda_1 d\lambda_2 d\alpha}$$

Since these equations cannot be obtained analytically, we adopt the Gibbs sampling technique to compute the Bayes estimate of R.

The posterior pdfs of λ_1, λ_2 and α are as follows:

$$f_{\lambda_1}(\lambda_1 \mid \lambda_2, \alpha, data) \propto \lambda_1^{m\alpha + a_1 - 1} e^{-b_1\lambda_1 - \lambda_1^\alpha \sum_{i=1}^m x_i^\alpha} \times$$

$$f_{\lambda_2}(\lambda_2 \mid \lambda_1, \alpha, data) \propto \lambda_2^{n\varepsilon + a_2 - 1} e^{-b_2\lambda_2 - k\lambda_2^\alpha \sum_{j=1}^n y_j^\alpha}$$

$$\prod_{i=1}^m \left[1 - e^{-(\lambda_1 x_i)^\varepsilon}\right]^{k-1}$$

and

$$f_\alpha(\alpha \mid \lambda_1, \lambda_2, data) \propto \alpha^{n+m+a_3-1} e^{-b_3\alpha} \lambda_1^{m\alpha} \times$$

$$\prod_{i=1}^m \left[1 - e^{-(\lambda_1 x_i)^\alpha}\right]^{k-1} \times \left[\prod_{i=1}^m x_i\right]^{\alpha-1} e^{-\lambda_1^\alpha \sum_{i=1}^m x_i^\alpha} \times$$

$$\lambda_2^{n\alpha} \left[\prod_{j=1}^n y_j\right]^{\alpha-1} e^{-k\lambda_2^\alpha \sum_{i=1}^n y_j^\alpha}$$

To generate random numbers from these distributions, we use the Metropolis-Hastings method with appropriate proposal distributions. Therefore, the algorithm of Gibbs sampling is as follows:

Step 1: Start with an initial guess $\left(\lambda_1^0, \lambda_2^0, \alpha^0\right)$.

Step 2: Set $t = 1$.

Step 3: Using Metropolis-Hastings, generate λ_1^t from f_{λ_1} with appropriate proposal distribution.

Step 4: Using Metropolis-Hastings, generate λ_2^t from f_{λ_2} with appropriate proposal distribution.

Step 5: Using Metropolis-Hastings, α^t from f_α with appropriate proposal distribution.

Step 6: Compute R^t from the expression of R.

Step 7: Set $t = t + 1$.

Step 8: Repeat step 3-6, T times.

Note that in steps 3-5, we use the Metropolis-Hastings algorithm with $q\left(\xi^{t-1}\right)$ proposal distribution as follows:

Let $x = \xi^{t-1}$.

Generate y from proposal distribution q.

Let $p(x,y) = \min\left\{1, f_\alpha(y) / f_\alpha(x).q(x) / q(y)\right\}$.

Accept y with probability $p(x,y)$ or accept x with probability $1 - p(x,y)$.

In case of exponential distributions (i.e. $\alpha = 1$), we have the posterior pdfs of λ_1 and λ_2 are as follows:

$$f_{\lambda_1}(\lambda_1 \mid \lambda_2, data) \propto \lambda_1^{m+a_1-1} e^{-\lambda_1\left(b_1 + \sum_{i=1}^m x_i\right)} \times \prod_{i=1}^m \left[1 - e^{-\lambda_i x_i}\right]^{k-1}$$

$$f_{\lambda_2}(\lambda_2 \mid \lambda_1, data) \propto \lambda_2^{n+a_2-1} e^{-\lambda_2\left(b_2 + k\sum_{j=1}^n y_j\right)}$$

Now the appropriate posterior mean and posterior variance of R become

$$\hat{E}(R \mid data) = \frac{1}{T}\sum_{t=1}^T R^t$$

and

$$MSE(R \mid data) = \frac{1}{T}\sum_{t=1}^T \left(R^t - R\right)^2 \text{ respectively.}$$

4. Simulation and discussion

In this section we present some results based on Monte Carlo simulations to compare the performance of R for different values of $k = 1,2,3$. Also the values of $m, n, \lambda_1, \lambda_2$ and α are mentioned under each table. All computations were performed using R-software and these are available on request from the corresponding author. We consider to draw inference on R when the baseline distribution of extended distribution is known. All the results are based on 1000 replications.

