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Common Fixed Point of $F-$ type Contractive Mappings in Generalized Orthogonal Metric Spaces

Qiancheng Wang[1], Hongyan Guan[1,*]

[1]School of Mathematics and Systems Science, Shenyang normal university, Shenyang 110034, China
*Corresponding author: guanhy8010@163.com

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Abstract In this paper, we propose a new class of orthogonal $F-$type contractive mappings, and prove one common fixed point theorem in complete orthogonal $b-$metric spaces. We also provide an example that supports our result.

Keywords: *O-b-metric space, fixed point, $O-\alpha-$admissible, orthogonal generalized contractive mapping, $F-$type function*

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1. Introduction

Fixed point theory is an important part of modern analysis. In particular, Banach contraction mapping principle [1] is an effective method to solve the problem of the existence and uniqueness of fixed points in complete metric space, and plays an important role in nonlinear analysis. About a century ago, Banach started as an abstract successive approximation method for solving differential equations, and later defined it as the concept of contraction mapping. Thus, the first fixed point theorem was produced. Later, many scholars gave some important generalizations of this result by changing the space type or contractive conditions. Istratescu [2,3] provided one of the most important ideas of convex contraction and proved some fixed point results. Another interesting extension of fixed point theory, known as "almost contraction map", was introduced by Berinde [4]. In contrast, there are multiple ways in which this concept of measurement has developed. In 1993, Czerwik [5] gave a generalized concept of metric spaces, called $b-$metric spaces, by changing the form of trigonometric inequality defined by metric spaces, and the author also proved some new fixed point theorems in this kind of spaces. Afterwards, many scholars carried out researches and got a lot of excellent results in this kind of space (see [6,7,8,9]), and the literatures cited therein. In 2012, Wardowski [9] gave a new type of compression mapping in complete metric space. That is, $F-$type contraction, and some sufficient conditions for the existence and uniqueness of fixed point of this type of mapping are obtained. Recently, Gordji et al. [10] introduced the concept of orthogonality, and proved the fixed point theorem in orthogonal complete metric space. In 2022, Eiman et al. [11] introduced the concept of orthogonal $L-$ contraction mappings and proved the fixed point theorem. Also in 2022, Dhanraj et al. [12] adopted the orthogonal Geraghty type for $\alpha-$admissible contraction mapping, fixed-point theorem are proved on orthogonal complete Branciari $b-$ metric spaces. In 2023, many researchers have deeply studied different types of contraction mapping based on complete orthogonal spaces, and have given applications (see [13,14,15,16]). In addition, many researchers have improved and generalized the concept of orthogonal metric spaces (see [17,18,19]).

In this paper, we propose a new class of contraction for double mappings of square and quadratic forms, and prove some fixed point theorems in an orthogonal complete $b-$metric space. Meanwhile, we provide a specific example to demonstrate the effectiveness of the result.

2. Preliminaries

Definition 2.1. Suppose $s\geq 1$ is a constant and X is a nonempty set. A function $d:X\times X\to[0,+\infty)$ is said to be a $b-$metric if for any $x,y,z\in X$,

(1)$d(x,y)=0$ if and only if $x=y$;

(2)$d(x,y)=d(y,x)$;

(3)$d(x,y)\leq s[d(x,y)+d(y,z)]$.

Generally, (X,d) is called a $b-$metric space.

Definition 2.2. Suppose (X,d) is a $b-$metric space, $x\in X$ and $\{x_n\}$ is a sequence in X .

(a) $\{x_n\}$ is convergent in (X, d) and converges to x, if for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_\varepsilon$. We denote this as $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(b) $\{x_n\}$ is a Cauchy sequence in (X, d), if for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$d(x_n, x_m) < \varepsilon$ for all $n, m > n_\varepsilon$.

Definition 2.3. Let X be a nonempty set and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ holds with the constraint

$$\exists x_0 \in X: \ (\forall x \in X,\ x \perp x_0) \ or \ (\forall x \in X,\ x_0 \perp x),$$

then $(X, \perp)$ is said to be an orthogonal set (briefly $O-$ set).

