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Wave Profile Investigation of the Higher Dimensional Nonlinear Evolution Equation through Nonlinear Auxiliary Equation

Hasibun Naher[1,*], Farah Aini Abdullah[2]

[1]Department of Mathematics and Natural Sciences, BRAC University, 66 Mohakhali, Dhaka 1212, Bangladesh
[2]School of Mathematical Sciences, Universiti Sains Malaysia, 11800 Penang, Malaysia
*Corresponding author: hasibun06tasauf@gmail.com

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Abstract In this article, more general and many new travelling wave solutions have been constructed through new extension of the (G′/G)-expansion method which is known as new generalized (G′/G)-expansion method. The key idea of this technique is to take full advantage of a higher ordinary nonlinear differential equation that has five different general solutions. The presentation of the travelling wave solutions is quite new and additional parameters are also used in the solution form. To illustrate the novelty and efficiency of this method, the (3+1)-dimensional Kadomstev-Petviashvili equation is desired to be investigated. The obtained solutions reveal the wider applicability to handle higher-dimensional nonlinear problems which arising in mathematical physics.

Keywords: *New generalized (G′/G)-expansion method, nonlinear auxiliary equation, (3+1)-dimensional Kadomstev-Petviashvili equation, travelling wave solutions, ordinary differential equations, and analytical solutions*

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1. Introduction

The investigation of nonlinear evolution equations (NLEEs) is one of the main themes in natural sciences, especially in the physical branches such as biophysics, plasma physics, solid state physics, nonlinear optics, quantum field theory, particle physics, fluid dynamics and so on. Due to importance of exact solutions of NLEEs in nonlinear science and engineering, it is required to construct new exact travelling wave solutions. In the recent past, various methods have been developed to produce explicit solutions by a diverse group of scientists. Such as, the Bäcklund transformation method [1], the Hirota’s bilinear method [2,3], the inverse scattering method [4], the Jacobi elliptic function method [5], the tanh-coth method [6,7], the F-expansion method [8], the exp- function method [9,10] and others [11-16].

Later on, Wang *et al.* [17] introduced a method with linear ordinary differential equation (LODE) which is called the (G'/G)-expansion method. Later on, many researchers [18-22] implemented this technique and proved that it is simple for producing travelling wave solutions.

In order to depict the effectiveness of the (G'/G)-expansion method, further research is carried out by a diverse group of scientists. For example, Zhang *et al.* [23] introduced an improved (G'/G)-expansion method. Therefore, a good number of researchers studied various nonlinear PDEs to produce analytical solutions [24-28]. Zayed [29] proposed another extension of (G'/G)-expansion method, where $G(\xi)$ satisfied the Jacobi elliptic equation: $\{G'(\xi)\}^2 = e_2 G^4(\xi) + e_1 G^2(\xi) + e_0$. Moreover, Zayed [30] extended the (G'/G)-method in which $G(\xi)$ satisfied the Riccati equation: $G'(\xi) = A + BG^2(\xi)$, A and B are arbitrary parameters. Akbar *et al.* [31] introduced a generalized and improved (G'/G)-expansion method and implemented to the KdV equation, the ZKBBM equation and the strain wave equation in microstructured solids for obtaing new travelling wave solutions. Consequently, Naher *et al.* [32] implemented this method to construct traveling wave solutions of the (3+1)-dimensional nonlinear PDE. In [33,34], Naher and Abdullah introduced another extended (G'/G)-expansion method to investigate several PDEs and produced various soliton solutions.

Very recently, Naher and Abdullah [35] proposed new generalized (G'/G)-expansion method. The significant of this method over the other methods are that it produces many new and more general solutions with some arbitrary parameters and it can handle NLEEs without boundary and initial conditions. Abundant exact and analytical solutions were produced with this novel and effective method by Naher and Abdullah [36].

The objectives of this work are: (i) to construct a rich class of new and more general exact travelling wave solutions, and (ii) to illustrate the comparison between newly generated results and the results obtained in the open literature. For this motivation, new generalized (G'/G) -expansion method is introduced and to exhibit the novelty and advantages of the method by implementing to the higher dimensional NLEEs, namely the (3+1)-dimensional Kadomstev-Petviashvili (KP) equation.

2. Algorithm of the New Generalized (G′/G)-expansion Method

Consider a general nonlinear partial differential equation:

$$P\left(u,u_t,u_x,u_y,u_z,u_{tt},u_{xt},u_{xx},u_{xxx}\ldots\right)=0, \tag{1}$$

where $u=u(x,y,z,t)$ is an unknown function, P is a polynomial in $u(x,y,z,t)$ and its derivatives in which the highest order derivatives and nonlinear terms are involved and the subscripts stand for the partial derivatives.

