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A New Approach to Fixed Point Theorems on a Metric Space Endowed with Graph

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Abstract In this paper, a new approach has been discussed to define the graph associated with the metric space and the iteration function is used to define its sub-graph. Subsequently, the fixed point theorems by Banach, Kannan, Chatterjea and Ciric are obtained using this new approach.

Keywords: *fixed point, iterated function, graph, sub-graph, w-sequence*

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1. Introduction

Banach [1] laid the foundation for the study of metric fixed point theory which instilled the interest of many mathematicians to develop results which had applications in mathematics as well as other branches of science. The contraction condition of the mapping in the Banach fixed point theorem was modified and some interesting results were exhibited by Kannan [2], Chatterjea [3], Ciric [4], Edelstein [5] to name a few. New results were obtained by studying fixed point theorems for different types of mappings such as multi-valued mappings [6], α-ψ - contractive mappings [7], (α,β)-(ψ,φ)-contractive mappings [8] in metric spaces. In the recent works of [9-15] new ideas were explored for these mappings in the setting of different metric spaces such as b-metric space, rectangular b-metric space , S-metric space.

Graph theoretical approach in Fixed point theory was initiated by the work of Espinola and Kirk [16] in the year 2006. Jachymski [17] took the lead by defining the graph associated with the metric space and subsequently proving results pertaining to the fixed point theorems on the metric space endowed with graph. He coined the term Banach G-contraction meaning Banach contraction on a metric space endowed with graph. His work invoked the interest of many to study fixed point theorems on various metric spaces endowed with graph. This led to a series of papers [18-27] in which different contraction principles has been proved in this background.

The concept of graphical metric space was introduced by Shukla, Radenovic and Vetro in their work [28] in the year 2017. In this paper, a new metric was defined for the metric space using graphs. The concept of convergence of a sequence and Cauchy sequence was studied in the context of graphical metric space and this was further extended to rectangular b-metric space in the year 2019 [29].

In the work done previously by many authors in the study of fixed point theorems on metric spaces endowed with graph, the graph was defined by taking the vertex set as the set X and the edge set contained the diagonal of the Cartesian product X x X, i.e. the graph was assumed to have loops at each and every vertex. In the present paper, the graph associated with the metric space contains edges joining a point with its image. Hence if the graph has a loop at a particular vertex, then that vertex is the fixed point of the mapping under consideration. To prove the various contraction principles, a sub-graph of the above graph is defined using the iterated function. The graph defined above is a weighted graph where the weights are the distance between the points. A sequence named as w-sequence is defined corresponding to the sequence of the edges of the sub-graph. The contraction principles by Banach [1], Kannan [2], Chatterjea [3] and Ciric [4] are proved using this approach on metric spaces endowed with graph.

2. Preliminaries

Let (X,d) be a metric space. In the following three sections the basic concepts related to sequences, definition of Graph and sub-graph, results connected with fixed points are exhibited.

2.1. Sequences

Definition 2.1.1. A sequence $\{x_n\}_{n=1}^{\infty}$ in the metric space (X,d) is said to be convergent if for a given ϵ>0 there exists $N_1 \in I$ such that $d(x_n, x) < \in, n \geq N_1$.

Definition 2.1.2. A sequence $\{x_n\}_{n=1}^{\infty}$ in the metric space (X,d) is said to be Cauchy if for every ϵ>0 there exists $N_2 \in I$ such that $d(x_n, x_m) < \in, n, m \geq N_2$.
Definition 2.2.3. A sequence $\{x_n\}_{n=1}^{\infty}$ in the metric space (X,d) is said to be monotone if either $x_n < x_{n+1}, n \in I$ (non-decreasing) or $x_n > x_{n+1}, n \in I$ (non-increasing).
Theorem 2.1.1 (Monotone Convergence theorem). If a sequence is monotone and is bounded then it converges.

2.2. Graphs

Definition 2.2.1 [30]. A graph G is an ordered pair (V, E) where V is the set of points called as vertices and E is the set of lines called as edges.
Definition 2.2.2 [30]. A Graph $G_o = (V_o, E_o)$ is called a sub-graph of $G = (V, E)$ if $V_o \subset V$ and $E_o \subset E$.
Definition 2.2.3. [30]. A weighted graph is a simple graph that has a number, termed as weight, associated with each of its edges. Hence a weighted graph consists of a vertex set, edge set together with the weights for each of its edges.
Definition 2.2.4 [30]. An edge of a graph G is called a loop if its initial vertex and terminal vertex are the same.