Definition 2.4. Let $(X, \perp, d)$ be an orthogonal metric space. Then, X is said to be $O-$ complete if every orthogonal Cauchy sequence is convergent.

Definition 2.5. A tripled $(X, \perp, d)$ is called an $O_b - MS$ if $(X, \perp)$ is an orthogonal set and (X, d) is a $b-$metric space.

Definition 2.6. Let $(X, \perp)$ be an orthogonal set. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called an orthogonal sequence ($O-$ sequence) if

$$(\forall n,\ x_n \perp x_{n+1}) \ or \ (\forall x,\ x_{n+1} \perp x_n).$$

Definition 2.7. Suppose $(X, \perp, d)$ is an $O_b - MS$. Then, $f: X \to X$ is said to be orthogonally continuous at $x \in X$ if, for each $O-$ sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with $x_n \to x$, we have $f(x_n) \to f(x)$. Also, f is said to be orthogonal continuous on X if f is orthogonal continuous at each $x \in X$.

Definition 2.8. Let X be a nonempty set, and f, g be two self mappings on X. f and g are called a pair of weakly compatible mappings, if they are commutative at each coincidence point, that is, $fx = gx \Rightarrow fgx = gfx.$

Definition 2.9. Let $(X, \perp)$ be an orthogonal set. A function $f: X \to X$ is called an orthogonal-preserving mapping if $fx \perp fy$ whenever $x \perp y$.

Definition 2.10. Let (X, d) be a complete $b-$metric space with parameter $s \ge 1$, $f, g: X \to X$ and $\alpha_s: X \times X \to \mathbb{R}^+$. Then, f is said to be $O - fg^{-1} - \alpha_s -$ admissible, if $\forall x, y \in X$ with $x \perp y$,

$$\alpha_s\left(fx, fy\right) \ge s^p \Rightarrow \alpha_s\left(fg^{-1}fx, fg^{-1}fy\right) \ge s^p.$$

Hypothesis 2.11. Let (X, d) be a complete $b-$metric space with parameter $s \ge 1$, let $\alpha_s: X \times X \to \mathbb{R}^+$ be a function.

(H_{s^p}) If $\{x_n\}$ is a sequence in X such that $gx_n \to gx$ as $n \to \infty$, then there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ with $\alpha(gx_{n_k}, gx) \ge s^p$ for all $k \in \mathbb{N}$.

(U_{s^p}) For all $u, v \in C(f, g)$, we have $\alpha(gu, gv) \ge s^p$, $\alpha(gv, gu) \ge s^p$.

(V_{s^p}) For all $u, v, w \in X$, $\alpha_s(u, v) \ge s^p$, $\alpha_s(v, w) \ge s^p$, we have $\alpha_s(u, w) \ge s^p$.

Definition 2.12. Let Δ denote the family of all functions $F: \mathbb{R}^+ \to \mathbb{R}$ satisfying the following properties:

(F_1) F is strictly increasing ;

(F_2) for each sequence $\{x_n\}_{n=1}^\infty$ of positive numbers, we have $\lim_{n\to\infty} x_n = 0, \lim_{n\to\infty} F(x_n) = -\infty;$

(F_3) there exists $k \in (0,1)$ such that $\lim_{x\to 0^+} x^k F(x) = 0;$

(F_4) If $\forall n \in \mathbb{N}, \tau + F(s^2 x_n) \le F(x_{n-1})$, we have

$$\tau + F(s^{2n} x_n) \le F(s^{2(n-1)} x_{n-1}).$$

Lemma 2.13. Let (X, d) be a $b-$metric space with parameter $s \ge 1$. Assume that $\{x_n\}$ and $\{y_n\}$ are $b-$ convergent to x and y, respectively. Then, we have

$$s^{-2} d(x, y) \le \liminf_{n\to\infty} d(x_n, y_n)$$
$$\le \limsup_{n\to\infty} d(x_n, y_n) \le s^2 d(x, y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$s^{-1} d(x, z) \le \liminf_{n\to\infty} d(x_n, z)$$
$$\le \limsup_{n\to\infty} d(x_n, z) \le s d(x, z).$$
$$s^{-1} d(x, z) \le \liminf_{n\to\infty} d(x_n, z)$$
$$\le \limsup_{n\to\infty} d(x_n, z) \le s d(x, z).$$