We report the average biases and mean squared errors (MSEs) over 1000 replications. We also compute the 95% lower confidence limit (lcl) of R based on asymptotic distribution of $\hat{R}$, using Boot-p and Boot-t methods. The bootstrap lcls are obtained using 1000 bootstrap replications in both cases. All the results are reported in Table 7, Table 8, Table 9.

Some of the points are quite clear from this experiment. The performances of the MLEs are quite satisfactory in terms of biases and MSEs. It is observed that when k increases then MSEs decreases for low value of R but increase for high value of R. High value of R is underestimated slightly (i.e. bias is negative), where as low value of R is overestimated generally. All lower confidence bounds are estimated satisfactorily. Particularly, Boot-t lcls perform very well. Based on all these, we recommend using the parametric percentile bootstrap lcls, particularly Boot-t lcls.

We do not have any prior information on R, therefore, we prefer to use the non-informative prior to compute different Bayes estimates. Since the non-informative prior, i.e. $a_1 = a_2 = b_1 = b_2 = 0$ provides prior distributions which are not proper, we adopt the suggestion of Congdon (2001, pp 20), i.e. choose $a_1 = a_2 = b_1 = b_2 = 0.0001$, which are almost like Jeffrey's prior, but they are proper. Under the same prior distributions, we compute Bayes estimate of λ_1 and λ_2 and have approximate Bayes estimates of R under squared error loss function. To generate random observations from the posterior distributions of λ_1 and λ_2, we use the Metropolis-Hastings method with proposal distributions $N(\lambda_1, 1)$ and $N(\lambda_2, 1)$ respectively. The algorithms of Gibbs sampling is described in section 3. The burn in sample in each case is taken 5000. The results are reported in Table 10, Table 11, Table 12 with the change of k the averages biases and the MSEs do not show clear picture. Therefore, if we do not have prior information about λ_1 and λ_2, then using Bayes estimates we may not gain much. Since the MLE is consistent and it can be used for constructing lower confidence limits also, we recommend using MLEs in this case.

5. Simulated Data Analysis

In this section we present the analysis of simulated data. The data set are presented in Table 1, Table 3 and Table 5. The results are summarized in Table 2, Table 4 and Table 6.

Table 1. Simulated Data Set $m = 15, n = 25, \lambda_1 = 2, \lambda_2 = 0.5, \alpha = 1, k = 1$

X=0.1816807, 0.5644605, 0.05036435, 0.00656449, 0.3189166, 0.1597823, 0.3605449, 0.6339824, 0.01011749, 0.2836022, 0.6458557, 1.217821, 0.6277124, 0.1900136, 1.014753
Y=1.298510, 0.7384054, 0.1103642, 2.444124, 0.3134927, 0.3663870, 0.1410308, 3.249928, 7.536066, 1.873777, 0.3976876, 4.933383, 5.022444, 1.379873, 1.016623, 0.2449651, 0.7953434, 5.571544, 1.658338, 0.1603021, 1.595187, 0.05412668, 0.7995375, 0.6618942, 0.2260054

Table 2. Estimates of R and lower Confidence Limits

MLE of R	0.803073
Bayes estimate of R	0.7959496
Delta Method estimate of R	0.8035658
Bootstrap-p estimate of R	0.8020293
Bootstrap-t estimate of R	0.8000846
LCL in Delta Method (LCL_d)	0.7816713
LCL in Bootstrap-p (LCL_p)	0.7406585
LCL in Bootstrap-t (LCL_t)	0.730892

Table 3. Simulated Data Set $m = 15, n = 25, \lambda_1 = 2, \lambda_2 = 0.5, \alpha = 1, k = 2$

X=0.9548599, 0.3978185, 0.8279778, 1.728254, 0.6092846, 2.193243, 1.505944, 0.7995457, 1.271949, 0.3518478, 0.393324, 0.4160646, 0.04846204, 0.1825408, 0.3132363
Y=0.4885969, 1.788655, 0.09735831, 1.317049, 3.85566, 1.874552, 0.2113092, 0.655856, 3.115124, 0.6412166, 0.7822648, 0.3472869, 0.8432377, 0.9068543, 3.370655, 0.5433124, 0.8411738, 1.958996,0.1115043, 0.1988442, 1.347204, 1.696238, 0.5008322, 0.2542893, 0.128458