The most important algorithms of the method as below:

Step 1. Suppose that the combination of real variables x and t by a variable ξ

$$u(x,y,z,t)=u(\xi),\ \xi=x+y+z\pm Vt, \tag{2}$$

where V denotes the speed of the travelling wave. Now using Eq. (2), Eq. (1) is transformed into an ODE for $u=u(\xi)$:

$$Q(u,u',u'',u''',\ldots)=0, \tag{3}$$

where Q is a function of $u(\xi)$ and the superscripts indicate the ordinary derivatives with respect to ξ.

Step 2. According to possibility, Eq. (3) can be integrated term by term one or more times, yields constant(s) of integration. The integral constant may be zero, for simplicity.

Step 3. Suppose that the travelling wave solution of Eq. (3) can be expressed as follows:

$$u(\xi)=\sum_{J=0}^{M}a_J\left[\beta(\xi)\right]^J+\sum_{J=1}^{M}b_J\left[\beta(\xi)\right]^{-J}, \tag{4}$$

where $\beta(\xi)=\left[d+\lambda(\xi)\right]$ and $\lambda(\xi)$ is:

$$\lambda(\xi)=\left(G'(\xi)/G(\xi)\right). \tag{5}$$

Here a_M or b_M may be zero, but both of them cannot be zero at a time, $a_J\ (J=0,1,2,\ldots,M)$, $b_J\ (J=1,2,\ldots,M)$ and d are arbitrary constants to be determined later and $G=G(\xi)$ satisfies the second order nonlinear ODE:

$$AGG''-BGG'-C(G')^2-EG^2=0, \tag{6}$$

where prime denotes the derivative with respect to ξ. A,B,C and E are real parameters.

Step 4. To determine the positive integer M, taking the homogeneous balance between the highest order nonlinear terms and the highest order derivatives appearing in Eq. (3).

Step 5. Substituting Eq. (4) and Eq. (6) along with Eq. (5) into Eq. (3) with the value of M obtained in Step 4 and yields polynomials in $(d+\lambda(\xi))^M\ (M=0,1,2,\ldots)$ and $(d+\lambda(\xi))^{-M}\ (M=1,2,3,\ldots)$. Then, each coefficient of the resulted polynomials to be zero, yields a set of algebraic equations for $a_J\ (J=0,1,2,\ldots,M)$, $b_J\ (J=1,2,\ldots,M)$, d and V.

Step 6. Suppose that the value of the constants can be found by solving the algebraic equations which are obtained in step 5. Substituting the values of $a_J\ (J=0,1,2,\ldots,M)$, $b_J\ (J=1,2,\ldots,M)$, d and V into Eq. (4), many new and more comprehensive exact travelling wave solutions of the nonlinear partial differential equation (1) can be obtained.

Using the general solution of Eq. (6), the following solutions of Eq. (5) are:

Family 1. When $B\neq 0, \Psi=A-C$ and

$$\Omega=B^2+4E(A-C)>0,$$

$$\lambda(\xi)=\left(\frac{G'}{G}\right)=\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{C_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+C_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+C_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)} \tag{7}$$

Family 2. When $B\neq 0, \Psi=A-C$ and

$$\Omega=B^2+4E(A-C)<0,$$

$$\lambda(\xi)=\left(\frac{G'}{G}\right)=\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-C_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+C_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{C_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+C_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)} \tag{8}$$

Family 3. When $B\neq 0, \Psi=A-C$ and

$$\Omega=B^2+4E(A-C)=0,$$

$$\lambda(\xi)=\left(\frac{G'}{G}\right)=\frac{B}{2\Psi}+\frac{C_2}{C_1+C_2\xi} \tag{9}$$

Family 4. When $B=0, \Psi = A-C$ and $\Delta = \Psi E > 0$,

$$\lambda(\xi) = \left(\frac{G'}{G}\right) = \frac{\sqrt{\Delta}}{\Psi} \frac{C_1 \sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{C_1 \cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)} \quad (10)$$

Family 5. $B=0, \Psi = A-C$ and $\Delta = \Psi E < 0$,

$$\lambda(\xi) = \left(\frac{G'}{G}\right) = \frac{\sqrt{-\Delta}}{\Psi} \frac{-C_1 \sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right) + C_2 \cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{C_1 \cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right) + C_2 \sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)} \quad (11)$$

3. Implementation of the New Generalized (G′/G)- Expansion Method

Let us consider the (3+1)-dimensional KP equation:

$$u_{xt} + 6u_x^2 + 6u\,u_{xx} - u_{xxxx} - u_{yy} - u_{zz} = 0. \quad (12)$$

Now, using the wave transformation Eq. (2) into the Eq. (12), which yields:

$$-(V+2)u'' + 6(u')^2 + 6uu' - u'''' = 0. \quad (13)$$

Eq. (13) is integrable, therefore, integrating twice with respect to ξ and setting the constants of integration to zero:

$$-(V+2)u + 3u^2 - u'' = 0, \quad (14)$$

Taking the homogeneous balance between nonlinear term u^2 and the highest order derivative u'' in Eq. (14), yields $M=2$.