2.3. Fixed Point Theorems

Definition 2.3.1 [1]. A mapping T: X→X is said to be a contraction if there exists a real number α such that 0≤α<1 and d(Tx,Ty)≤αd(x,y).
Defintion 2.3.2 [2]. Let (X,d) be a metric space. A mapping T: X→X is called a Kannan mapping if there exists $\alpha \epsilon \left[0, \frac{1}{2}\right)$ such that:

$$d(Tx, Ty) \leq \alpha \left[d(x, Tx) + d(y, Ty) \right].$$

Definition 2.3.3 [3]. Let (X,d) be a metric space. A mapping T:X→X is called a Chatterjea mapping if there exists $\alpha \epsilon \left[0, \frac{1}{2}\right)$ such that:

$$d(Tx, Ty) \leq \alpha \left[d(x, Ty) + d(y, Tx) \right].$$

Definition 2.3.4 [4]. Let (X,d) be a metric space. A mapping T:X→X is said to be a λ-generalized contraction if and only if for every x, y ε X there exist non-negative numbers q(x,y), r(x,y), s(x,y) and t(x,y) such that

$$\sup_{x,y \in X} \{ q(x,y) + r(x,y) + s(x,y) + 2t(x,y) \} = \lambda < 1$$

and

$$d(Tx, Ty) \leq q(x,y) d(x,y) + r(x,y) d(x, Tx) + s(x,y) d(y, Ty) + t(x,y) \left[d(x, Ty) + d(y, Tx) \right]$$

holds for every x,y ε X.
Definition 2.3.5 [4]. Let (X,d) be a metric space. A mapping T:X→X is said to be T-orbitally complete if every Cauchy sequence $\{T^{n_i}x : i \in N\}, x \in X$ has a limit point in X.
Definition 2.3.6 [4]. Let (X,d) be a metric space. A mapping T:X→X is said to be T-orbitally continuous if for u$\in X$ such that $u = \lim_{i \to \infty} T^{n_i}x$ for some x$\in X$, we have $Tu = \lim_{i \to \infty} TT^{n_i}x$.

Theorem 2.3.1 [4]. Let T be a λ-generalized contraction of T-orbitally complete metric space X into itself. Then
i) There is in X a unique fixed point u under T,
ii) $T^n x \to u$ for every x ε X and
iii) $d(T^n x, u) \leq \frac{\lambda^n}{1-\lambda} d(x, Tx)$.

3. Main Result

Let (X,d) be a metric space. Let T: X→X. We now define the graph associated with the metric space as below:
Definition 3.1. Let (X,d) be a metric space. Let T: X→X. Define a weighted graph G associated with X as below: Let G=(V,E) where V=X and E={(x,Tx)/x ε X}. The weights of the edges are the distance between the endpoints of the edges. Now (X,d) becomes a metric space endowed with the graph G.
Definition 3.2. The sub-graph G_o of G is defined as below: Let y_o be any arbitrary point of X. Let $G_o = (V_o, E_o)$ where $V_o = \{y_o, Ty_o, T^2y_o, \ldots\}$ and let

$$E_o = \left\{ (y_o, Ty_o), (Ty_o, T^2 y_o), \ldots \right\}$$

Then $V_o \subset V$ and $E_o \subset E$. Hence G_o is a sub-graph of G.
Definition 3.3. Let (X,d) be a metric space endowed with the graph G. Let G_o be the sub-graph of G defined as in Definition 3.2. Let $w_n = d(T^{n-1}y_o, T^n y_o)$. Then the sequence $\{w_n\}_{n=1}^{\infty}$ is called the w-sequence of real numbers associated with the graph G_o.
Example 3.1 Let X={0,1,2,3}. The metric on X is defined as d(x,y)=|x-y|, x, y ε X. Let T: X→X be defined as below:

$$Tx = \begin{cases} 0, & x \in \{0,1,2\} \\ 1, & x = 3 \end{cases}$$

Following is the the graph associated with X defined as in Definition 3.1. G=(V,E) where V={0,1,2,3}, E={(0,0),(1,0),(2,0),(3,1)}.