Table 4. Estimates of R and lower Confidence Limits

MLE of R	0.5343442
Bayes estimate of R	0.5178546
Delta Method estimate of R	0.5412183
Bootstrap-p estimate of R	0.546389
Bootstrap-t estimate of R	0.5437109
LCL in Delta Method (LCL_d)	0.5081122
LCL in Bootstrap-p (LCL_p)	0.372174
LCL in Bootstrap-t (LCL_t)	0.3623463

Table 5. Simulated Data Set $m = 15, n = 25, \lambda_1 = 2, \lambda_2 = 0.5, \alpha = 1, k = 3$

X=1.418822, 1.100097, 0.4044287, 0.1266725, 1.411248, 1.556682, 0.5193744, 0.7009385, 0.982689, 1.353991, 1.916082, 0.424731, 0.2788498, 0.3117678, 0.2348663
Y=2.046273, 0.3752036, 0.8128688, 0.1674414, 0.5964549, 1.252022,0.1706026, 0.7781672, 0.06740291, 1.193519, 0.2799181, 1.848790, 0.2382824, 0.3149041, 0.3736654,0.05219408, 0.3164043, 0.02093941, 1.118972, 0.08428241, 0.09862126, 0.0548317,0.04357614,0.01655727,0.7900938

Table 6. Estimates of R and lower Confidence Limits

MLE of R	0.331045
Bayes estimate of R	0.2942824
Delta Method estimate of R	0.3369187
Bootstrap-p estimate of R	0.3550662
Bootstrap-t estimate of R	0.3582895
LCL in Delta Method (LCL_d)	0.3039611
LCL in Bootstrap-p (LCL_p)	0.15292558
LCL in Bootstrap-t (LCL_t)	0.1024406

The true values of R for the simulated data sets in Table 1, Table 3 and Table 5 are 0.8, 0.5333333 and 0.3324675 respectively (see Table 7, Table 8 and Table 9 for $\lambda_1 = 2$, $\lambda_2 = 0.5$). We observe that, in all the cases, the MLE of R is very close to the true value. One should note that in real life data situation, the true value of R is not possible to get and hence comparison of biases and MSEs are not possible. However, in the present scenario, one can get almost the true picture from the simulation results presented in section 4. From Table 7, Table 8 and Table 9 , it is ensured that the MLEs of R has minimum biases and MSEs comparing the values corresponding to $\lambda_1 = 2$, $\lambda_2 = 0.5$. It is evident from the analysis of data sets and the results presented in Table 2, Table 4 and Table 6 that the MLE of R is fairly good compared to the Bayes estimate – the fact was also reported in simulation study. Some improvements in the case of Bayes estimate of R may be expected if the appropriate prior distributions are selected when it is available besides non-informative prior. In all the data sets, lcl's in Bootstrap-t are better from maximum coverage probability point of view.

6. Concluding Remark

In this article, we have discussed inference problem of $R = P(X < Y)$ with $X = \max(X_1, X_2, ...X_k)$ and $Y = \min(Y_1, Y_2, ...Y_k)$. The X and Y distributions have been considered Weibull. We have considered maximum likelihood estimate and Bayes estimate of R. Comparing these two, we recommend to use MLE for R. An emphasis has been given on lower confidence limits as this is the one of practical importance-practitioners want to assert that the system is at least attained this limit. To construct lcls, we consider Delta method and two bootstrap methods -percentile (Boot-p) and bootstrap-t (Boot-t). We recommend using the parametric bootstrap lcls, particularly Boot-t lcls.

Table 7. Simulation results of Extreme distribution. $m = 25, n = 25, \alpha = 1, k = 1$