Therefore, the solution of Eq. (14) is of the form:

$$u(\xi) = a_0 + a_1(d+\lambda) + a_2(d+\lambda)^2 + b_1(d+\lambda)^{-1} + b_2(d+\lambda)^{-2}, \quad (15)$$

where a_0, a_1, a_2, b_1, b_2 and d are constants to be determined.

Substituting Eq. (15) together with Eqs. (5) and (6) into Eq. (14), the left-hand side is converted into polynomials in $(d+\lambda)^M \; (M=0,1,2,...)$ and $(d+\lambda)^{-M} \; (M=1,2,3,...)$. Collecting each coefficient of these resulted polynomials to zero, yields a set of algebraic equations (for simplicity, the algebraic equations are not presented) for $a_0, a_1, a_2, b_1, b_2, d$ and V. Solving these algebraic equations with the help of symbolic computation software Maple, the following set of results are obtained:

Case 1:

$$\begin{aligned} a_0 &= \frac{2\Psi\left(d^2\Psi + Bd - E\right)}{A^2}, \\ a_1 &= \frac{-2\Psi(B+2d\Psi)}{A^2}, \\ a_2 &= \frac{2\Psi^2}{A^2}, b_1 = 0, b_2 = 0, \\ V &= \frac{-\left(2A^2 + B^2 + 4E\Psi\right)}{A^2}, d = d, \end{aligned} \quad (16)$$

where $\Psi = A-C, A, B, C$, E and d are free parameters.

Case 2:

$$\begin{aligned} a_0 &= \frac{\left(B^2 + 6d^2\Psi^2 + 6Bd\Psi - 2E\Psi\right)}{3A^2}, \\ a_1 &= \frac{-2\Psi(B+2d\Psi)}{A^2}, \\ a_2 &= \frac{2\Psi^2}{A^2}, b_1 = 0, b_2 = 0, \\ V &= \frac{-\left(2A^2 - B^2 - 4E\Psi\right)}{A^2}, d = d, \end{aligned} \quad (17)$$

where $\Psi = A-C, A, B, C$, E and d are free parameters.

Case 3:

$$\begin{aligned} a_0 &= \frac{-\left(B^2 + 4E\Psi\right)}{6A^2}, \\ a_1 &= 0, a_2 = 0, b_1 = 0, \\ V &= \frac{-\left(2A^2 - B^2 - 4E\Psi\right)}{A^2}, d = \frac{-B}{2\Psi}, \\ b_2 &= \frac{B^4 + 8EB^2\Psi + 16E^2\Psi^2}{8A^2\Psi^2}, \end{aligned} \quad (18)$$

where $\Psi = A-C, A, B, C$ and E are free parameters.

Case 4:

$$\begin{aligned} a_0 &= \frac{2\Psi\left(d^2\Psi + Bd - E\right)}{A^2}, a_1 = 0, a_2 = 0, \\ b_1 &= \frac{-2\left(2d^3\Psi^2 + 3d^2B\Psi - 2Ed\Psi + dB^2 - BE\right)}{A^2}, \\ V &= \frac{-\left(2A^2 + B^2 + 4E\Psi\right)}{A^2}, d = d, \\ b_2 &= \frac{2\left(\begin{array}{l} d^4\Psi^2 + 2d^3B\Psi - 2Ed^2\Psi \\ + d^2B^2 - 2dBE + E^2 \end{array}\right)}{A^2}, \end{aligned} \quad (19)$$

where $\Psi = A-C, A, B, C$, E and d are free parameters.

Case 5:

$$a_0=\frac{-\left(B^2+4E\Psi\right)}{A^2},$$
$$a_1=0,a_2=\frac{2\Psi^2}{A^2},b_1=0,\ d=\frac{-B}{2\Psi},$$
$$V=\frac{-2\left(A^2+2B^2+8E\Psi\right)}{A^2}, \tag{20}$$
$$b_2=\frac{B^4+8EB^2\Psi+16E^2\Psi^2}{8A^2\Psi^2},$$

where $\Psi=A-C, A, B, C$ and E are free parameters.

Case 6:

$$a_0=\frac{\left(B^2+4E\Psi\right)}{3A^2},$$
$$a_1=0,a_2=\frac{2\Psi^2}{A^2},b_1=0,\ d=\frac{-B}{2\Psi},$$
$$V=\frac{-2\left(A^2-2B^2-8E\Psi\right)}{A^2}, \tag{21}$$
$$b_2=\frac{B^4+8EB^2\Psi+16E^2\Psi^2}{8A^2\Psi^2},$$

where $\Psi=A-C, A, B, C$ and E are free parameters.

