Wavelet-Galerkin Method and Some Numerical Method for Lane-Emden Type Differential Equation

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Abstract In this paper, we will compare the performance of Adomian decomposition method and the wavelet-Galerkin method applied to the Lane-Emden type differential equation. The Galerkin Wavelet method (GWM), which is known as a numerical approach is used for the Lane- Emden equation, as an initial value problem. This approach consists of using integral operator, to convert the Lane- Emden equation in to an integral equation, then applying Galerkin Wavelet method to solve the resulted integral equation. The properties of Galerkin Wavelet method (GWM) and the Adomian Decomposition Method are also addressed. Although the Adomian decomposition solution required slightly more computational effort than the wavelet-Galerkin solution, it resulted in more accurate results than the wavelet-Galerkin method. To illustrate the methods two examples are provided and the results are in good agreement with exact solution.

Keywords: Galerkin Wavelet, Adomian Decomposition method, Lane-Emden equation, integral equations

1. Introduction

In recent years, the studies of initial value problems, presented as second order ordinary differential equations (ODEs) have attached the attention of many mathematicians and physicists. One of the known equations of this type is the Lane- Emden equation, formulated as the fallowing

$$u''(x) + \frac{\alpha}{x}u'(x) + f(x,u) = g(x), \ 0 < x \le 1, \alpha \ge 0 \quad (1)$$

With initial conditions

$$u(0) = A, \ u'(0) = B, \quad (2)$$

Where A, B are constant, $f(x,u)$ is a continuous real valued function, and $g(x) \in C[0,1]$. Eq. (1) has been used to model several phenomena in mathematical physics and astrophysics such as the theory of staller structure, the thermal behavior of a spherical cloud of gas [1,2,3].

Several methods, for solving the Lane-Emden equation are known. A discussion of formulation of some of these models and the physical structure of the solutions can be found in [4,5,6]. Most algorithms currently in use for handling the Lane-Emden type problems are based on series solutions. Wavelet theory is a relatively new and an emerging area in mathematical research. It has been applied in a wide range of engineering disciplines; particularly, wavelets are successfully used in signal analysis for wave form representation and segmentations, time-frequency analysis and fast algorithms for easy implementation [7].

2. Wavelets and Galerkin Wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b, vary continuously we have the following family of continuous wavelets [8]:

$$\phi_{a,b}(t) = |a|^{-\frac{1}{2}} \phi(\frac{t-b}{a}), \ a,b \in R, \ a \ne 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}, \ b = nb_0a_0^{-k}, \ a_0 > 1, \ b_0 > 0$ where n,k are positive integers, we have the following family of discrete wavelets.

$$\phi_{k,n}(t) = |a_0|^{\frac{k}{2}} \phi(a_0^k t - nb_0).$$

Where $\phi_{k,n}(t)$, a wavelet basis in $L^2(R)$. In particular, if $a_0 = 2, b_0 = 1$, then $\phi_{k,n}(t)$ forms an orthonormal basis, [8].

A Multi-resolution analysis of $L^2(R)$ is defined as a sequence of closed subspaces v_j with the following properties, [4]

1) $v_j \subset v_{j+1} \quad \forall j \in Z,$

2) $f(x) \in v_j \Leftrightarrow f(2x) \in v_{j+1}$,

3) $f(x) \in v_0 \Leftrightarrow f(x+1) \in v_0$,

4) $\bigcup\limits_{j \in z} v_j$ Are dens in $L^2(R)$, $\bigcap\limits_{j \in z} v_j = 0$.

The existence of a scaling function $\varphi(x)$ is required for which the translate generate a basis in each v_j i.e.

$$v_j = \overline{span} \left\{ \varphi_{ji} \right\}_{i \in z}$$

With

$$\varphi_{ji} = 2^{\frac{j}{2}} \varphi(2^j x - i) \quad i, j \in Z.$$

In the classical case this basis is orthonormal, so that

$$\left\langle \varphi_{ji}, \varphi_{jk} \right\rangle_R = \delta_{ik}, \tag{3}$$

Where $\left\langle f, g \right\rangle = \int\limits_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx$, is the usual inner product.

Let w_j denote a subspace complementing the subspace v_j in v_{j+1} i.e. $v_{j+1} = v_j \oplus w_j$.

Each element of v_{j+1} can be uniquely written as the sum of two elements, one in v_j and the other in w_j, which contains the details needed to pass from an approximation at the level j to an approximation at the level $j+1$.