3. Main Results

Theorem 3.1. Let (X, d) be an orthogonal complete $b-$metric space, with parameter $s \ge 2$. Suppose $f, g: X \to X$ satisfy the following conditions:

(1) f is orthogonal continuous, f, g are weakly compatible ;

(2) $f(X) \subset g(X)$, and $g(X)$ is closed ;

(3) f is a $O - fg^{-1} - \alpha_{s^p} -$ admissible mapping ;

(4) f, g^{-1} are orthogonal preserving ;

(5) there is an orthogonal element $x_0 \in X$ satisfying $\alpha(fx_0, fx_0) \geq s^p$;

(6) If $x, y \in X, x \perp y, d(x,y) \geq 0$,we have :

$$\tau + F(\alpha_s(fx, gy)d^2(fx, fy)) \leq F(N(x,y)), \quad (1)$$

where

$$N(x,y) = \max\{d^2(fx,gx), d^2(gx,gy), d^2(fy,gy), \frac{1}{s}d(fx,gx)d(fx,fy), d(fx,gx)d(gx,gy)\},$$

$F: \mathbb{R}^+ \to \mathbb{R}$ is a function such that $F_1 - F_4$, $\tau > 0$, $p \geq 3$ is a constant, $\alpha_s : X \times X \to \mathbb{R}$ satisfying that $\alpha_s(x,y) \geq s^p$, and properties $(H_{s^p}), (U_{s^p})$ and (V_{s^p}). Then f and g possess a common fixed point in X. Moreover, f, g possess a unique common fixed point in

$$y^{*\perp} = \{x \mid x \perp y^* (or) y^* \perp x, x \in X\}.$$

Proof : By the definition of orthogonality, we find that x_0 with $x_0 \perp y$ or $y \perp x_0$, for all $y \in X$. Since $f(X) \subset g(X)$, there exists $x_1 \in X$, such that $fx_0 = gx_1$. In turn define sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_n = fx_n = gx_{n+1}$ for $n \in \mathbb{N}$.

Since $x_1 \in X$ and f is orthogonal-preserving, without loss of generality, then we obtain $x_0 \perp x_1$, $fx_0 \perp fx_1$, and $y_0 \perp y_1, gx_1 \perp gx_2$. It follows from g^{-1} and f are orthogonal-preserving that $g^{-1}gx_1 \perp g^{-1}gx_2, x_1 \perp x_2$. Thus, we have $fx_1 \perp fx_2, y_1 \perp y_2$, which imply that $\{x_n\}, \{y_n\}$ are orthogonal sequences.

For orthogonal element $x_0 \in X$, in light of condition (3),(5), we obtain $\alpha(fx_0, fx_0) \geq s^p$,

$$\alpha(fx_0, fx_0) = \alpha(y_0, y_0) = \alpha(fx_0, gx_1) \geq s^p,$$
$$\alpha(fg^{-1}fx_0, fg^{-1}gx_1) = \alpha(y_1, y_1) \geq s^p,$$
$$\alpha(y_1, y_1) = \alpha(fx_1, gx_2) \geq s^p,$$
$$\alpha(fg^{-1}fx_1, fg^{-1}gx_2) = \alpha(y_2, y_2) \geq s^p.$$

Hence, for all $n \in \mathbb{N}$, we deduce $\alpha(y_n, y_n) \geq s^p$. Replacing x by x_n and y by x_{n+1} in (1), we have

$$\tau + F(\alpha_s(fx_n, gx_{n+1})d^2(fx_n, fx_{n+1})) \leq F(N(x_n, x_{n+1})).$$

