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Recurrence Relations for Single and Product Moments of Generalized Order Statistics from Extreme Value Distribution

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Abstract In this paper, we establish some recurrence relations satisfied by single and product moments of Generalized Order Statistics from Extreme Value Distribution. These recurrence relations are independent of left truncated point and therefore are also applicable for Logistic as well as for half Logistic distributions studied in Balakrishnan (1985) and Saran and Pandey (2012). For a particular case these results verify the corresponding results of Saran and Pandey (2004) and Kumar (2010).

Keywords: *order statistics, record values, generalized order statistics, single moment, product moments, recurrence relations, extreme value distribution*

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1. Introduction

Generalized order statistics (GOS) have been introduced and extensively studied in Kamps (1995 a,b) as a unified theoretical set-up which contains a variety of models of ordered random variables with different interpretations. Examples of such models are: Ordinary order statistics, Sequential order statistics, Progressive type II censored order statistics, Record values, k^{th} record value and Pfeifer's records. There is no natural interpretation of generalized order statistics in terms of observed random samples but these models can be effectively applied in life testing and reliability analysis, medical and life time data, and models related to software reliability analysis, etc. The common approach makes it possible to define several distributional properties at once. The structural similarities of these models are based on the similarity of their joint density function.

2. Standard Extreme Value Distribution

A random variable X is said to have an standard extreme value distribution if its probability density function is of the form

$$f\left(x\right)=\alpha\, e^{\alpha x}\times e^{-\,e^{\alpha x}}, \quad -\infty<x<\infty,\ \alpha>0 \qquad (2.1)$$

and the cumulative distribution is given by

$$F\left(x\right)=1-e^{-\,e^{\alpha x}}, \quad -\infty<x<\infty,\ \alpha>0 \qquad (2.2)$$

The extreme value distribution is used in the analysis of data concerning floods, extreme sea level and air pollution problems.

The cumulative distribution function and probability density function of random variable X, respectively, takes the form

$$f(x)=\left(\sum_{j=0}^{\infty}\frac{\alpha^{j+1}x^{j}}{j!}\right)(1-F(x)). \qquad (2.3)$$

$$\frac{f(x)}{1-F(x)}=\left(\sum_{j=0}^{\infty}\frac{\alpha^{j+1}x^{j}}{j!}\right)=\sum_{j=0}^{\infty}\alpha_j x^j, \textit{where } \alpha_j=\frac{\alpha^{j+1}}{j!}$$

The mathematical form of pdf, as given in (2.3), is very useful to derive the expression for recurrence relations for single and product moments of GOS.

3. Generalized Order Statistics

Let $\{X_n, n\ge 1\}$ be a sequence of absolutely continuous, independent and identically distributed random variables with cdf $F(x)=P\left(X\le x\right)$ and pdf f(x). Assume k > 0, $n\in\{2,3,\ldots\}$, $\tilde{m}=\left(m_1,m_2,\ldots,m_{n-1}\right)\in R^{n-1}$, $M_r=\sum_{j=r}^{n-1}m_j$, such that $\gamma_r=k+n-r+M_r>0$ for all $r\in\{1,2,\ldots,n-1\}$. Then $X\left(r,n,\tilde{m},k\right)$, r = 1, 2,…,n, are called GOS if their joint pdf is given by

$$f^{X(1,n,\tilde{m},k),\,X(2,n,\tilde{m},k),\,\dots,\,X(n,n,\tilde{m},k)}(x_1,x_2,\dots,x_n)$$
$$= k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left(1-F(x_i)\right)^{m_i} f(x_i)\right) \quad (3.1)$$
$$\times\left(1-F(x_n)\right)^{k-1} f(x_n),$$

where $F^{-1}(0+) < x_1 \le x_2 \le \dots \le x_n < F^{-1}(1)$.

By choosing appropriate values of parameters, we get the distribution of a few very common statistics as shown in the table given below.