Substituting Eq. (16) to Eq. (21) into Eq. (15), along with Eq. (7) and simplifying, yields following travelling wave solutions (if $C_1=0$ but $C_2\neq 0$) respectively:

$$u1_1(x,y,z,t)=\frac{1}{2A^2}\left\{\Omega\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)-\left(B^2+4E\Psi\right)\right\},$$

$$u2_1(x,y,z,t)=\frac{1}{6A^2}\left\{3\Omega\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)-\left(B^2+4E\Psi\right)\right\},$$

$$u3_1(x,y,z,t)=\frac{1}{6A^2}\left(\begin{array}{l}\frac{3}{\Omega}\left(B^4+8B^2E\Psi+16E^2\Psi^2\right)\tanh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\\ -\left(B^2+4E\Psi\right)\end{array}\right),$$

$$u4_1(x,y,z,t)=\frac{2\Psi\left(d^2\Psi+Bd-E\right)}{A^2}+b_1\left(d+\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\coth\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)^{-1}+b_2\left(d+\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\coth\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)^{-2},$$

where

$$b_1=\frac{-2\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{A^2},$$
$$V=\frac{-\left(2A^2+B^2+4E\Psi\right)}{A^2},$$
$$b_2=\frac{2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{A^2},$$

$$u5_1(x,y,z,t)=\frac{-\left(B^2+4E\Psi\right)}{A^2}+\left(\frac{B^4+8EB^2\Psi+16E^2\Psi^2}{8A^2\Psi^2}\right)\left\{\frac{4\Psi^2}{\Omega}\left(1-\text{sech}^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)\right\}+\frac{\Omega\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{2A^2},$$

$$u6_1(x,y,z,t)=\frac{1}{6A^2}\left\{\begin{array}{l}2\left(B^2+4E\Psi\right)+3\Omega\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\\ +\frac{3}{\Omega}\left(B^4+8EB^2\Psi+16E^2\Psi^2\right)\tanh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\end{array}\right\},$$

Substituting Eq. (16) to Eq. (21) into Eq. (15) along with Eq. (7) and simplifying, the exact solutions become (if $C_2=0$ but $C_1\neq 0$) correspondingly:

$$u1_2(x,y,z,t)=\frac{1}{2A^2}\left\{\Omega\left(1-\text{sech}^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)-\left(B^2+4E\Psi\right)\right\},$$

$$u2_2(x,y,z,t)=\frac{1}{6A^2}\left\{3\Omega\tanh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)-\left(B^2+4E\Psi\right)\right\},$$

$$u3_2(x,y,z,t)=\frac{1}{6A^2}\left(\begin{array}{l}\frac{3}{\Omega}\left(B^4+8B^2E\Psi+16E^2\Psi^2\right)\\ \times\left\{1-\text{sech}^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right\}-\left(B^2+4E\Psi\right)\end{array}\right),$$

$$u4_2(x,y,z,t)=\frac{2\Psi\left(d^2\Psi+Bd-E\right)}{A^2}+b_1\left(d+\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\left(\frac{\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)\right)^{-1}+b_2\left(d+\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\left(\frac{\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)\right)^{-2},$$

$$u5_2(x,y,z,t) = \frac{-\left(B^2+4E\Psi\right)}{A^2} + \left(\frac{B^4+8EB^2\Psi+16E^2\Psi^2}{8A^2\Psi^2}\right) \times \left\{\frac{4\Psi^2}{\Omega}\left(\operatorname{csch}^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+1\right)\right\} + \frac{\Omega\left\{\cosh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)-1\right\}}{2A^2\cosh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)},$$

$$u6_2(x,y,z,t) = \frac{B^2+4E\Psi}{3A^2} + \frac{\Omega}{2A^2}\left\{1-\operatorname{sech}^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right\} + \frac{\left(B^4+8EB^2\Psi+16E^2\Psi^2\right)}{2A^2\Omega}\left\{\operatorname{csch}^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+1\right\},$$

Substituting Eq. (16) to Eq. (21) into Eq. (15), together with Eq. (8) and simplifying, the travelling wave solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$)) respectively:

$$u1_3(x,y,z,t) = \frac{-1}{2A^2}\left\{\left(B^2+4E\Psi\right)+\Omega\left(\csc^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)-1\right)\right\},$$

$$u2_3(x,y,z,t) = \frac{-\left(B^2+4E\Psi\right)}{6A^2} - \frac{\Omega}{2A^2}\left\{\csc^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)-1\right\},$$

$$u3_3(x,y,z,t) = \frac{-\left(B^2+4E\Psi\right)}{6A^2} - \frac{\left(B^4+8B^2E\Psi+16E^2\Psi^2\right)}{2A^2\Omega}\left\{\sec^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)-1\right\},$$

$$u4_3(x,y,z,t) = \frac{2\Psi\left(d^2\Psi+Bd-E\right)}{A^2} + b_1\left(\frac{2\Psi}{2d\Psi+B+\sqrt{-\Omega}\cot\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right) + b_2\left(\frac{4\Psi^2}{\left(2d\Psi+B+\sqrt{-\Omega}\cot\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)^2}\right),$$

$$u5_3(x,y,z,t) = \frac{-\left(B^2+4E\Psi\right)}{A^2} - \frac{\Omega}{2A^2}\cot^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right) - \left(\frac{B^4+8EB^2\Psi+16E^2\Psi^2}{2\Omega A^2}\right)\tan^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right),$$