That is,

$$\tau + F(\alpha_s(fx_n, gx_{n+1})d^2(y_n, y_{n+1})) \leq F(N(x_n, x_{n+1})).$$

Since $s^2 d^2(y_n, y_{n+1}) \leq \alpha_s(fx_n, gx_{n+1})d^2(y_n, y_{n+1})$, then

$$\tau + F(s^2 d^2(y_n, y_{n+1})) \leq \tau + F(\alpha_s(fx_n, gx_{n+1})d^2(y_n, y_{n+1})) \leq F(N(x_n, x_{n+1})),$$

where

$$N(x_n, x_{n+1}) = \max\{d^2(y_n, y_{n-1}), d^2(y_{n-1}, y_n), d^2(y_{n+1}, y_n), \frac{1}{s}d(y_n, y_{n-1})d(y_n, y_{n+1}), d^2(y_n, y_{n-1})\}.$$

If $d(y_{n-1}, y_n) < d(y_{n+1}, y_n)$, then we have

$$\tau + F(s^2 d^2(y_n, y_{n+1})) \leq F(d^2(y_n, y_{n+1})).$$

Since $\tau > 0, F$ is strictly increasing, and $s \geq 1$, this is a contradiction. Thus, $d(y_{n-1}, y_n) \geq d(y_{n+1}, y_n)$, and the inequality becomes

$$\tau + F(s^2 d^2(y_n, y_{n+1})) \leq F(d^2(y_{n-1}, y_n)).$$

According to (F_4), we get

$$\tau + F(s^{2n} d^2(y_n, y_{n+1})) \leq F(s^{2(n-1)} d^2(y_{n-1}, y_n)). \quad (2)$$

By calculation, we get

$$\tau + F(s^{2n} d^2(y_n, y_{n+1})) \leq F(s^{2(n-1)} d^2(y_{n-1}, y_n)),$$
$$\tau + F(s^{2(n-1)} d^2(y_{n-1}, y_n)) \leq F(s^{2(n-2)} d^2(y_{n-2}, y_{n-1})),$$
$$\vdots$$
$$\tau + F(s^2 d^2(y_1, y_2)) \leq F(d^2(y_0, y_1)).$$

Obtained through organization

$$F(s^{2n} d^2(y_n, y_{n+1})) \leq F(d^2(y_0, y_1)) - n\tau. \quad (3)$$

In (3), letting $n \to \infty$,we have

$$\lim_{n\to\infty} F(s^{2n} d^2(y_n, y_{n+1})) = -\infty.$$

Thus, $\lim_{n\to\infty} s^{2n} d^2(y_n, y_{n+1}) = 0$.

According to (F_3), there exists $k \in (0, \frac{1}{2})$ such that

$$\lim_{n\to\infty} (s^{2n} d^2(y_n, y_{n+1}))^k F(s^{2n} d^2(y_n, y_{n+1})) = 0.$$

In (3), multiplying $(s^{2n} d^2(y_n, y_{n+1}))^k$, we have

$$(s^{2n} d^2(y_n, y_{n+1}))^k F(s^{2n} d^2(y_n, y_{n+1})) - (s^{2n} d^2(y_n, y_{n+1}))^k F(d^2(y_0, y_1)) \leq -(s^{2n} d^2(y_n, y_{n+1}))^k n\tau. \quad (4)$$

Taking $n \to \infty$ in (4), we have

$$\lim_{n\to\infty} n(s^{2n} d^2(y_n, y_{n+1}))^k = 0. \quad (5)$$

Hence there exists $n_1 \in \mathbb{N}$ such that

$n(s^{2n}d^2(y_n, y_{n+1}))^k \le 1,\ s^{2n}d^2(y_n, y_{n+1}) \le \frac{1}{n^{\frac{1}{k}}}$, and

$s^n d(y_n, y_{n+1}) \le \frac{1}{n^{\frac{1}{k}}}$ as $n \ge n_1$.