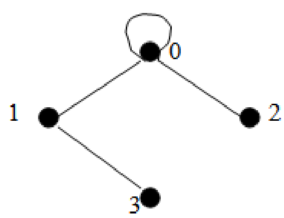

Figure 3.1. The Graph G associated with X

The sub-graph G_o corresponding to every element of X and the w-sequence in each case are illustrated below:
Case (i): $y_o = 0$
Let
$G_o = (V_o, E_o)$
where $V_o = \{0, T0, T^2 0, \ldots\}$={0}, $E_o = \{(0,0)\}$.

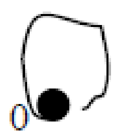

Figure 3.2. Graph G_o corresponding to the point $y_o = 0$

The w-sequence in this case is as follows:

$$\left\{d\left(y_o, Ty_o\right), d\left(Ty_o, T^2 y_o\right), d\left(T^2 y_o, T^3 y_o\right), \ldots\ldots\right\}$$
$$= \{0, 0, 0, \ldots.\}.$$

Case(ii): $\boldsymbol{y_o = 1}$

Let

$$G_o = \left(V_o, E_o\right)$$

where $V_o = \{1, T1, T^2 1, \ldots\} = \{1, 0\}$, $E_o = \{(0,0), (0,1)\}$.

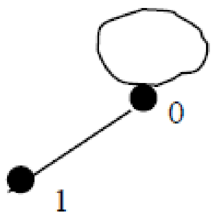

Figure 3.3. Graph G_o corresponding to the point $y_o = 1$

The w-sequence in this case is given by

$$\left\{d\left(y_o, Ty_o\right), d\left(Ty_o, T^2 y_o\right), d\left(T^2 y_o, T^3 y_o\right), \ldots\ldots\right\}$$
$$= \{1, 0, 0, \ldots\ldots\}.$$

Case (iii): $\boldsymbol{y_o = 2}$

Let

$$G_o = \left(V_o, E_o\right)$$

where $V_o = \{2, T2, T^2 2, \ldots\} = \{2, 0\}$, $E_o = \{(0,0), (0,2)\}$.

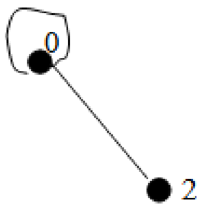

Figure 3.4. Graph G_o corresponding to the point $y_o = 2$

The w-sequence in this case is given below:

$$\left\{d\left(y_o, Ty_o\right), d\left(Ty_o, T^2 y_o\right), d\left(T^2 y_o, T^3 y_o\right), \ldots\ldots\right\}$$
$$= \{2, 0, 0, \ldots\}.$$

Case (iii): $\boldsymbol{y_o = 3}$

Let

$$G_o = \left(V_o, E_o\right)$$

where $V_o = \{3, T3, T^2 3 \quad , \ldots\} = \{3,1,0\} = \{0,1,3\}$, $E_o = \{(0,0), (0,1), (1,3)\}$.

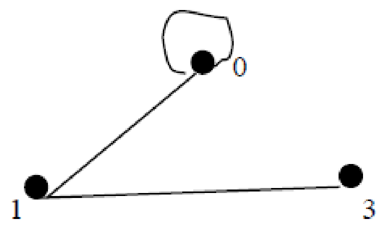

Figure 3.5. Graph G_o corresponding to the point $y_o = 3$

The w-sequence in this case is given as below:

$$\left\{d\left(y_o, Ty_o\right), d\left(Ty_o, T^2 y_o\right), d\left(T^2 y_o, T^3 y_o\right), \ldots\ldots\right\}$$
$$= \{2, 1, 0, 0, \ldots.\}.$$

The following two lemmas are useful in proving the fixed point theorems on the metric space (X,d) endowed with the graph G.

Lemma 3.1. Let (X,d) be a metric space and let T:X→X. Let G be graph associated with X. Let y_o be any arbitrary point of X. Let G_o be the sub-graph of G defined as in Definition 3.2. Then the sequence $\{y_o, Ty_o, T^2 y_o, \ldots\ldots\}$ is Cauchy if and only if the w-sequence associated with the graph G_o is non-increasing.