$$u6_3(x,y,z,t) = \frac{\left(B^2+4E\Psi\right)}{3A^2} - \frac{\Omega}{2A^2}\left\{\csc^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)-1\right\} - \frac{\left(B^4+8EB^2\Psi+16E^2\Psi^2\right)}{2A^2\Omega}\left\{\sec^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)-1\right\},$$

$$u1_4(x,y,z,t) = \frac{-1}{2A^2\cos^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\left\{\begin{matrix}\left(B^2+4E\Psi\right)\cos^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\\ +\Omega\sin^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\end{matrix}\right\},$$

$$u2_4(x,y,z,t) = a_0 + a_1\left\{d+\frac{B}{2\Psi}-\frac{\sqrt{-\Omega}\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{2\Psi\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right\} + a_2\left\{d+\frac{B}{2\Psi}-\frac{\sqrt{-\Omega}\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{2\Psi\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right\}^2,$$

$$u3_4(x,y,z,t) = \frac{a_0\Omega\tan^2\left(\frac{i\sqrt{\Omega}}{2\Psi}\xi\right)-4b_2\Psi^2}{\Omega\tan^2\left(\frac{i\sqrt{\Omega}}{2\Psi}\xi\right)},$$

$$u4_4(x,y,z,t) = a_0 + b_1\left(d+\frac{B}{2\Psi}-\frac{i\sqrt{\Omega}}{2\Psi}\tan\left(\frac{i\sqrt{\Omega}}{2\Psi}\xi\right)\right)^{-1} + b_2\left(d+\frac{B}{2\Psi}-\frac{i\sqrt{\Omega}}{2\Psi}\tan\left(\frac{i\sqrt{\Omega}}{2\Psi}\xi\right)\right)^{-2},$$

where

$$b_1 = \frac{-2\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{A^2},$$

$$V = \frac{-\left(2A^2+B^2+4E\Psi\right)}{A^2},$$

$$b_2=\frac{2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{A^2},$$

$$u5_4(x,y,z,t)=\frac{\begin{bmatrix}4a_0\Psi^2\Omega\tan^2\left(\frac{i\sqrt{\Omega}}{2\Psi}\xi\right)\\ -a_2\Omega^2\tan^4\left(\frac{i\sqrt{\Omega}}{2\Psi}\xi\right)-16b_2\Psi^4\end{bmatrix}}{4\Psi^2\Omega\tan^2\left(\frac{i\sqrt{\Omega}}{2\Psi}\xi\right)},$$

$$u6_4(x,y,z,t)=a_0-\frac{a_2\Omega}{4\Psi^2}\left\{\sec^2\left(\frac{i\sqrt{\Omega}}{2\Psi}\xi\right)-1\right\}-\frac{4b_2\Psi^2}{\Omega\left\{\sec^2\left(\frac{i\sqrt{\Omega}}{2\Psi}\xi\right)-1\right\}},$$

Substituting Eq. (16) to Eq. (21) into Eq. (15), along with Eq. (9) and simplifying, yields exact solutions respectively:

$$u1_5(x,y,z,t)=\frac{1}{2A^2}\left\{4\Psi^2\left(\frac{C_2}{C_1+C_2\xi}\right)^2-\left(4E\Psi+B^2\right)\right\}$$

$$u2_5(x,y,z,t)=\frac{B^2+6d^2\Psi^2+6Bd\Psi-2E\Psi}{3A^2}-\frac{2\Psi(B+2d\Psi)}{A^2}\left\{d+\frac{B}{2\Psi}+\left(\frac{C_2}{C_1+C_2\xi}\right)\right\}+\frac{2\Psi^2}{A^2}\left\{d+\frac{B}{2\Psi}+\left(\frac{C_2}{C_1+C_2\xi}\right)\right\}^2$$

$$u3_5(x,y,z,t)=\frac{-\left(4E\Psi+B^2\right)}{6A^2}+\left(\frac{B^4+8EB^2\Psi+16E^2\Psi^2}{8A^2\Psi^2}\right)\left(\frac{C_1+C_2\xi}{C_2}\right)^2$$

$$u4_5(x,y,z,t)=a_0+b_1\left(d+\frac{B}{2\Psi}+\left(\frac{C_2}{C_1+C_2\xi}\right)\right)^{-1}+b_2\left(d+\frac{B}{2\Psi}+\left(\frac{C_2}{C_1+C_2\xi}\right)\right)^{-2},$$

$$u5_5(x,y,z,t)=\frac{-\left(4E\Psi+B^2\right)}{A^2}+\frac{2\Psi^2}{A^2}\left(\frac{C_2}{C_1+C_2\xi}\right)^2+\left(\frac{B^4+8EB^2\Psi+16E^2\Psi^2}{8A^2\Psi^2}\right)\left(\frac{C_1+C_2\xi}{C_2}\right)^2$$

$$u6_5(x,y,z,t)=\frac{1}{24A^2\Psi^2}\left\{\begin{aligned}&8\Psi^2\left(B^2+4E\Psi\right)+\left(\frac{4\sqrt{3}C_2\Psi^2}{C_1+C_2\xi}\right)^2\\&+3\left(B^4+8EB^2\Psi+16E^2\Psi^2\right)\left(\frac{C_1+C_2\xi}{C_2}\right)^2\end{aligned}\right\}$$