Next, we are going to prove $\{y_n\}$ is Cauchy. For ease of use, set $d_n = d(y_n, y_{n+1})$.

So

$$d^2(y_n, y_{n+i}) \le (sd_n + s^2 d_{n+1} + \cdots + s^{i-1}d_{n+i-2} + s^{i-1}d_{n+i-1})^2$$

and

$$\begin{aligned} & sd_n + s^2 d_{n+1} + \cdots + s^{i-1}d_{n+i-2} + s^{i-1}d_{n+i-1} \\ & \le s^n d_n + s^{n+1}d_{n+1} + \cdots + s^{n+i-2}d_{n+i-2} + s^{n+i-2}d_{n+i-1} \\ & \le s^n d_n + s^{n+1}d_{n+1} + \cdots + s^{n+i-2}d_{n+i-2} + s^{n+i-1}d_{n+i-1} \\ & = \sum_{i=n}^{n+i-1} s^i d_i \le \sum_{i=n}^{\infty} s^i d_i \le \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{2k}}}. \end{aligned}$$

Since $k \in (0, \frac{1}{2})$,and $\frac{1}{2k} > 1$, $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{2k}}} = 0$, then

$$\lim_{n\to\infty}(sd_n + s^2 d_{n+1} + \cdots + s^{i-1}d_{n+i-2} + s^{i-1}d_{n+i-1}) = 0$$

and $\lim_{n\to\infty} d(y_n, y_{n+i}) = 0$. Therefore, there exists $y^* \in X$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = y^*.$$

Since $g(X)$ is closed, there is a $u \in X$ satisfying $gu = y^*$. Next, we will prove that $fu = y^*$. In view of the property (H_{s^p}), one can get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $\alpha(gx_{n_k}, y^*) \ge s^p, \alpha(fx_{n_k}, y^*) \ge s^p$ for all $k \in \mathbb{N}$.

Since $\lim_{n\to\infty} fx_n = y^*$, and f is orthogonal continuous, we have $y^* = \lim_{n\to\infty} fx_n = f \lim_{n\to\infty} fx_{n-1} = fy^*$. In view of the condition $x_0 \perp y^*$ and f is orthogonal preserving, one can deduce that

$$f^{n_k-1}x_0 \perp f^{n_k-1}y^*, f^{n_k-1}y^* \perp y^*, f^{n_k-1}x_0 \perp y^*.$$

Because g^{-1} is orthogonal preserving and $y_{n_k-1} \perp y^*$, we have $g^{-1}y_{n_k-1} \perp g^{-1}y^*$. Since $y^* = gu$, thus $x_{n_k} \perp u$.

Replacing x by x_{n_k} and y by u in (1), we have

$$\begin{aligned} & F(s^3 d^2(fx_{n_k}, fu)) \\ & \le F(\alpha_s(fx_{n_k}, gu)d^2(fx_{n_k}, fu)) \\ & < \tau + F(\alpha_s(fx_{n_k}, gu)d^2(fx_{n_k}, fu)) \\ & \le F(\max\{d^2(fx_{n_k}, gx_{n_k}), d^2(gx_{n_k}, gu), d^2(fu, gu), \\ & \quad \frac{1}{s}d(fx_{n_k}, gx_{n_k})d(fx_{n_k}, fu), d(fx_{n_k}, gx_{n_k})d(gx_{n_k}, gu)\}). \end{aligned}$$

Since F is strictly increasing, $\tau > 0$, we get

$$\begin{aligned} & s^3 d^2(fx_{n_k}, fu) \\ & \le \alpha_s(fx_{n_k}, gu)d^2(fx_{n_k}, fu) \\ & \le \max\{d^2(fx_{n_k}, gx_{n_k}), d^2(gx_{n_k}, gu), d^2(fu, gu), \\ & \quad \frac{1}{s}d(fx_{n_k}, gx_{n_k})d(fx_{n_k}, fu), d(fx_{n_k}, gx_{n_k})d(gx_{n_k}, gu)\}. \end{aligned}$$

Letting $n \to \infty$, from Lemma 2.13, we obtain

$$\begin{aligned} s^3 \frac{1}{s^2} d^2(fu, y^*) & \le \alpha_s(fx_{n_k}, gu)\frac{1}{s^2} d^2(fu, y^*) \\ & \le d^2(fu, y^*). \end{aligned}$$

This is a contradiction. So $d^2(fu, y^*) = 0$ and $fu = y^*$.