S.No.	Choice of parameters for i = 1, 2,…, n	GOS becomes
1	$\gamma_i = n-i+1,\ m_1 = m_2 = \dots = m_{n-1} = 0$ and k = 1	Joint distribution of n order statistics
2	$\gamma_i = k,\ m_1 = m_2 = \dots = m_{n-1} = -1,\ k \in N$	k[th] record value
3	$\gamma_i = (n-i+1)\alpha_i,\ \alpha_i > 0$	Sequential order statistics
4	$\gamma_i = \alpha - i + 1,\ \alpha > 0$	Order statistics with non integer sample size
5	$\gamma_i = \beta_i,\ \beta_i > 0$	Pfeifer's record values
6	$m_i \in N_o,\ k \in N$	Progressively type-II right censored order statistics

The joint pdf of first r, GOS is given by:

$$f^{X(1,n,\tilde{m},k),X(2,n,\tilde{m},k),\dots,X(r,n,\tilde{m},k)}(x_1,x_2,\dots,x_r)$$
$$= c_{r-1}\left(\prod_{i=1}^{r-1}\left(1-F(x_i)\right)^{m_i} f(x_i)\right) \quad (3.2)$$
$$\times\left(1-F(x_r)\right)^{k+n-r+M_r-1} f(x_r),$$

where, $F^{-1}(0+) < x_1 \le x_2 \le \dots \le x_r < F^{-1}(1)$.

We now consider two cases:

Case I: $m_1 = m_2 = \dots = m_{n-1} = m$

Case II: $\gamma_i \ne \gamma_j$; i ≠ j, i, j = 1, 2, …, n - 1.

For case I, the GOS will be denoted by X(r, n, m, k). The pdf of X(r, n, m, k) is given by

$$f^{X(r,n,m,k)}(x)$$
$$= \frac{c_{r-1}}{(r-1)!}\left(1-F(x)\right)^{\gamma_r-1} f(x)\, g_m^{r-1}\left(F(x)\right),\quad x \in R \quad (3.3)$$

and the joint pdf of X(r, n, m, k) and X(s, n, m, k), 1 ≤ r < s ≤ n is given by:

$$f^{X(r,n,m,k),\,X(s,n,m,k)}(x,y)$$
$$= \frac{c_{s-1}}{(r-1)!\,(s-r-1)!}\left(\left(1-F(x)\right)^m f(x)\right)$$
$$\cdot g_m^{r-1}\left(F(x)\right)\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1} \quad (3.4)$$
$$\cdot\left(1-F(y)\right)^{\gamma_s-1} f(y),\ x<y,$$

where, $c_{r-1} = \prod_{j=1}^{r}\gamma_j,\ \gamma_j = k+(n-j)(m+1),\ r = 1,2,\dots,n,$

$g_m(x) = h_m(x) - h_m(0),\ x \in (0,1)$

and

$$h_m(x) = \begin{cases} -\dfrac{(1-x)^{m+1}}{m+1}, & m \ne -1, \\ -\log(1-x), & m = -1. \end{cases} \quad (3.5)$$

For case II, the pdf of $X(r,n,\tilde{m},k)$ is given by

$$f^{X(r,n,\tilde{m},k)}(x)$$
$$= c_{r-1}\ f(x)\sum_{i=1}^{r} a_i(r)\left(1-F(x)\right)^{\gamma_i-1} f(x),\ x \in R. \quad (3.6)$$

Also, the joint pdf of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, 1 ≤ r < s ≤ n is given by

$$f^{X(r,n,\tilde{m},k),\,X(s,n,\tilde{m},k)}(x,y)$$
$$= c_{s-1}\left\{\sum_{i=r+1}^{s} a_i^r(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right\} \quad (3.7)$$
$$\times\left\{\sum_{i=1}^{r} a_i(r)\left(1-F(x)\right)^{\gamma_i}\right\}\frac{f(x)}{1-F(x)}\,\frac{f(y)}{1-F(y)},$$

Where, $c_{s-1} = \prod_{j=1}^{s}\gamma_j,\ \gamma_j = k+n-j+M_j,\ s = 1,2,\dots,n.$.

Further it can be proved that

(i) $a_i(r) = \prod_{j(\ne i)=1}^{r}\left(\gamma_j-\gamma_i\right)^{-1},\quad 1 \le i \le r \le n$

(ii) $a_i^r(s) = \prod_{j(\ne i)=r+1}^{s}\left(\gamma_j-\gamma_i\right)^{-1},\quad r+1 \le i \le s \le n$.