Substituting Eq. (16) to Eq. (21) into Eq. (15), together with Eq. (10) and simplifying, the obtained travelling wave solutions become (if $C_1=0$ but $C_2\neq 0$; $C_2=0$ but $C_1\neq 0$)) respectively:

$$u1_6(x,y,z,t)=\frac{2\Delta}{A^2}\left\{\coth^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)-1\right\}$$

$$u2_6(x,y,z,t)=\frac{2E(A-C)}{3A^2}\left\{2\coth^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\operatorname{csch}^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right\}$$

$$u3_6(x,y,z,t)=\frac{-2E(A-C)}{3A^2}\left\{4-3\operatorname{sech}^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right\}$$

$$u4_6(x,y,z,t)=\frac{2}{A^2}\left\{\begin{aligned}&\left(d^2\Psi^2-E\Psi\right)-\left(\frac{2d^3\Psi^3-2Ed\Psi^2}{d\Psi+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)\\&+\left(\frac{d^4\Psi^4-2Ed^2\Psi^3+E^2\Psi^2}{\left(d\Psi+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2}\right)\end{aligned}\right\}$$

$$u5_6(x,y,z,t)=\frac{-4E\Psi}{A^2}+\frac{2E\Psi\coth^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{A^2}+\frac{2E\Psi}{A^2\coth^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}$$

$$u6_6(x,y,z,t)=\frac{4E(A-C)}{3A^2}+\frac{2E(A-C)}{A^2}\left\{2+\operatorname{csch}^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)-\operatorname{sech}^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right\}$$

$$u1_7(x,y,z,t)=\frac{2\Psi}{A^2}\left\{\frac{\Delta}{\Psi}\left(1-\sec h^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)-E\right\}$$

$$u2_7(x,y,z,t)=\frac{2}{3A^2}\left\{3\Delta\tanh^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)-E\Psi\right\}$$

$$u3_7(x,y,z,t) = \frac{6E^2(A-C)^2\cosh^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right) - 2E\Psi\Delta\sinh^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{3A^2\Delta\sinh^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}$$

$$u4_7(x,y,z,t) = \frac{2\left\{\begin{array}{l} d^2\Psi^2\Delta\tanh^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right) \\ -E\Psi\Delta\tanh^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right) - Ed^2\Psi^3 + E^2\Psi^2 \end{array}\right\}}{A^2\left(d\Psi + \sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2}$$

$$u5_7(x,y,z,t) = \frac{\left[\begin{array}{l} -4E\Psi\Delta\tanh^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right) \\ +2\Delta^2\tanh^4\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right) + 2E^2\Psi^2 \end{array}\right]}{A^2\Delta\tanh^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}$$

$$u6_7(x,y,z,t) = \frac{1}{3A^2}\left\{\begin{array}{l} 4E\Psi + 6\Delta\tanh^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right) \\ +\frac{6E^2\Psi^2}{\Delta}\coth^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right) \end{array}\right\}$$

Substituting Eq. (16) to Eq. (21) into Eq. (15), along with Eq. (11) and simplifying, yields following exact solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$)) respectively:

$$u1_8(x,y,z,t) = \frac{-2}{A^2\sin^2\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\left\{\begin{array}{l} E\Psi\sin^2\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right) \\ +\Delta\left(1-\sin^2\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right) \end{array}\right\}$$

$$u2_8(x,y,z,t) = \frac{6d^2\Psi^2 - 2E\Psi}{3A^2} - \frac{4d\Psi^2}{A^2}\left\{d + \frac{\sqrt{-\Delta}}{\Psi}\cot\left(\frac{i\sqrt{\Delta}}{\Psi}\xi\right)\right\} + \frac{2\Psi^2}{A^2}\left\{d + \frac{i\sqrt{\Delta}}{\Psi}\cot\left(\frac{i\sqrt{\Delta}}{\Psi}\xi\right)\right\}^2,$$

$$u3_8(x,y,z,t) = \frac{-2E\Psi}{3A^2} - \frac{2E^2(A-C)^2}{A^2\Delta}\left\{\sec^2\left(\frac{i\sqrt{\Delta}}{\Psi}\xi\right) - 1\right\}$$

$$u4_8(x,y,z,t) = \frac{2}{A^2}\left[\begin{array}{l} d^2\Psi^2 - E\Psi \\ -\left(2d^3\Psi^2 - 2Ed\Psi\right) \\ \times\left\{d + \frac{\sqrt{-\Delta}}{\Psi}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right\}^{-1} \\ +\left(d^4\Psi^2 - 2Ed^2\Psi + E^2\right) \\ \times\left\{d + \frac{\sqrt{-\Delta}}{\Psi}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right\}^{-2} \end{array}\right]$$