Proof. Suppose the w-sequence associated with the graph G_o is non-increasing. Then

$$w_1 > w_2 > w_3 > \ldots\ldots > w_n > w_{n+1} > \ldots\ldots$$

i.e the w-sequence, being the sequence of the length of the edges between the terms of the sequence $\{y_o, Ty_o, T^2 y_o, \ldots\ldots\}$ is a sequence of non-negative real numbers bounded below by zero. Hence by Theorem 2.1.1, the w-sequence is convergent. This implies the terms of the sequence $\{y_o, Ty_o, T^2 y_o, \ldots\ldots\}$ are closer as n approaches infinity. i.e. the iterated sequence $\{y_o, Ty_o, T^2 y_o, \ldots\ldots\}$ is Cauchy.

Lemma 3.2. Let (X,d) be a metric space. Let T: X→X. Let G be the graph associated with X. The point y^* of X is a fixed point of T if and only if the graph G has a loop at y^*.

Proof. Let y^* be the fixed point of T. Then $Ty^* = y^*$. According to the Definition 3.1 of G, (Ty^*, y^*)εG. i.e.(y^*, y^*)εG.⇒ G has a loop at y^*.

We now proceed to prove the Fixed point theorems by Banach, Kannan, Chatterjea and λ-generalized contraction by Ciric on a metric space (X,d) endowed with Graph G.

Theorem 3.1. Let (X,d) be a complete metric space and let T: X→X be a contraction on X. Let G be the graph associated with X. Then T has a unique fixed point $y^* \in X$.

Proof. Let y_o be any arbitrary point of X. The graph G and its sub-graph G_o are defined as in Definition 3.1 and Definition 3.2. Consider the iterated sequence $\{y_o, Ty_o, T^2 y_o, \ldots\ldots\}$ in X. According to Lemma 3.1, to prove that this sequence is Cauchy, it is enough to prove that the w-sequence associated with the graph G_o is non-increasing.

Since T is a contraction on X we have,

$$w_{n+1} = d\left(T^n y_o, T^{n+1} y_o\right)$$
$$\leq \alpha d\left(T^{n-1} y_o, T^n y_o\right) = \alpha w_n.$$

i.e. $w_{n+1} < w_n$ since $0 \leq \alpha < 1, n \in I$.

Hence the w-sequence associated with G_o is non-increasing. This implies, from Lemma 3.1, the iterated sequence $\{y_o, Ty_o, T^2 y_o, \ldots\ldots\}$ is Cauchy. But X is complete. Hence this sequence converges to say, y^* in X. The mapping T being a contraction is continuous. Hence the sequence $\{TT^{n-1} y_o\}_{n=1}^{\infty}$ converges to Ty^*. But the sequence $\{TT^n y_o\}_{n=1}^{\infty}$ is a subsequence of the sequence $\{T^{n-1} y_o\}_{n=1}^{\infty}$. Hence the subsequence must have the same limit as the parent sequence. But the limit of a sequence is unique. Hence we must have $Ty^* = y^*$. This implies $(y^*, y^*) \in G$. i.e. G has a loop at y^*. Hence by Lemma 3.2, y^* is a fixed point of T.

To prove uniqueness, let if possible, z^* be any other fixed point of T. Then $Tz^* = z^*$.

Since T is a contraction on X, we have,

$$d\left(Ty^*, Tz^*\right) \le \alpha d\left(y^*, z^*\right), 0 \le \alpha < 1$$

$$d\left(y^*, z^*\right) < d\left(y^*, z^*\right)$$

which is a contradiction. Hence the fixed point of T is unique.

Theorem 3.2. Let (X,d) be a complete metric space. Let T: X→X and let G be the graph associated with X. If T satisfies

$$d(Tx, Ty) \le \alpha\left(d(x, Tx) + d(y, Ty)\right) \quad (1)$$

for all x, y ε X, where $\alpha \in [0, \frac{1}{2})$, then T has a unique fixed point.

Proof. Let y_o be any arbitrary point of X. The graph G and its sub-graph G_o are defined as in Definition 3.1 and Definition 3.2. Consider the iterated sequence $\{y_o, Ty_o, T^2y_o, \ldots\ldots\}$ in X. According to Lemma 3.1, to prove that this sequence is Cauchy, it is enough to prove that the w-sequence associated with the graph G_o is non-increasing.