λ_1, λ_2	R	$\hat{R}$	MSE	Bias	(LCL_d)	(LCL_p)	(LCL_t)
0.5, 4	0.1111111	0.1135962	0.0008171	0.0024851	0.06717577	0.06881608	0.05249087
0.5, 2	0.2	0.2045146	0.0021304	0.0045146	0.1293211	0.1154656	0.1258536
0.5, 1	0.3333333	0.3342373	0.0040516	0.0009040	0.2361792	0.2383113	0.2302903
0.5, 0.7	0.4166667	0.4176976	0.0044618	0.0010309	0.3062456	0.3481601	0.3765411
0.5, 0.5	0.5	0.5014158	0.0049875	0.0014158	0.3866008	0.3564049	0.4408366
0.7, 0.5	0.5833333	0.5856562	0.0047464	0.0023228	0.4720385	0.3897911	0.5320068
1, 0.5	0.6666667	0.6637099	0.0039245	-0.0029567	0.5641103	0.5805	0.0.6340184
2, 0.5	0.8	0.7976575	0.0020609	_0.0023425	0.7236657	0.7168892	0.7600102
4, 0.5	0.888889	0.8868172	0.0007826	_0.0020717	0.8405416	0.7950288	0.8201139
6, 0.5	0.923077	0.9203894	0.0004453	_0.0026875	0.884047	0.8878481	0.8917386
8, 0.5	0.9411765	0.9387401	0.0002814	-0.0024363	0.912866	0.9282181	0.934747
10, 0.5	0.952381	0.950444	0.00001817	-0.0019369	0.9491219	0.951334	0.9487624
12, 0.5	0.96	0.9593142	0.0001324	-0.000685	0.9543751	0.953653	0.958791

Table 8. Simulation results of Extreme distribution. $m = 25, n = 25, \alpha = 1, k = 2$

λ_1, λ_2	R	$\hat{R}$	MSE	Bias	(LCL_d)	(LCL_p)	(LCL_t)
0.5, 4	0.0065359	0.0083097	4.783×10^{-5}	0.0017738	0.002268227	0.001085347	0.001474738
0.5, 2	0.0222222	0.0269159	0.0021304	0.0003441	0.008973157	0.005397917	0.004357397
0.5, 1	0.0666667	0.0752649	0.0015669	0.0085983	0.03303469	0.04988657	0.03440165
0.5, 0.7	0.1096491	0.1177737	0.0030139	0.0081246	0.06027919	0.09364642	0.07319867
0.5, 0.5	0.1666667	0.1754231	0.0048939	0.0087564	0.09791531	0.1247929	0.1208384
0.7, 0.5	0.2401961	0.2573227	0.0080461	0.0171266	0.1523984	0.1780068	0.1876356
1, 0.5	0.3333333	0.3448201	0.0093009	0.0114868	0.2292687	0.2923863	0.3107781
2, 0.5	0.5333333	0.5370427	0.0090017	0.0037094	0.4270614	0.4326371	0.3696069
4, 0.5	0.7111111	0.70733867	0.0054977	-0.0037244	0.6173199	0.5958696	0.5419874
6, 0.5	0.7912088	0.7878744	0.0034205	-0.0033344	0.715463	0.7222103	0.7254289
8, 0.5	0.8366013	0.832172	0.0024920	-0.0044293	0.8068216	0.8223224	0.8188247
10, 0.5	0.8658009	0.8628244	0.0017320	-0.0029765	0.8515252	0.8497543	0.8428659
12, 0.5	0.8861538	0.8820422	0.0014025	-0.0041117	0.880598	0.8751033	0.879289

Table 9. Simulation results of Extreme distribution. $m = 25, n = 25, \alpha = 1, k = 3$

λ_1, λ_2	R	$\hat{R}$	MSE	Bias	(LCL_d)	(LCL_p)	(LCL_t)
0.5, 4	0.0003418	0.0007536	2.319×10^{-6}	0.0004118	9.798×10^{-6}	0.0002123167	5.342×10^{-5}
0.5, 2	0.0021978	0.0049632	0.0001557	0.0027653	0.001604674	0.001061839	0.0001979209
0.5. 1	0.0119047	0.0186784	0.0004833	0.0067736	0.004293301	0.004584146	0.002540799
0.5, 0.7	0.0258478	0.0377235	0.0020825	0.0118757	0.01189554	0.02185004	0.01466073
0.5, 0.5	0.05	0.0609959	0.0027359	0.0109959	0.02455426	0.0348501	0.04705408
0.7, 0.5	0.0896029	0.1035574	0.0046264	0.0139545	0.04642489	0.06371596	0.05418988
1, 0.5	0.1523810	0.1740531	0.0097857	0.216722	0.09244965	0.09359726	0.01240031
2, 0.5	0.3324675	0.3545781	0.0154544	0.0221105	0.2296966	0.1717405	0.2039296
4, 0.5	0.5443913	0.543986	0.01197589	-0.0119759	0.439521	0.495666	0.3509279
6, 0.5	0.6564103	0.653129	0.0093561	-0.00328128	0.5611863	0.6457864	0.6114739
8, 0.5	0.7246351	0.7235012	0.0066902	-0.0011339	0.6384436	0.6682031	0.6618039
10, 0.5	0.770379	0.7665545	0.0053469	-0.0038245	0.6978315	0.7339302	0.769091
12, 0.5	0.8031373	0.79752	0.0045354	-0.0056172	0.7751819	0.7884733	0.7729683