$$u5_8(x,y,z,t) = \frac{\left[\begin{array}{l} -4E\Psi\Delta\sin^2\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\cos^2\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right) \\ -2\Delta^2\cos^4\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right) - 2E^2\Psi^2\sin^4\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right) \end{array}\right]}{A^2\Delta\sin^2\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\cos^2\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}$$

$$u6_8(x,y,z,t) = \frac{2\left\{\begin{array}{l} 2E(A-C)\Delta\cot^2\left(\frac{i\sqrt{\Delta}}{\Psi}\xi\right) \\ -3\Delta^2\cot^4\left(\frac{i\sqrt{\Delta}}{\Psi}\xi\right) - 3E^2(A-C)^2 \end{array}\right\}}{3A^2\Delta\cot^2\left(\frac{i\sqrt{\Delta}}{\Psi}\xi\right)}$$

$$u1_9(x,y,z,t) = \frac{-2}{A^2}\left\{E(A-C) + \Delta\left(\sec^2\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right) - 1\right)\right\}$$

$$u2_9(x,y,z,t) = \frac{1}{3A^2}\left[\begin{array}{l} 6d^2\Psi^2 - 2E\Psi - \frac{12d\Psi\left\{\begin{array}{l} d\Psi\cos\left(\frac{i\sqrt{\Delta}}{\Psi}\xi\right) \\ -\sqrt{-\Delta}\sin\left(\frac{i\sqrt{\Delta}}{\Psi}\xi\right) \end{array}\right\}}{\cos\left(\frac{i\sqrt{\Delta}}{\Psi}\xi\right)} \\ +\frac{6\left\{d\Psi\cos\left(\frac{i\sqrt{\Delta}}{\Psi}\xi\right) - \sqrt{-\Delta}\sin\left(\frac{i\sqrt{\Delta}}{\Psi}\xi\right)\right\}^2}{\cos^2\left(\frac{i\sqrt{\Delta}}{\Psi}\xi\right)}, \end{array}\right]$$

$$u3_9(x,y,z,t) = \frac{-2E\Delta\Psi\sin^2\left(\frac{i\sqrt{\Delta}}{\Psi}\xi\right) + 6E^2\Psi^2\cos^2\left(\frac{i\sqrt{\Delta}}{\Psi}\xi\right)}{3A^2\Delta\sin^2\left(\frac{i\sqrt{\Delta}}{\Psi}\xi\right)}$$

$$u4_9(x,y,z,t) = \frac{\left\{\begin{array}{l} 2\left(d^2\Psi^2 - E\Psi\right)\left\{d\Psi - i\sqrt{\Delta}\tan\left(\frac{i\sqrt{\Delta}}{\Psi}\xi\right)\right\}^2 \\ -\left(2d^3\Psi^3 - 2Ed\Psi^2\right)\left\{d\Psi - i\sqrt{\Delta}\tan\left(\frac{i\sqrt{\Delta}}{\Psi}\xi\right)\right\} \\ +\left(d^4\Psi^4 - 2Ed^2\Psi^3 + E^2\Psi^2\right) \end{array}\right\}}{A^2\left\{d\Psi - i\sqrt{\Delta}\tan\left(\frac{i\sqrt{\Delta}}{\Psi}\xi\right)\right\}^2}$$

$$u5_9(x,y,z,t) = \frac{1}{A^2\Delta\tan^2\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\left\{\begin{array}{l} -4E\Psi\Delta\tan^2\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right) \\ -2\Delta^2\tan^4\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right) - 2E^2\Psi^2 \end{array}\right\}$$

$$u6_9(x,y,z,t) = \frac{\left\{\begin{array}{l} 4E\Delta\Psi - 6\Delta^2\left\{\sec^2\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right) - 1\right\} \\ -6E^2\Psi^2\left\{\csc^2\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right) - 1\right\} \end{array}\right\}}{3A^2\Delta}$$

4. Discussions

The advantages and validity of this executed method over the basic (G'/G) -expansion method have been examined as below.

Advantages: The key advantage of new generalized (G'/G) -expansion method over the basic (G'/G) -expansion method is that this method provides more general and huge number of new exact travelling wave solutions with various arbitrary parameters. The analytical solutions of NLEEs have its vital importance to disclose the internal mechanism of the complex physical phenomena. Apart from the physical application, the exact solutions help the numerical solvers to compare the exactness of their results and assist them in the stability analysis.