From (1) we have,

$$w_{n+1} = d\left(T^n y_o, T^{n+1} y_o\right)$$

$$\le \alpha\left[d\left(T^{n-1} y_o, T^n y_o\right) + d\left(T^n y_o, T^{n+1} y_o\right)\right]$$

$$w_{n+1} \le \alpha\left[w_n + w_{n+1}\right]$$

$$w_{n+1} \le \frac{\propto}{1-\propto} w_n < w_n \ since\ 0 \le \alpha < \frac{1}{2}.$$

Hence the w-sequence associated with G_o is non-increasing. From Lemma 3.1, the iterated sequence $\{y_o, Ty_o, T^2y_o, \ldots\ldots\}$ is a Cauchy sequence in X. But X is complete. Therefore this sequence converges in X. Let $y^* = \lim_{n\to\infty} T^n y_o$.

Consider

$$d\left(y^*, Ty^*\right) \le d\left(y^*, T^n y_o\right) + d\left(T^n y_o, Ty^*\right)$$

$$\le d\left(y^*, T^n y_o\right) + \alpha\left[d\left(T^{n-1} y_o, T^n y_o\right) + d\left(y^*, Ty^*\right)\right]$$

$$(1-\alpha) d\left(y^*, Ty^*\right) \le d\left(y^*, T^n y_o\right) + \alpha d\left(T^{n-1} y_o, T^n y_o\right).$$

Allow n→∞ on both sides. Then we have,

$$(1-\alpha) d\left(y^*, Ty^*\right) \le d\left(y^*, y^*\right) + \alpha d\left(y^*, y^*\right).$$

Hence $d(y^*, Ty^*) = 0$.

$$\Rightarrow Ty^* = y^*.$$

i.e. $y^*$is a fixed point of T.

To prove uniqueness, let if possible, z^* be any other fixed point of T. Then $Tz^* = z^*$.

From (1) we have,

$$d\left(Ty^*, Tz^*\right) \le \alpha[d\left(y^*, Ty^*\right) + d\left(z^*, Tz^*\right)$$

where $0 \le \alpha < \frac{1}{2}$.

$$d\left(y^*, z^*\right) \le \alpha\left[d\left(y^*, y^*\right) + d\left(z^*, z^*\right)\right].$$

This implies $d(y^*, z^*) = 0. i.e\ y^* = z^*$.

Hence the fixed point of T is unique.

Theorem 3.3. Let (X,d) be a complete metric space. Let T: X→X and let G be the graph associated with X. If T satisfies

$$d(Tx, Ty) \le \alpha\left(d(x, Ty) + d(y, Tx)\right) (2)$$

for all x, y ε X , where $\alpha \in [0, \frac{1}{2})$, then T has a unique fixed point.

Proof. Let y_o be any arbitrary point of X. The graph G and its sub-graph G_o are defined as in Definition 3.1 and Definition 3.2. Consider the iterated sequence $\{y_o, Ty_o, T^2y_o, \ldots\ldots\}$ in X. According to Lemma 3.1, to prove that this sequence is Cauchy, it is enough to prove that the w-sequence associated with the graph G_o is non-increasing.

From (2) we have,

$$w_{n+1} = d\left(T^n y_o, T^{n+1} y_o\right)$$

$$\le \alpha\left[d\left(T^{n-1} y_o, T^{n+1} y_o\right) + d\left(T^n y_o, T^n y_o\right)\right]$$

$$w_{n+1} \le \alpha[d\left(T^{n-1} y_o, T^n y_o\right) + d\left(T^n y_o, T^{n+1} y_o\right)$$

$$\le \alpha\left[w_n + w_{n+1}\right]$$

$$w_{n+1} \le \frac{\propto}{1-\propto} w_n < w_n \ since\ 0 \le \alpha < \frac{1}{2}.$$

Hence the w-sequence associated with G_o is non-increasing. From Lemma 3.1 the iterated sequence $\{y_o, Ty_o, T^2y_o, \ldots\ldots\}$ is a Cauchy sequence in X. But X is complete. Therefore this sequence converges in X. Let $y^* = \lim_{n\to\infty} T^n y_o$.

Consider

$$d\left(y^*, Ty^*\right) \le d\left(y^*, T^n y_o\right) + d\left(T^n y_o, Ty^*\right)$$

$$\le d\left(y^*, T^n y_o\right) + \alpha[d\left(T^{n-1} y_o, Ty^*\right) + d\left(y^*, T^n y_o\right)]$$

Allow n→∞ on both sides. Then we have,

$$d\left(y^*, Ty^*\right) \le d\left(y^*, y^*\right) + \alpha[d\left(y^*, Ty^*\right) + d\left(y^*, y^*\right)]$$

$$(1-\alpha) d\left(y^*, Ty^*\right) \le 0$$

Hence $d(y^*, Ty^*) = 0$.

$$\Rightarrow Ty^* = y^*.$$

i.e. y^* is a fixed point of T.

