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Characterization of Distribution by Conditional Expectation of Lower Record Values

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Abstract It is widely known that the problem of characterizing a distribution an important problem which has recently attracted the attention of many researchers. Thus various characterizations have been established in many directions. In this paper, a general form of continuous probability distribution is characterized through conditional expectation of contrast of lower record statistics, conditioned on a non-adjacent record statistics and some of its deductions are also discussed.

Keywords: *characterization, conditional expectation, continuous distributions, lower record values*

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1. Introduction

The record values were introduced by [1]. Suppose that $\{X_i\}_{i\geq 1}$ is a sequence of independent and identically distributed random variables with common distribution function $F(x)$ and common probability density function $f(x)$. Set $Y_n = \min(\max)\{X_1, X_2, \dots, X_n\}$ for $n \geq 1$. We say X_j is a lower (upper) record values of this sequence if $Y_j < (>) Y_{j-1}$ for $j > 1$. By definition X_1 is a lower as well as upper record values and $L_n = \min\{j \mid j > L_{n-1}, Y_j < Y\, L_{n-1}, n \geq 2\}$ with $L_1 = 1$ denote the times of lower record values.

Record values are found in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records, e.g. Olympic records. It is also useful in reliability theory, meteorology, hydrology, seismology, mining. For a more specific example, consider the situation of testing the breaking strength of wooden beams as described by [2].

For comprehensive accounts of the theory and applications of record values, we refer the readers to [3,4,5,6].

2. Objective

Characterizing the distributions via their record statistics has a long history. For excellent review one may refer to [7-15] amongst others.

The aim of this paper is to characterize a general class of distributions via the contrast of the conditional expectation of function of lower record statistics, conditioned on non-adjacent lower record statistics.

3. Method

Let $X_{L(1)}, X_{L(2)}, \dots, X_{L(r)}$ be the first r lower record statistics from a population whose probability density function *pdf* is $f(x)$ and the distribution function (df) is $F(x)$. Let $H(x) = -\log F(x)$ Then the *pdf* of $X_{L(r)}$, $r = 1, 2, \dots$ is

$$f_r(x) = \frac{[H(x)]^{r-1} f(x)}{\Gamma(r)}, -\infty < x < \infty \tag{3.1}$$

and the joint *pdf* of two lower records $X_{L(r)}$ and $X_{L(s)}$, $r < s$, $r, s = 1, 2, \dots$ is

$$f_{rs}(x, y) = \frac{1}{\Gamma(r)\Gamma(s-r)}[H(x)]^{r-1} \bullet [H(y) - H(x)]^{s-r-1} h^*(x) f(y) \tag{3.2}$$

where $h^*(x) = -\frac{d\, H(x)}{dx}$.

Then the conditional *pdf* of $X_{L(s)}$ given $X_{L(r)} = x$, $1 \leq r < s$ is

$$f(X_{L(s)} \mid X_{L(r)} = x) = \frac{1}{\Gamma(s-r)}[-\ln F(y) + \ln F(x)]^{s-r-1} \frac{f(y)}{F(x)} \tag{3.3}$$

4. Characterization Result

Theorem: Let X be an absolutely continuous random variable with the *df* $F(x)$ and the *pdf* $f(x)$ on the support (α,β), where α and β may be finite or infinite. Then for $1 \le m < r < s \le n$,

$$\sum_{i=r}^{m} c_i E[h(X_{L(l)}) \mid X_{L(s)} = x] = \frac{1}{a}\sum_{i=r}^{m} lc_i, \quad l = i-1, i \tag{4.1}$$

if and only if

$$F(x) = e^{-a\,h(x)}, x \in (\alpha,\beta), a > 0 \tag{4.2}$$

where c_i are real numbers $r \le i \le m$, satisfying $\sum_{i=r}^{m} c_i = 0$, $c_i \ne 0$ for some i and $h(x)$ is a non-increasing and differentiable function of x such that $F(x)$ is a *df* .

Proof: First we will prove (4.2) implies (4.1). We have from [14] $E[h(X_{L(i)}) \mid X_{L(s)} = x] = h(x) + \frac{i-s}{a}$ for $F(x) = e^{-a\,h(x)}, x \in (\alpha,\beta), a > 0$

Therefore,

$$\begin{aligned}\sum_{i=r}^{m} c_i E[h(X_{L(i)}) \mid X_{L(s)} = x] &= \sum_{i=r}^{m} c_i \left[h(x) + \frac{i-s}{a} \right] \\ &= \frac{1}{a}\sum_{i=r}^{m} i\,c_i \quad as \quad \sum_{i=r}^{m} c_i = 0\end{aligned} \tag{4.3}$$

hence the ‘if’ part.