Validity: A good agreement is found between our obtained solutions and published results in the earlier literature, if the parameters take particular values, which validate the obtained solutions. Bekir and Uygun [37] used the basic (G'/G) -expansion method to the (3+1)-dimensional KP equation and obtained only six solutions (F.1) to (F.6) (see Appendix). On the other hand, fifty-four solutions have been generated via the new generalized (G'/G) -expansion method. Bekir and Uygun [37] presented the solution form as $u(\xi) = \sum_{i=0}^{m} a_i \left(G'(\xi)/G(\xi)\right)^i$, where $a_m \neq 0$ and LODE is used as an auxiliary equation: $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$. In this case, there are only three solutions with the general solution of LODE and also has a very few options of solution style. On the other hand, the solution form of this article is $u(\xi) = \sum_{J=0}^{M} a_J \left[d + \left(\frac{G'(\xi)}{G(\xi)}\right)\right]^J + \sum_{J=1}^{M} b_J \left[d + \left(\frac{G'(\xi)}{G(\xi)}\right)\right]^{-J}$, where a_M or b_M may be zero, but both of them cannot be zero at a time and second order nonlinear ODE (SONLODE) is used as an auxiliary equation:

$$AG(\xi)G''(\xi) - BG(\xi)G'(\xi) - C\left(G'(\xi)\right)^2 - E\left(G(\xi)\right)^2 = 0,$$

where A, B, C and E are arbitrary parameters. It is important to point out that there are five solutions with the general solutions of SONLODE. Moreover, several choices of multipattern solutions are available, and those could be used to investigate the real-world problems through considering various values of arbitrary parameters.

5. Conclusions

In this article, the new generalized (G'/G) -expansion method has successfully been applied to the (3+1)-dimensional KP equation. In the basic (G'/G) -expansion method, the auxiliary equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, has three different general solutions. But in the new generalized (G'/G) -expansion method, the second order nonlinear ODE as the auxiliary equation $AG(\xi)G''(\xi) - BG(\xi)G'(\xi) - C\{G'(\xi)\}^2 - E\{G(\xi)\}^2 = 0,$ and has five different general solutions. Due to investigation with the higher order nonlinear ODE of the higher dimensional evolution equation many new and more explicit soliton solutions are constructed with several arbitrary parameters. These parameters might be important to demonstrate more complex physical phenomena. This study also shows that new generalized $\left(G'(\xi)/G(\xi)\right)$ -expansion method is quite efficient and well suited to be implemented for constructing new exact solutions of various NLEEs which frequently arise in mathematical physics, engineering sciences and many scientific real-world problems. Furthermore, the obtained solutions could be used as models in real world problems, such as tsunami waves and earthquake etc.

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Appendix

Bekir and Uygun's solutions [37]

Bekir and Uygun's [37] produced exact solutions of the (3+1)-dimensional KP equation via the basic $\left(G'/G\right)$-expansion method which are as follows:

When $\lambda^2 - 4\mu > 0$,

$$u_1(\xi) = \left(\frac{\lambda^2 - 4\mu}{2}\right)\left\{\left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)}\right)^2 - 1\right\}, \quad \text{(F.1)}$$

where $\xi = x + y + z + \left(\lambda^2 - 4\mu + 2\right)t$ and C_1, C_2 are arbitrary constants.

$$u_2(\xi) = \left(\frac{\lambda^2 - 4\mu}{2}\right)\left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)}\right)^2 + \frac{5\lambda^2}{6} + \frac{2\mu}{3}, \quad \text{(F.2)}$$

where $\xi = x + y + z + \left(\lambda^2 - 4\mu - 2\right)t$ and C_1, C_2 are arbitrary constants.

When $\lambda^2 - 4\mu < 0$,

$$u_3(\xi) = \left(\frac{4\mu - \lambda^2}{2}\right)\left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right) + C_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right)}{C_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right) + C_2 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right)}\right)^2 - \frac{\lambda^2}{2} + 2\mu, \quad \text{(F.3)}$$

where $\xi = x + y + z + \left(\lambda^2 - 4\mu + 2\right)t$ and C_1, C_2 are arbitrary constants.

$$u_4(\xi) = \left(\frac{4\mu - \lambda^2}{2}\right)\left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right) + C_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right)}{C_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right) + C_2 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right)}\right)^2 + \frac{6\lambda^2}{6} + \frac{2\mu}{3}, \quad \text{(F.4)}$$

where $\xi = x + y + z + \left(\lambda^2 - 4\mu - 2\right)t$ and C_1, C_2 are arbitrary constants.

When $\lambda^2 - 4\mu = 0$,

$$u_5(\xi) = \frac{2C_2^2}{(C_1 + C_2\xi)^2} - \frac{\lambda^2}{2} + 2\mu, \quad \text{(F.5)}$$

where $\xi = x + y + z + \left(\lambda^2 - 4\mu + 2\right)t$ and C_1, C_2 are arbitrary constants.

$$u_6(\xi) = \frac{2C_2^2}{(C_1 + C_2\xi)^2} + \frac{5\lambda^2}{6} + \frac{2\mu}{3}, \quad \text{(F.6)}$$

where $\xi = x + y + z + \left(\lambda^2 - 4\mu - 2\right)t$ and C_1, C_2 are arbitrary constants.