To prove uniqueness, let if possible, z^* be any other fixed point of T. Then $Tz^* = z^*$.

From (2) we have,

$$d\left(Ty^*, Tz^*\right) \le \alpha\left[d\left(y^*, Tz^*\right) + d\left(z^*, Ty^*\right)\right]$$

where $0 \le \alpha < \frac{1}{2}$.

$$d\left(y^*, z^*\right) \le \alpha\left[d\left(y^*, z^*\right) + d\left(z^*, y^*\right)\right]$$

$$d\left(y^*, z^*\right) \le 2\alpha d\left(y^*, z^*\right)$$

$\Rightarrow \alpha \ge \frac{1}{2}$. This is a contradiction.

Hence $y^* = z^*$. i.e the fixed point of T is unique.

Following theorem is the fixed point theorem for λ-generalized contraction in a metric space endowed with a graph G.

Theorem 3.4. Let T be a λ-generalized contraction of T-orbitally complete metric space (X,d) into itself. Let G be the graph associated with X. Then

i) There is a unique fixed point y^* in X,

ii) $\lim_{n\to\infty} T^n x = y^*$ for every x ∈ X and

iii) $d(T^n x, y^*) \le \frac{\lambda^n}{1-\lambda} d(x, Tx)$.

Proof. Let y_o be any arbitrary point in X. Consider the iterated sequence $\{y_o, Ty_o, T^2 y_o, \ldots\}$ in X. Define $x_n = T^n y_o$, $n = 0,1,2,3,\ldots$. Hence we have $x_{n+1} = Tx_n, n = 0,1,2,3\ldots$ .The graph G and its sub-graph G_o are defined as in Definition 3.1 and Definition 3.2. According to Lemma 3.1, to prove that the iterated sequence is Cauchy, it is enough to prove that the w-sequence associated with the graph G_o is non-increasing.

Since T is a λ-generalized contraction we have,

$$\begin{aligned}
d\left(x_n, x_{n+1}\right) &= d\left(Tx_{n-1}, Tx_n\right) \\
&\le q(x_{n-1}, x_n) d\left(x_{n-1}, x_n\right) \\
&+ r\left(x_{n-1}, x_n\right) d\left(x_{n-1}, Tx_{n-1}\right) \\
&+ s\left(x_{n-1}, x_n\right) d\left(x_n, Tx_n\right) \\
&+ t\left(x_{n-1}, x_n\right)\left[d\left(x_{n-1}, Tx_n\right) + d\left(x_n, Tx_{n-1}\right)\right] \\
&= q\left(x_{n-1}, x_n\right) d\left(x_{n-1}, x_n\right) \\
&+ r\left(x_{n-1}, x_n\right) d\left(x_{n-1}, x_n\right) \\
&+ s\left(x_{n-1}, x_n\right) d\left(x_n, x_{n+1}\right) \\
&+ t\left(x_{n-1}, x_n\right)\left[d\left(x_{n-1}, x_{n+1}\right) + d\left(x_n, x_n\right)\right]
\end{aligned}$$

$$\begin{aligned}
d\left(x_n, x_{n+1}\right) &\le q\left(x_{n-1}, x_n\right) d\left(x_{n-1}, x_n\right) \\
&+ r\left(x_{n-1}, x_n\right) d\left(x_{n-1}, x_n\right) \\
&+ s\left(x_{n-1}, x_n\right) d\left(x_n, x_{n+1}\right) \\
&+ t\left(x_{n-1}, x_n\right)\left[d\left(x_{n-1}, x_n\right) + d\left(x_n, x_{n+1}\right)\right].
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
&\left(1 - s\left(x_{n-1}, x_n\right) - t\left(x_{n-1}, x_n\right)\right) d\left(x_n, x_{n+1}\right) \\
&\le \left(q\left(x_{n-1}, x_n\right) + r\left(x_{n-1}, x_n\right) + t\left(x_{n-1}, x_n\right)\right) d(x_{n-1}, x_n)
\end{aligned}$$

$$\begin{aligned}
&d(x_n, x_{n+1}) \\
&\le \frac{q\left(x_{n-1}, x_n\right) + r\left(x_{n-1}, x_n\right) + t\left(x_{n-1}, x_n\right)}{\left(1 - s\left(x_{n-1}, x_n\right) - t\left(x_{n-1}, x_n\right)\right)} d\left(x_{n-1}, x_n\right).
\end{aligned} \tag{3}$$