(iii) $a_i(r) = \left(\gamma_{r+1}-\gamma_i\right) a_i(r+1)$

(iv) $c_r = c_{r-1}\gamma_{r+1}$

(v) $\sum_{i=1}^{r+1} a_i(r+1) = 0$

(vi)

$$\sum_{i=1}^{r} a_i(r)\left(1-F(x)\right)^{\gamma_i} = \frac{\left(1-F(x)\right)^{\gamma_r}}{(r-1)!} g_m^{r-1}\left(F(x)\right) \quad (3.9)$$

(vii)

$$\sum_{i=r+1}^{s} a_i^r(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}$$
$$= \frac{\left(1-F(x)\right)^{-(m+1)(s-r-1)}}{(s-r-1)!}\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_s} \quad (3.10)$$
$$\times\left(h_m\left(F(y)\right)-h_m\left(F(x)\right)\right)^{s-r-1}$$

The moments of order statistics have generated considerable interest in the recent years. The expressions for several recurrence relations and identities satisfied by single as well as product moments of order statistics have been obtained by several authors in the past. These relations help in reducing the quantum of computations involved. Joshi (1978, 1982) established recurrence relations for exponential distribution with unit mean and were further extended by Balakrishnan and Joshi (1984) for doubly truncated exponential distribution. For linear-exponential distribution, Balakrishnan and Malik(1986) derived the similar type of relations which were extended to doubly truncated linear exponential distribution by Mohie El-Din et al. (1997) and Saran and Pushkarna (1999). Nain (2010 a, b) obtained recurrence relations for ordinary order statistics and k[th] record values from p[th] order exponential and generalized weibull distributions, respectively.

The recurrence relations for the moments of generalized order statistics based on non identically distributed random variables were developed by Kamps (1995 a, b). Pawlas and Szynal (2001) obtained recurrence relations for single and product moments of generalized order statistics from Pareto, generalized Pareto and Burr distributions. Saran and Pandey (2004, 2009) obtained recurrence relations for single and product moments of generalized order statistics from linear-exponential and Burr distributions. Saran and Nain (2012) obtained recurrence relations for single and product moments generalized order statistics from doubly truncated p[th] order exponential distribution. Saran and Nain (2013) also obtained explicit expressions for single and product moments of Generalized Order Statistics from a new class of exponential distribution.

In this paper, we have established recurrence relations for single and product moments of GOS from Extreme Value Distribution. This distribution has many applications in analysis of data concerning floods, extreme sea levels and air pollution problems.

The results so obtained are generalized versions of some of the recurrence relations obtained by Kumar (2010), Saran and Pandey (2004).

Notations

For n = 1, 2, 3, ..., $1 \le r < s \le n,\ k \ge 1$ and $u, v \in \{\ 0,1,2,...\}$, we denote by

(i) $\mu^{u}_{r:\,m,\,n,\,k} = E\left(\mathrm{X}^{u}\left(\text{ r, n, m, k }\right)\right)$

(ii) $\mu^{u,\,v}_{r,\,s:\,m,\,n,\,k} = E\left(\mathrm{X}^{u}\left(\text{ r, n, m, k }\right)\mathrm{X}^{v}\left(\text{ s, n, m, k }\right)\right)$

(iii) $\mu^{u}_{r:\,\tilde{m},\,n,\,k} = E\left(\mathrm{X}^{u}\left(\text{ r, n, } \tilde{m}\text{, k }\right)\right)$

(iv) $\mu^{u,\,v}_{r,\,s:\,\tilde{m},\,n,\,k} = E\left(\mathrm{X}^{u}\left(\text{ r, n, } \tilde{m}\text{, k }\right)\mathrm{X}^{v}\left(\text{ s, n, } \tilde{m}\text{, k }\right)\right)$

4. Recurrence Relations For Single and Product Moments

Case I: $m_1 = m_2 = ... = m_{n-1} = m$

Theorem 1.