To prove the sufficiency part, we have

$$\sum_{i=r}^{m} c_i E[h(X_{L(i)}) \mid X_{L(s)} = x] = \frac{1}{a}\sum_{i=r}^{m} i\,c_i \tag{4.4}$$

or,

$$\sum_{i=r}^{m} c_i \frac{a}{\Gamma(i-s)} \int_{\alpha}^{s} h(y) \begin{bmatrix} -\ln F(y) \\ +\ln F(x) \end{bmatrix}^{i-s-1} \frac{f(y)}{F(x)} dy = \sum_{i=r}^{m} i\,c_i \tag{4.5}$$

Integrating left hand side of (4.5) by parts, we get

$$\begin{aligned}&-\sum_{i=r}^{m} c_i \frac{a}{\Gamma(i-s)} \int_{\alpha}^{x} h'(y) [-\ln F(y) + \ln F(x)]^{i-s-1} \frac{F(y)}{F(x)} dy \\ &+\sum_{i=r}^{m} c_i \frac{a}{\Gamma(i-s-1)} \int_{\alpha}^{x} h(y) \begin{bmatrix} -\ln F(y) \\ +\ln F(x) \end{bmatrix}^{i-s-2} \frac{f(y)}{F(x)} dy \\ &= \sum_{i=r}^{m} i\,c_i\end{aligned} \tag{4.6}$$

That is,

$$\begin{aligned}&-\sum_{i=r}^{m} c_i \frac{a}{\Gamma(i-s)} \int_{\alpha}^{x} h'(y) \begin{bmatrix} -\ln F(y) \\ +\ln F(x) \end{bmatrix}^{i-s-1} \frac{F(y)}{F(x)} dy \\ &= \sum_{i=r}^{m} c_i (i-i+1) = 0\end{aligned} \tag{4.7}$$

Now from (3.3), we have

$$\frac{1}{\Gamma(i-s)} \int_{\alpha}^{x} [-\ln F(y) + \ln F(x)]^{i-s-1} \frac{f(y)}{F(x)} dy = 1$$

Therefore,

$$\sum_{i=r}^{m} c_i \frac{1}{\Gamma(i-s)} \int_{\alpha}^{x} \begin{bmatrix} -\ln F(y) \\ +\ln F(x) \end{bmatrix}^{i-s-1} \frac{f(y)}{F(x)} dy = 0 \tag{4.8}$$

Comparing (4.7) and (4.8), we get

$$-a\,h'(y) F(y) = f(y)$$

implying

$$F(y) = e^{-a\,h(y)}, a > 0$$

and hence the Theorem.

Remark: Putting $c_m = 1$ and $c_r = -1$ in Theorem , we get the characterizing result as obtained by [14].

Table 4.1. Examples based on the distribution function $F(x) = e^{-a\,h(x)}, \quad a > 0$

Distribution	$F(x)$	a	$h(x)$
Inverse Weibull	$e^{-\theta x^{-p}} \quad 0 < x < \infty$	θ	x^{-p}
Power function	$\left(\frac{x}{a}\right)^p \quad 0 < x < a$	p	$-\log(x/a)$
Logistic	$(1+e^{-x})^{-1}$, $-\infty < x < \infty$	1	$\log(1+e^{-x})$
Burr Type II	$(1+e^{-x})^{-k}$, $-\infty < x < \infty$	k	$\log(1+e^{-x})$
Burr Type III	$(1+x^{-c})^{-k}$, $0 < x < \infty$	k	$\log(1+x^{-c})$
Burr Type IV	$\left[1+\left(\frac{c-x}{x}\right)^{1/c}\right]^{-k}$, $0 < x < c$	k	$\log\left[1+\left(\frac{c-x}{x}\right)^{1/c}\right]$
Burr Type V	$(1+ce^{-\tan x})^{-k}$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$	k	$\log(1+ce^{-\tan x})$
Burr Type VI	$(1+ce^{-k\sinh x})^{-k}$, $-\infty < x < \infty$	k	$\log(1+ce^{-k\sinh x})$
Burr Type VII	$\left(\frac{1+\tanh x}{2}\right)^k$, $-\infty < x < \infty$	k	$-\log\left(\frac{1+\tanh x}{2}\right)$
Burr Type VIII	$\left(\frac{2}{\pi}\tan^{-1} e^x\right)^k$, $-\infty < x < \infty$	k	$-\log\left(\frac{2}{\pi}\tan^{-1} e^x\right)$
Burr Type X	$(1-e^{-x2})^k$, $0 < x < \infty$	k	$-\log(1-e^{-x2})$
Burr Type XI	$(x-\frac{1}{2\pi}\sin 2\pi x)^k$, $0 < x < 1$	k	$-\log(x-\frac{1}{2\pi}\sin 2\pi x)$
Gumbel	$\exp[-e^{-x}]$, $-\infty < x < \infty$	1	e^{-x}
Extreme value II	$e^{-\left(\frac{\theta}{x}\right)^p} \quad 0 < x < \infty$	θ^p	x^{-p}

5. Discussion

The purpose of this paper was to characterize a general classs of probability distribution through the conditional expectation based on lower record statistics conditioned on non-adjacent lower record statistics using the contrast technique. We hope that findings of this paper will useful for the researcher in various fields. Further advancement of research in distribution theory, lower record theory and their application.

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