From Definition 2.3.4, we have,

$$\sup_{x_{n-1}, x_n \in X} \left\{\begin{array}{l} q\left(x_{n-1}, x_n\right) + r(x_{n-1}, x_n) \\ + s\left(x_{n-1}, x_n\right) + 2t\left(x_{n-1}, x_n\right) \end{array}\right\} = \lambda < 1$$

$$\begin{aligned}
&q\left(x_{n-1}, x_n\right) + r\left(x_{n-1}, x_n\right) \\
&+ s\left(x_{n-1}, x_n\right) + 2t\left(x_{n-1}, x_n\right) \le \lambda < 1
\end{aligned}$$

$$\begin{aligned}
&q\left(x_{n-1}, x_n\right) + r\left(x_{n-1}, x_n\right) + t\left(x_{n-1}, x_n\right) \\
&< 1 - s\left(x_{n-1}, x_n\right) - t\left(x_{n-1}, x_n\right)
\end{aligned}$$

$\Rightarrow$

$$\frac{q\left(x_{n-1}, x_n\right) + r\left(x_{n-1}, x_n\right) + t\left(x_{n-1}, x_n\right)}{1 - s\left(x_{n-1}, x_n\right) - t\left(x_{n-1}, x_n\right)} < 1.$$

Using this in (3) we have,

$$d\left(x_n, x_{n+1}\right) < d\left(x_{n-1}, x_n\right) \text{ (A)}$$

From Definition 3.3,

$$w_n = d\left(T^{n-1} y_o, T^n y_o\right) = d\left(x_{n-1}, x_n\right).$$

Using this in (A) we have,

$$w_{n+1} < w_n, n \in I.$$

Hence the w-sequence is non-increasing and this implies the iterated sequence $\{y_o, Ty_o, T^2 y_o, \ldots\}$ is Cauchy in X. But X is complete. Therefore the iterated sequence $\{y_o, Ty_o, T^2 y_o, \ldots\}$ converges to say, y^* in X.

i.e.

$$\lim_{n\to\infty} x_n = y^* \tag{4}$$

This proves the condition (ii) of the theorem.

Now to prove that y^* is the fixed point of T.

Since T is a λ-generalized contraction we have,

$$\begin{aligned}
d\left(Ty^*, Tx_n\right) &\le q\left(y^*, x_n\right) d\left(y^*, x_n\right) \\
&+ r\left(y^*, x_n\right) d\left(y^*, Ty^*\right) \\
&+ s\left(y^*, x_n\right) d\left(x_n, Tx_n\right) + \\
&t\left(y^*, x_n\right)\left[d\left(y^*, Tx_n\right) + d\left(x_n, Ty^*\right)\right]
\end{aligned}$$

From Definition 2.3.4 we have,

$$\sup_{x,y\in X} \left\{q(x,y) + r(x,y) + s(x,y) + 2t(x,y)\right\} = \lambda < 1$$

This implies each of the non-negative numbers q(x,y),r(x,y),s(x,y),t(x,y) must be less than λ.

Hence we have,

$$d\left(Ty^*, Tx_n\right) \le \lambda \left(\begin{array}{l} d\left(y^*, x_n\right) + d\left(y^*, Ty^*\right) \\ + d\left(x_n, Tx_n\right) + d\left(y^*, Tx_n\right) + d\left(x_n, Ty^*\right) \end{array}\right)$$

$$\leq \lambda \begin{pmatrix} d\left(y^*, x_n\right) + d\left(y^*, Tx_n\right) + d\left(Tx_n, Ty^*\right) \\ +d\left(x_n, Tx_n\right) + d\left(y^*, Tx_n\right) \\ +d\left(x_n, Tx_n\right) + d\left(Tx_n, Ty^*\right) \end{pmatrix}$$

$$(1-2\lambda) d\left(Ty^*, Tx_n\right)$$
$$\leq \lambda \begin{pmatrix} d\left(y^*, x_n\right) + d\left(y^*, x_{n+1}\right) + d\left(x_n, x_{n+1}\right) \\ +d\left(y^*, x_{n+1}\right) + d\left(x_n, x_{n+1}\right) \end{pmatrix}$$

$$d\left(Ty^*, Tx_n\right)$$
$$\leq \frac{\lambda}{1-2\lambda}\left(d\left(y^*, x_n\right) + 2d\left(x_n, x_{n+1}\right) + 2d\left(y^*, x_{n+!}\right)\right)$$