Based on the function $\varphi(x)$. One can find $\phi(x)$, the so-called mother wavelet, of which the translates and dilates constitute orthonormal bases of the spaces w_j, $w_j = \overline{span}\left\{ \varphi_{ji} \right\}_{i \in z}$, generated by the following wavelets

$$\varphi_{ji} = 2^{\frac{j}{2}} \phi(2^j x - i) \quad i, j \in Z.$$

Each function $f \in L^2(R)$, can now be expressed as

$$f(x) = \sum\limits_{i \in z} c_{j0i}\, \varphi_{j0\,i} + \sum\limits_{j=j_0}^{\infty} \sum\limits_{i \in z} d_{ji}\, \phi_{ji}(x), \tag{4}$$

Where

$$c_{ji} = \left\langle f, \varphi_{ji} \right\rangle_R, \quad d_{ji} = \left\langle f, \phi_{ji} \right\rangle_R.$$

Of course, in numerical application the summation (4) are truncated, which corresponds to the projection of f into a subspace $v_j \subset L^2(R)$. as otherwise by construction,

$$\left\langle \phi_{ji}, \phi_{lk} \right\rangle = \delta_{jl}\delta_{ik},$$
$$\left\langle \phi_{ji}, \varphi_{lk} \right\rangle = 0 \quad j \geq l.$$

In addition to (3).

3. Solution of Lane-Emden Equation

In the Wavelet- Galerkin method, the solution $u(x)$ of the equation can be approximated by the J th level wavelet series on the interval $[0,1]$, by

$$u(x) = 2^{\frac{J}{2}} \sum\limits_{k=-l+2}^{2^J -1} u_{J,k}\phi(2^J x - k), \tag{5}$$

Therefore, the Galerkin discritization scheme to Eq. (1) gives a nonlinear system of equations that involves the coefficients of connection. For details about the GWM and its application for solving problem, we refer the reader to [9].

Introducing Eq. (5) into Eq. (1) we obtain

$$
\begin{aligned}
& 2^{\frac{5}{2}J} \sum\limits_{k=2-l}^{2^J -1} u_k \phi''(2^J x - k) \\
& + \frac{\alpha}{x} 2^{\frac{3}{2}J} \sum\limits_{k=2-l}^{2^J -1} u_k \phi'(2^J x - k) \\
& + f(x, 2^{\frac{J}{2}} \sum\limits_{k=2-l}^{2^J -1} u_k \phi(2^J x - k)) \\
& = g(x),
\end{aligned}
\tag{6}
$$

In the other expression

$$\sum\limits_{k=2-l}^{2^J -1} u_k \phi_{Jk}'' + \frac{\alpha}{x} \sum\limits_{k=2-l}^{2^J -1} u_k \phi_{Jk}' + f(x,u_k) = g(x), \tag{7}$$

To determine the coefficient u_k, we take the inner product of both sides of Eq. (7) with ϕ_{Jl}, as

$$
\begin{aligned}
& \sum\limits_{k=2-l}^{2^J -1} u_k \int\limits_0^1 \phi_{JL}\, \phi_{Jk}''\, dx \\
& + \frac{\alpha}{x} \sum\limits_{k=2-l}^{2^J -1} u_k \int\limits_0^1 \phi_{JL}\, \phi_{Jk}'\, dx + \int\limits_0^1 \phi_{JL}\, f(x,u_k)\, dx \\
& = \int\limits_0^1 \phi_{JL}\, g(x)\,dx,
\end{aligned}
\tag{8}
$$

$$L = 2-l, 3-l, ..., 2^J - l.$$

Or

$$
\begin{aligned}
& \sum\limits_{k=2-l}^{2^J -1} u_k \int\limits_0^1 \phi_{JL}\, \phi_{Jk}''\, dx + \frac{\alpha}{x} \sum\limits_{k=2-l}^{2^J -1} u_k \int\limits_0^1 \phi_{JL}\, \phi_{Jk}'\, dx \\
& = \int\limits_0^1 \phi_{JL}\, (g(x) - f(x,u_k))\,dx,
\end{aligned}
\tag{9}
$$

$$L = 2-l, 3-l, ..., 2^J - 1.$$

We assume that $(g(x) - f(x,u_k)) = \sum\limits_{i=0}^{m} a_i x^i$, is a polynomial of degree m in x.