Since f, g are weakly compatible, one can get $fu = y^* = gu,\ fgu = gfu,\ fy^* = gy^*$. By the continuity of f , we have $fy^* = y^*$. Therefore, $fy^* = y^* = gy^*$, that is, f and g possess a common fixed point in X .

Next, we will prove that f and g possess a unique common fixed point in

$$y^{*\perp} = \{x \mid x \perp y^* (or) y^* \perp x, x \in X\}.$$

First, $y^{*\perp}$ is nonempty set, because $\{y_n\} \subseteq y^{*\perp}$.

If there exists $t \perp y^*$, and $t \ne y^*, t$ is a common fixed point of f, g , then $ft = gt = t \ne y^*$. Replacing x by y^* and y by t in (1),

$$\begin{aligned} & F(s^3 d^2(fy^*, ft)) \\ & \le F(\alpha_s(fy^*, gt)d^2(fy^*, ft)) \\ & < \tau + F(\alpha_s(fy^*, gt)d^2(fy^*, ft)) \\ & \le F(\max\{d^2(fy^*, gy^*), d^2(gy^*, gt), d^2(ft, gt), \\ & \quad \frac{1}{s}d(fy^*, gy^*)d(fy^*, ft), d(fy^*, gy^*)d(gy^*, gt)\}). \end{aligned}$$

We have

$$F(s^3 d^2(fy^*, ft)) \le F(\alpha_s(fy^*, gt) d^2(fy^*, ft))$$
$$< \tau + F(\alpha_s(fy^*, gt) d^2(fy^*, ft)) \le F(d^2(gy^*, gt))$$

Since $ft = gt = t \ne y^*, \tau > 0$ and F is strictly increasing, then $s^3 d^2(y^*, t) \le d^2(y^*, t),$ a contradiction. It follows that $y^* = t.$ That is, f, g possess a unique common fixed point in $y^{*\perp} = \{x \mid x \perp y^* (or) y^* \perp x, x \in X\}$.

Example 3.2 Let $X = [0, 2]$ and $d : X \times X \to [0, +\infty)$ be a mapping defined by $d(x, y) = |x - y|^2$, for all $x, y \in X$. Define the binary relation $\perp$ on X by $x \perp y$ if $xy \le (x + 2 \vee y + 2),$ where

$$x + 2 \vee y + 2 = x + 2 \quad or \quad y + 2,$$

Then (X, d) is an $O-$ complete $b-$ metric space. Define the mappings $f, g : X \to X$ by

$$f(x) = \begin{cases} x^2, x \in [0,1) \\ \dfrac{1}{x}, x \in [1,2] \end{cases}, g(x) = \begin{cases} x, x \in [0,1) \\ 2x - 1, x \in [1,2] \end{cases},$$

$$\alpha(x, y) = \begin{cases} 2^3 + 1, x, y \in [0,1) \\ 2^3 + 1, x, y \in [1,3]. \\ 0, otherwise \end{cases}$$