Using (4) we have, $d(Ty^*, y^*) = 0 \Rightarrow Ty^* = y^*. i.e. (y^*, Ty^*) \in G$. Hence G has a loop at y^*.

Therefore y^* is the fixed point of T and condition (i) of the theorem is proved.

To prove uniqueness, let if possible, z^* be any other fixed point of T. Then $Tz^* = z^*$.

Since T is a λ-generalized contraction we have,

$$d\left(Ty^*, Tz^*\right)$$
$$\leq q\left(y^*, z^*\right) d\left(y^*, z^*\right) + r\left(y^*, z^*\right) d\left(y^*, Ty^*\right)$$
$$+s\left(y^*, z^*\right) d\left(z^*, Tz^*\right)$$
$$+t\left(y^*, z^*\right)\left[d\left(y^*, Tz^*\right) + d\left(z^*, Ty^*\right)\right].$$

Since $Ty^* = y^*$ and $Tz^* = z^*$ we have,

$$d(y^*, z^*) = 0 \Rightarrow y^* = z^*.$$

Hence the fixed point of T is unique and the condition (i) of the theorem is proved.

We now proceed to prove the condition (iii) of the theorem.

From Definition 2.3.4 we have,

$$\sup_{x,y\in X}\left\{q(x,y) + r(x,y) + s(x,y) + 2t(x,y)\right\} = \lambda$$
$$\Rightarrow q(x,y) + r(x,y) + s(x,y) + 2t(x,y) \leq \lambda,$$

Since λ<1 we have,

$$q(x,y) + r(x,y) + t(x,y) + \lambda s(x,y) + \lambda t(x,y)$$
$$\leq q(x,y) + r(x,y) + s(x,y) + 2t(x,y) \leq \lambda$$
$$\rightarrow \frac{q(x,y) + r(x,y) + t(x,y)}{1 - s(x,y) - t(x,y)} \leq \lambda$$

Using this in (3) we have,

I,e. $d\left(x_n, x_{n+1}\right) \leq \lambda d\left(x_{n-1}, x_n\right)$

Repeating this argument we have,

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n)$$
$$\leq \lambda^2 d(x_{n-2}, x_{n-1}) \leq \ldots\ldots \lambda^n d\left(x_o, x_1\right).$$

Hence for some positive integer p, we have,

$$d(x_n, x_{n+p}) \leq \sum_{i=0}^{p-1} d\left(x_{n+i}, x_{n+i+1}\right)$$

$$d(x_n, x_{n+p}) \leq \sum_{i=0}^{p-1} \lambda^{n+i} d\left(x_o, x_1\right)$$

$$d\left(x_n, x_{n+p}\right) \leq \lambda^n \sum_{i=0}^{\infty} \lambda^i d\left(x_o, x_1\right)$$

$$d\left(x_n, x_{n+p}\right) \leq \frac{\lambda^n}{1-\lambda} d\left(x_o, x_1\right)$$

$$d\left(T^n y_o, T^{n+p} y_o\right) \leq \frac{\lambda^n}{1-\lambda} d\left(y_o, Ty_o\right)$$

Allow n+p→∞ then we have,

$$d\left(T^n y_o, y^*\right) \leq \frac{\lambda^n}{1-\lambda} d\left(y_o, Ty_o\right)$$

Hence the condition (iii) of the theorem is also proved.

4. Conclusion

In this paper, the graph associated with the metric space is defined using a new approach and a sequence corresponding to the weights of the edges of the graph, namely w-sequence is defined. Using this sequence, the sequence of iterated functions is proved to be Cauchy. This methodology is followed for proving the contraction principles by Banach, Kannan, Chatterjea and λ-generalized contraction by Ciric. This approach can be used in proving the other contraction principles also.

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