For n = 1, 2, 3, ..., $1 \le r < s \le n,\ k \ge 1$ and $u, v \in \{\ 0,1,2,...\}$

(a)

$$\mu^{u}_{r:\,m,\,n,\,k} = \alpha\gamma_r \sum_{j=0}^{\infty} \frac{\alpha^{j}}{u+j+1}\left[\mu^{u+j+1}_{r:\,m,\,n,\,k} - \mu^{u+j+1}_{r-1:\,m,\,n,\,k}\right] \quad (4.1)$$

(b)

$$\mu^{u,\,v}_{r,s:\,m,\,n,\,k}(r,s) = \alpha\gamma_s \sum_{j=0}^{\infty} \frac{\alpha^{j}}{v+j+1}\left[\mu^{u,\,v+j+1}_{r,s:\,m,\,n,\,k} - \mu^{u,\,v+j+1}_{r,s-1:\,m,\,n,\,k}\right] \quad (4.2)$$

Proof (a):

The u[th] order moment of X(r, n, m, k) is given by

$$\mu^{u}_{r:\,m,\,n,\,k} = \frac{c_{r-1}}{(r-1)!}\int_{\beta}^{\infty} x^{u}\left(1-F(x)\right)^{\gamma_r - 1} f(x)\, g_m^{r-1}\left(F(x)\right)dx. \quad (4.3)$$

Substituting f(x) from (2.3) we have

$$\mu^{u}_{r:\,m,\,n,\,k} = \frac{c_{r-1}}{(r-1)!}\sum_{j=0}^{\infty}\alpha_j \int_{\beta}^{\infty} x^{u+j}\left(1-F(x)\right)^{\gamma_r}\, g_m^{r-1}\left(F(x)\right)dx. \quad (4.4)$$

Integrating by parts, taking x^{u+j} as the part to be integrated, we obtain

$$\mu^{u}_{r:\,m,\,n,\,k} = \frac{c_{r-1}}{(r-1)!}\sum_{j=0}^{\infty}\frac{\alpha_j}{u+j+1}\int_{\beta}^{\infty} x^{u+j+1}\Big\{\gamma_r\left(1-F(x)\right)^{\gamma_r-1} g_m^{r-1}\left(F(x)\right) f(x) - (r-1)\left(1-F(x)\right)^{\gamma_r+m} g_m^{r-2}\left(F(x)\right) f(x)\Big\} \quad (4.5)$$

On substituting $\gamma_r + m = \gamma_{r-1} - 1,\ c_{r-1} = \gamma_r c_{r-2}$ and $\alpha_j = \alpha^{j+1}$ in (4.5), we shall derive the recurrence relation as stated in (4.1).

Proof (b). By definition

$$\mu^{u,\,v}_{r,s:\,m,\,n,\,k} = \frac{1}{(r-1)!\,(s-r-1)!}\int_{\beta}^{\infty} x^{u}\left(\left(1-F(x)\right)^{m} f(x)\right) \times g_m^{r-1}\left(F(x)\right) J\left(x:v,r,s,m\right) dx, \quad (4.6)$$

where,

$$J\left(x:v,r,s,m\right) = c_{s-1}\int_{x}^{\infty} y^{v}\left[h_m\left(F(y)\right) - h_m\left(F(x)\right)\right]^{s-r-1} \times \left(1-F(y)\right)^{\gamma_s - 1} f(y)\, dy. \quad (4.7)$$

Substituting f(y) from (2.3) we have:

$$J\left(x:v,r,s,m\right) = c_{s-1}\sum_{j=0}^{\infty}\alpha_j\int_{x}^{\infty} y^{v+j}\begin{bmatrix} h_m\left(F(y)\right) \\ -h_m\left(F(x)\right)\end{bmatrix}^{s-r-1} \times \left(1-F(y)\right)^{\gamma_s} f(y)\, dy. \quad (4.8)$$

Integrating by parts, taking y^{v+j} as the part to be integrated, we obtain:

$$J\left(x:v,r,s,m\right)=c_{s-1}\sum_{j=0}^{\infty}\frac{\alpha_j}{v+j+1}$$

$$\int_{x}^{\infty} y^{v+j+1}\Big\{\gamma_s\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}$$
$$\times\left(1-F(y)\right)^{\gamma_s-1}f(y)-(s-r-1)$$
$$\times\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-2}$$
$$\times\left(1-F(y)\right)^{\gamma_s+m}f(y)\ \Big\}dy$$