Table 10. Simulation results on Bayes estimate of R. $m = 25, n = 25, \alpha = 1, k = 1$

λ_1, λ_2	R	$\hat{R}_{Bayes}$	MSE_{Bayes}	$Bias_{Bayes}$
0.5, 4	0.00653594	0.0097507	0.0012219	-0.0269989
0.5, 2	0.2	0.1716103	0.0026374	-0.02838968
0.5, 1	0.3333333	0.4184339	0.0410979	0.0851006
0.5, 0.7	0.4166667	0.467789	0.0419940	0.0511223
0.5, 0.5	0.5	0.5822145	0.0491516	0.0822145
0.7, 0.5	0.5833333	0.5365633	0.0423323	-0.0467700
1, 0.5	0.6666667	0.6567367	0.0044769	-0.0099299
2, 0.5	0.8	0.7203618	0.0252081	-0.0796638
4, 0.5	0.8888889	0.8726364	0.0014182	-0.0162525
6, 0.5	0.923077	0.9107002	0.0006347	-0.0123763
8, 0.5	0.9411765	0.9331756	0.0003161	-0.0080009
10, 0.5	0.952381	0.9113743	0.0022611	-0.0410066
12, 0.5	0.96	0.9518523	0.0003059	-0.0081477

Table 11. Simulation results on Bayes estimate of R. $m = 25, n = 25, \alpha = 1, k = 2$

λ_1, λ_2	R	$\hat{R}_{Bayes}$	MSE_{Bayes}	$Bias_{Bayes}$
0.5, 4	0.00653594	0.0097507	3.569×10^{-5}	0.0032147
0.5, 2	0.0222222	0.0279071	0.0005166	0.0056849
0.5, 1	0.0666667	0.0796617	0.0012755	0.0129951
0.5, 0.7	0.1096491	0.1545251	0.0053302	0.0448759
0.5, 0.5	0.1666667	0.249138	0.0151225	0.0824713
0.7, 0.5	0.2401961	0.2938248	0.0076047	0.0536287
1, 0.5	0.3333333	0.2986629	0.0407104	-0.0346704
2, 0.5	0.5333333	0.5174908	0.0075395	-0.0158425
4, 0.5	0.7111111	0.6814342	0.0072425	-0.0296769
6, 0.5	0.7912088	0.7813415	0.0037524	-0.0098672
8, 0.5	0.8366013	0.8042431	0.0027298	-0.0323582
10, 0.5	0.8658009	0.8863284	0.0009485	0.0205275
12, 0.5	0.8861538	0.8729647	0.0010199	-0.0131891

Table 12. Simulation results on Bayes estimate of R. $m = 25, n = 25, \alpha = 1, k = 3$

λ_1, λ_2	R	$\hat{R}_{Bayes}$	MSE_{Bayes}	$Bias_{Bayes}$
0.5, 4	0.0003419	0.0049051	8.674×10^{-5}	0.0045632
0.5, 2	0.0021978	0.0037606	8.674×10^{-5}	0.0015628
0.5, 1	0.0119047	0.0086153	0.0001116	-0.0032895
0.5, 0.7	0.0258478	0.0258799	0.0002330	3.218×10^{-5}
0.5, 0.5	0.05	0.0551663	0.0006893	0.0051663
0.7, 0.5	0.0896029	0.0577642	0.0016864	-0.0318387
1, 0.5	0.1523810	0.1183781	0.0035415	-0.0340028
2, 0.5	0.3324675	0.4490644	0.0201559	0.1165969
4, 0.5	0.5443913	0.548878	0.0057507	0.0044868
6, 0.5	0.6564103	0.644167	0.0039817	-0.0122433
8, 0.5	0.7246351	0.6831095	0.0108551	-0.0415256
10, 0.5	0.770379	0.7478266	0.0026877	-0.0225523
12, 0.5	0.8031373	0.7780665	0.0069134	-0.0250707

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