Clearly, f, g^{-1} are orthogonal preserving, f is orthogonal continuous, f, g are weakly compatible, $f(X) \subset g(X)$,and $g(X)$ is closed. Now, let us consider the mapping F defined by

$$F(t) = \ln t, \quad \tau = \ln(\frac{1}{2^7}).$$

Let $x_0 = \dfrac{1}{2}$. If $x \in [0,1)$, we have

$$\frac{1}{2} x \le \frac{1}{2} + 2 \Rightarrow \frac{1}{2} \perp x.$$

If $x \in [1, 2]$, we have $\dfrac{1}{2} x \le x + 2 \Rightarrow \dfrac{1}{2} \perp x.$ So $\dfrac{1}{2}$ is orthogonal element in X . It is easy to show that

$$\alpha(f(\frac{1}{2}), f(\frac{1}{2})) = \alpha(\frac{1}{4}, \frac{1}{4}) \ge 2^3,$$

$$\alpha(fg^{-1}f(\frac{1}{2}), fg^{-1}f(\frac{1}{2})) = \alpha(f(\frac{1}{4}), f(\frac{1}{4}))$$
$$= \alpha(\frac{1}{16}, \frac{1}{16}) \ge 2^3, \quad \text{which}$$

$$\alpha(fg^{-1}f(\frac{1}{4}), fg^{-1}f(\frac{1}{4})) = \alpha(f(\frac{1}{16}), f(\frac{1}{16})$$
$$= \alpha(\frac{1}{16^2}, \frac{1}{16^2}) \ge 2^3,$$

imply that f is an $O - fg^{-1} - \alpha_s -$ admissible mapping. Next we show that f, g^{-1} are orthogonal preserving.

Case 1: $x \in [0,1), y \in [0,1)$. We have

$$f(x) \cdot f(y) = x^2 \cdot y^2 \le x + 2,$$
$$g^{-1}(x) \cdot g^{-1}(y) = x \cdot y \le x + 2$$

Case 2: $x \in [1, 2], y \in [1, 2]$. We obtain

$$f(x) \cdot f(y) = \frac{1}{x} \cdot \frac{1}{y} \le \frac{1}{x} + 2,$$

$$g^{-1}(x) \cdot g^{-1}(y) = \frac{1}{2}(x+1) \cdot \frac{1}{2}(y+1) \le \frac{1}{2}(x+1) + 2.$$

Case 3: $x \in [0,1), y \in [1, 2]$. Clearly,

$$f(x) \cdot f(y) = x^2 \cdot \frac{1}{y} \le x^2 + 2,$$

$$g^{-1}(x) \cdot g^{-1}(y) = x \cdot \frac{1}{2}(y+1) \le \frac{1}{2}(y+1) + 2.$$

Case 4: $x \in [1, 2], y \in [0,1)$. It is obvious that

$$f(x) \cdot f(y) = y^2 \cdot \frac{1}{x} \le y^2 + 2,$$

$$g^{-1}(x) \cdot g^{-1}(y) = y \cdot \frac{1}{2}(x+1) \le \frac{1}{2}(x+1) + 2.$$

Hence, f, g^{-1} are orthogonal preserving.

Consider

Case 1: $f(x) \in [0,1), g(y) \in [0,1)$. Obviously,

$$\tau + \ln(2^3 |x^2 - y^2|^4) \le \ln(|x - y|^4),$$

$$\tau \le \ln(\frac{1}{2^3 |x + y|^4}) \le \ln(\frac{1}{2^7}).$$

It is clear that (1) is satisfied.

Case 2: $f(x) \in [1, 2], g(y) \in [1, 2]$. It is easy to show

$$\tau + \ln(2^3 \left|\frac{1}{x} - \frac{1}{y}\right|^4) \le \ln(|(2x-1) - (2y-1)|^4),$$

$$\tau \le \ln(2|xy|^4) \le \ln 2.$$

That is condition (1) holds.

Case 3: $f(x) \in [0,1), g(y) \in [1, 2],$ or $f(x) \in [1, 2], g(y) \in [0,1)$. Then $\alpha(fx, gy) = 0$.

Hence, (1) fulfills. Therefore, all the conditions of Theorem 3.1 are satisfied. Therefore, one can conclude that f and g possess a common fixed point in X . Obviously, $x = 1$ a common fixed point.

4. Conclusions

In this paper, we proved a fixed point theorem of a new class of orthogonal $F-$ type contractive mappings, in orthogonal $b-$ metric space. In addition, we also provided an example to explain in detail the practicality of the obtained results.