After using $\gamma_s+m=\gamma_{s-1}-1$ *and* $c_{s-1}=\gamma_s c_{s-2}$, we get:

$$J\left(x:v,r,s,m\right)$$
$$=\gamma_s\sum_{j=0}^{\infty}\frac{\alpha_j}{v+j+1}\begin{Bmatrix}J\left(x:v+j+1,r,s,m\right)\\-(s-r-1)\ J\left(x:v+j+1,r,s-1,m\right)\end{Bmatrix}.$$

On substituting J(x : v, r, s, m) so obtained in (4.6), we shall derive the recurrence relation as stated in (4.2).

Case II: $\gamma_i\neq\gamma_j$; i ≠ j, i, j = 1, 2, …, n - 1.

Lemma 1.

(a)

$$c_{r-1}\sum_{i=1}^{r}a_i(r)\ \gamma_i\left(1-F(x)\right)^{\gamma_i}$$
$$=\gamma_r\begin{bmatrix}c_{r-1}\sum_{i=1}^{r}a_i(r)\left(1-F(x)\right)^{\gamma_i}\\-c_{r-2}\sum_{i=1}^{r-1}a_i(r-1)\left(1-F(x)\right)^{\gamma_i}\end{bmatrix}\qquad(4.9)$$

(b)

$$c_{s-1}\sum_{i=r+1}^{s}a_i^r(s)\ \gamma_i\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}$$
$$=\gamma_s\begin{bmatrix}c_{s-1}\sum_{i=r+1}^{s}a_i^r(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\\-c_{s-2}\sum_{i=r+1}^{s-1}a_i^r(s-1)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\end{bmatrix}\qquad(4.10)$$

Proof (a): Differentiating both sides of (3.9), with respect to x, we get:

$$\sum_{i=1}^{r}a_i(r)\ \gamma_i\left(1-F(x)\right)^{\gamma_i-1}$$
$$=\frac{1}{(r-1)!}\begin{bmatrix}\gamma_r\left(1-F(x)\right)^{\gamma_r-1}g_m^{r-1}\left(F(x)\right)\\-(r-1)\left(1-F(x)\right)^{\gamma_r}g_m^{r-2}\left(F(x)\right)g_m^{/}\left(F(x)\right)\end{bmatrix}$$
$$=\frac{1}{(r-1)!}\begin{bmatrix}\gamma_r\left(1-F(x)\right)^{\gamma_r-1}g_m^{r-1}\left(F(x)\right)\\-(r-1)\left(1-F(x)\right)^{\gamma_r+m}g_m^{r-2}\left(F(x)\right)\end{bmatrix}$$
$$\left(\because\ g_m^{/}\left(F(x)\right)=\left(1-F(x)\right)^m\right)$$
$$=\frac{1}{(r-1)!}\begin{bmatrix}\gamma_r\left(1-F(x)\right)^{\gamma_r-1}g_m^{r-1}\left(F(x)\right)\\-(r-1)\left(1-F(x)\right)^{\gamma_{r-1}-1}g_m^{r-2}\left(F(x)\right)\end{bmatrix}$$
$$=\frac{1}{1-F(x)}\begin{bmatrix}\gamma_r\sum_{i=1}^{r}a_i(r)\left(1-F(x)\right)^{\gamma_i}\\-\sum_{i=1}^{r-1}a_i(r-1)\left(1-F(x)\right)^{\gamma_i}\end{bmatrix},$$

which on using the relation $c_{r-1}=\gamma_r c_{r-2}$ leads to (4.9).

Proof (b): Differentiating both sides of (3.10), with respect to y, we get:

$$\sum_{i=r+1}^{s}a_i^r(s)\gamma_i\frac{\left(1-F(y)\right)^{\gamma_i-1}}{\left(1-F(x)\right)^{\gamma_i}}$$
$$=\frac{\left(1-F(x)\right)^{-(m+1)(s-r-1)}}{(s-r-1)!\left(1-F(x)\right)^{\gamma_s}}$$
$$\times\begin{Bmatrix}\gamma_s\left(1-F(y)\right)^{\gamma_s-1}\left(h_m\left(F(y)\right)-h_m\left(F(x)\right)\right)^{s-r-1}\\-(s-r-1)\left(1-F(y)\right)^{\gamma_s}\\\times\left(h_m\left(F(y)\right)-h_m\left(F(x)\right)\right)^{s-r-2}h_m^{/}\left(F(y)\right)\end{Bmatrix}$$
$$=\frac{\left(1-F(x)\right)^{-(m+1)(s-r-1)}}{(s-r-1)!\left(1-F(x)\right)^{\gamma_s}}$$
$$\times\begin{Bmatrix}\gamma_s\left(1-F(y)\right)^{\gamma_s-1}\left(h_m\left(F(y)\right)-h_m\left(F(x)\right)\right)^{s-r-1}\\-(s-r-1)\left(1-F(y)\right)^{\gamma_s+m}\begin{pmatrix}h_m\left(F(y)\right)\\-h_m\left(F(x)\right)\end{pmatrix}^{s-r-2}\end{Bmatrix}$$
$$\left(\because\ h_m^{/}\left(F(y)\right)=\left(1-F(y)\right)^m\right)$$
$$=\begin{Bmatrix}\frac{\gamma_s\left(1-F(x)\right)^{-(m+1)(s-r-1)}}{(s-r-1)!\left(1-F(y)\right)}\\\times\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_s}\left(h_m\left(F(y)\right)-h_m\left(F(x)\right)\right)^{s-r-1}\\-\frac{\left(1-F(x)\right)^{-(m+1)(s-r-2)}}{(s-r-2)!\left(1-F(y)\right)}\\\times\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_s-1}\left(h_m\left(F(y)\right)-h_m\left(F(x)\right)\right)^{s-r-2}\end{Bmatrix}$$
$$=\frac{1}{\left(1-F(y)\right)}\begin{bmatrix}\gamma_s\sum_{i=r+1}^{s}a_i^r(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\\-\sum_{i=r+1}^{s-1}a_i^r(s-1)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\end{bmatrix},$$

which on using the relation $c_{s-1}=\gamma_s c_{s-2}$ leads to (4.10).

Theorem 2.

For n = 1, 2, 3,..., $1\le r<s\le n$, $k\ge1$ and $u,v\in\{\ 0,1,2,...\}$

(a)

$$\mu^{u}_{r:\tilde{m},n,k} = \alpha\gamma_r \sum_{j=0}^{\infty} \frac{\alpha^j}{u+j+1}\left[\mu^{u+j+1}_{r:\tilde{m},n,k} - \mu^{u+j+1}_{r-1:\tilde{m},n,k}\right] \quad (4.11)$$

(b)

$$\mu^{u,v}_{r,s:\tilde{m},n,k} = \alpha\gamma_s \sum_{j=0}^{\infty} \frac{\alpha^j}{v+j+1}\left[\begin{array}{l}\mu^{u,\,v+j+1}_{r,s:\tilde{m},n,k} \\ -\mu^{u,\,v+j+1}_{r,s-1:\tilde{m},n,k}\end{array}\right] \quad (4.12)$$

Proof (a):

The u[th] order moment of $X(r,n,\tilde{m},k)$ is given by:

$$\mu^{u}_{r:\tilde{m},n,k} = c_{r-1} \int_{\beta}^{\infty} x^u \sum_{i=1}^{r} a_i(r)\left(1-F(x)\right)^{\gamma_i - 1} f(x)dx \quad (4.13)$$

Substituting the value of f(x) from (2.3), we have

$$\mu^{u}_{r:\tilde{m},n,k} = c_{r-1} \sum_{j=0}^{\infty} \alpha_j \left\{ \int_{\beta}^{\infty} x^{u+j} \sum_{i=1}^{r} a_i(r)\left(1-F(x)\right)^{\gamma_i} dx \right\}. \quad (4.14)$$

Integrating by parts, taking x^{u+j} as the part to be integrated, we obtain:

$$\mu^{u}_{r:\tilde{m},n,k} = c_{r-1} \sum_{j=0}^{\infty} \frac{\alpha_j}{u+j+1} \times \left\{ \int_{\beta}^{\infty} x^{u+j+1} \sum_{i=1}^{r} a_i(r)\gamma_i \left(1-F(x)\right)^{\gamma_i-1} f(x) \right\} \quad (4.15)$$

After using (4.9), we shall derive the recurrence relation given in (4.11).