References

[1] Banach, S. Sur les operations dans les ensembles abstraits et leurs applications aux equations integrals [J]. Fundam. Math., 1992, 3, 133–181.

[2] Istratescu, V. Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters (I) [J]. Ann. Mat. Pure Appl, 1982, 130, 89–104.

[3] Istratescu, V. Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters (II) [J]. Ann. Mat. Pura Appl, 1983, 134, 327–362.

[4] Berinde, V. Approximating fixed points of weak contractions using the Picard iteration [J]. Nonlinear Anal Forum, 2004, 9, 43–53.

[5] Czerwik, S. Contraction mappings in *b*- metric spaces [J]. Acta Math Inform Univ Ostrav,1993,1:5-11.

[6] Abbas, J. Common fixed point of four maps in *b*- metric spaces.[J]. Hacet. J Math. Stat., 2014, 43(4): 613-624.

[7] Aydi, H., Bota, M.F., Karapinar, E., et al. A commonfixed point for weak ϕ- contractions on *b*- metric spaces [J]. Fixed Point Theory, 2012, 13(2): 337-346.

[8] Suzuki, T. A new type of fixed point theorem in metric spaces [J]. Nonlinear Anal., 2009, 71(11): 5313-5317.

[9] Wardowski, D. Fixed points of a new type of contractive mappings in complete metric spaces [J]. Fixed Point Theory Appl., 2012, 2012(1): 1-6.

[10] Gordji, M.E., Habibi, H. Fixed point theory in generalized orthogonal metric space. [J]. Linear Topol. Algebra, 2017, 6, 251–260.

[11] Aiman, M., Arul Joseph, G., Absar, U.H., Senthil Kumar, P., Gunaseelan, M., Imran, A.B. Solving an integral equation via orthogonal Brianciari metric spaces. [J]. J Funct. Spaces, 2022, 2022, 7251823.

[12] Dhanraj, M., Gnanaprakasam, A.J., Mani, G., Ege, O., De la Sen, M. Solution to integral equation in an *o*- complete Branciari *b*-metric spaces, Axioms, 2022,11(12): 728, 1-14.

[13] Al-Mazrooei, A.E., Ahmad, J. Fixed point approach to solve nonlinear fractional differential equations in orthogonal *F*- metric spaces, Aims Math., 2023,8(3): 5080-5098.

[14] Gardasevic-Filipovic, M., Kukic, K., Gardasevic, D., Mitrovic, Z.D. Some best proximity point results in the orthogonal *o*-complete *b*- metric - like spaces, J Contemp. Math. Anal. , 2023,58, 105-115.

[15] Gnanaprakasam, A.J., Mani, G., Ege, O., Aloqaily, A., Mlaiki, N. New fixed point results in orthogonal *b*- metric spaces with related applications, Mathematics, 2023,11(3): 677, 1-18.

[16] Prakasam, S.K., Gnanaprakasam, A.J., Mani, G., Jarad, F. Solving an integral equation via orthogonal generalized *a*- Ψ- Geraghty contractions, Aims Mathematics, 2023, 8(3): 5899-5917.

[17] Gnanaprakasam, A.J., Nallaselli, G., Haq, A.U., Mani, G., Baloch, I.A., Nonlaopon, K. Common fixed-points technique for the existence of a solution to fractional integro differential equations via orthogonal branciari metric spaces [J]. Symmetry, 2022, 14, 1859.

[18] Prakasam, S.K., Gnanaprakasam, A.J., Kausar, N., Mani, G., Munir, M. Solution of integral equation via orthogonally modified *F*- contraction mappings on *o*- complete metric-like space [J]. Int. J. Fuzzy Log. Intell. Syst, 2022, 22, 287–295.

[19] Senthil Kumar, P., Arul Joseph, G., Ege, O., Gunaseelan, M., Haque, S., Mlaiki, N. Fixed point for an *OgF-c* in *o*- complete *b*-metric-like spaces [J]. Aims Math., 2022, 8, 1022–1039.