Proof (b): We know that

$$\mu^{u,v}_{r,s:\tilde{m},n,k} = \int_{\beta}^{\infty} x^u \sum_{i=1}^{r} a_i(r)\left(1-F(x)\right)^{\gamma_i} \times \frac{f(x)}{1-F(x)} J\left(x:v,r,s,\tilde{m}\right)dx, \quad (4.16)$$

where

$$J\left(x:v,r,s,\tilde{m}\right) = c_{s-1}\int_{x}^{\infty} y^v \sum_{i=r+1}^{s} a_i^r(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i} \frac{f(y)}{1-F(y)}dy.$$

Substituting f(y) from (2.3) we have:

$$J\left(x:v,r,s,\tilde{m}\right) = \sum_{j=0}^{\infty} \alpha_j \left\{ c_{s-1}\int_{x}^{\infty} y^{v+j} \sum_{i=r+1}^{s} a_i^r(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i} dy \right\}. \quad (4.17)$$

Integrating by parts, taking y^{v+j} as the part to be integrated, we obtain:

$$J\left(x:v,r,s,\tilde{m}\right) = \sum_{j=0}^{\infty} \frac{\alpha_j}{v+j+1} \left\{ c_{s-1}\int_{x}^{\infty} y^{v+j+1} \times \sum_{i=r+1}^{s} a_i^r(s) \frac{\gamma_i\left(1-F(y)\right)^{\gamma_i-1}}{\left(1-F(x)\right)^{\gamma_i}} f(y)\, dy \right\}$$

$$= \sum_{j=0}^{\infty} \frac{\alpha_j}{v+j+1} \left\{ \int_{x}^{\infty} y^{v+j+1}\frac{f(y)}{1-F(y)} \times \left[c_{s-1}\sum_{i=r+1}^{s} a_i^r(s)\frac{\gamma_i\left(1-F(y)\right)^{\gamma_i}}{\left(1-F(x)\right)^{\gamma_i}}\right] dy \right\}$$

$$= \gamma_s \sum_{j=0}^{\infty} \frac{\alpha_j}{v+j+1} \left\{ \left[c_{s-1}\int_{x}^{\infty} y^{v+j+1} \sum_{i=r+1}^{s} a_i^r(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i} - c_{s-2}\int_{x}^{\infty} y^{v+j+1} \sum_{i=r+1}^{s-1} a_i^r(s-1)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i} \right] \frac{f(y)}{1-F(y)}dy \right\}$$

$$\left[\text{by using } (22)\right]$$

$$= \gamma_s \sum_{j=0}^{\infty} \frac{\alpha_j}{v+j+1} \left[\begin{array}{l} J\left(x:v+j+1,r,s,\tilde{m}\right) \\ -J\left(x:v+j+1,r,s-1,\tilde{m}\right)\end{array}\right]. \quad (4.18)$$

On substituting the above expression of $J\left(x:v,r,s,\tilde{m}\right)$ in (4.16) we get

$$\mu^{u,v}_{r,s:\tilde{m},n,k} = \gamma_s \sum_{j=0}^{\infty} \frac{\alpha_j}{v+j+1} \int_{\beta}^{\infty} x^u \sum_{i=1}^{r} a_i(r)\left(1-F(x)\right)^{\gamma_i} \frac{f(x)}{1-F(x)} \times \left\{\begin{array}{l} J\left(x:v+j+1,r,s,\tilde{m}\right) \\ -J\left(x:v+j+1,r,s-1,\tilde{m}\right)\end{array}\right\} dx$$

which on using (4.16), leads to (4.12).

5. Conclusion

In the study presented above, we demonstrate the recurrence relations for single and product moments of GOS from Extreme Value Distribution. These results generalize the corresponding results of Kumar (2010) and Saran and Pandey (2004).

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