We write the Eq. (9) as

$$\sum\limits_{k=2-l}^{2^J -1} c_{kL}^J u_k + \frac{\alpha}{x} \sum\limits_{k=2-l}^{2^J -1} b_{kL}^J u_k = d_{mL}^J, \tag{10}$$

$$L = 2-l, 3-l, ..., 2^J - 1,$$

Where

$$c_{kL}^{J} = \int_0^1 \phi_{Jk}''(x)\phi_{JL}(x)\,dx,$$

$$b_{kL}^{J} = \int_0^1 \phi_{Jk}'(x)\phi_{JL}(x)\,dx, \tag{11}$$

$$d_{kL}^{J} = \int_0^1 \sum_{i=0}^{m} a_i x^i \, \phi_{JL}(x)\,dx.$$

Equation (10) can be further put into the matrix-vector form

$$A_1 U = D, \tag{12}$$

Where

$$A_1 = C + \alpha B + \beta A, B = [b_{kl}^{J}]_{2-L \le k,l \le 2^{J}-1},$$

$$C = [c_{kl}^{J}]_{2-L \le k,l \le 2^{J}-1}, \quad D = [d_{kl}^{J}]_{2-L \le k,l \le 2^{J}-1} \tag{13}$$

And

$$U = [u_{l-2}, u_{l-3}, ..., u_{2^{J}-1}]^{t},$$

Where t, denotes the matrix transpose. Now we have a linear system of $2^{J} + L - 2$ equations of the $2^{J} + L - 2$ unknown coefficients. We can obtain the coefficients of the approximate solution by solving this linear system. The solution U, gives the coefficients in the Wavelet-Galerkin approximation $u_J(x)$ of $u(x)$.

4. Adomian Decomposition Method

Consider the following equation

$$y - N(y) = f, \tag{14}$$

Where N, is a nonlinear operator from a Hilbert space H in to H and f is a given function in H. Now, we are looking for $y \in H$ satisfying (14).

At the beginning of the 1980, Adomian developed a very powerful method to solve Equation (14) in which the solution y, was considered as the sum of decomposition series:

$$y = \sum_{i=0}^{\infty} y_i, \tag{15}$$

And $N(y)$ as the sum of the decomposition series:

$$N(y) = \sum_{n=0}^{\infty} A_n. \tag{16}$$

The method consist of the following recursion scheme

$$y_0 = f, \ y_{n+1} = A_n(y_0, y_1, ..., y_n), \ n = 0, 1, ... \tag{17}$$

Where the A_n' s is polynomials depending on $y_0, y_1, ..., y_n$, are called the Adomian polynomials; these are defined as

$$n! A_n = \frac{d^n}{d\lambda^n} [N(\sum_{i=0}^{n} \lambda^i y_i)] \big|_{\lambda=0}, \ n = 0, 1, ... \tag{18}$$

The Adomian technique is equivalent to determining the sequence:

$$S_n = y_1 + y_2 + ... + y_n \tag{19}$$

By using the iterative scheme

$$S_{n+1} = N(y_0 + S_n), \quad S_0 = 0 \tag{20}$$

Associated with the functional equation

$$S = N(y_0 + s) \tag{21}$$

For the study of the numerical resolution (21), Cherruault used fixed point theorem [10,11,12]. In particular, we know that if N, is a contraction ($\|N\| < 1$) then the sequence $\{S_n\}$ defined by (20) converges to the only solution S of (21).

Furthermore we have $y_n = S_n - S_{n-1} \to_{n \to \infty} 0$.

5. Illustrative Examples

To illustrate the methods two examples are provided

5.1. Example 1

Consider the following Lane-Emden equation

$$y''(x) + \frac{\alpha}{x} y'(x) + y(x) = 6 + 12x + x^2 + x^3,$$

$$0 < x \le 1, \ 0 < \alpha \le 1, \ y(0) = 0, \ y'(0) = 0. \tag{22}$$

The presented method is applied for J=6.

Table 5.1. The absolute errors at different times and space locations for example 1

(X, α)	E W	E decomposition
(0.25, 0.1)	8.5436E_005	3.7747E_015
(0.5, 0.1)	9.3299E_005	4.8849E_015
(0.75, 0.1)	1.7983E_005	6.2172E_015
(0.25, 0.5)	5.4634E_004	7.2729E_009
(0.5, 0.5)	7.6003E_004	9.3386E_009
(0.75, 0.5)	5.7020E_004	1.1991E_008
(0.25, 0.1)	1.3965E_004	3.9273E_006
(0.5, 0.1)	1.7219E_004	5.0428E_006
(0.75. 0.1)	9.1179E_005	6.4750E_006

5.2. Example 2

Let's consider the following form of the Lane-Emden equation;

$$y''(x) + \frac{\alpha}{x} y'(x) = 2(2x+3)y(x),$$

$$0 < x \le 1, \ 0 < \alpha \le 1. \ y(0) = 1, \ y'(0) = 0. \tag{23}$$

The presented method is applied for J=6.

Table 5.2. The absolute errors at different times and space locations for example 2

(X, α)	E W	E decomposition
(0.25, 0.1)	4.7799E_003	1.6077E_009
(0.5, 0.1)	3.8654E_003	2.2736E_009
(0.75, 0.1)	2.7143E_003	1.6077E_009
(0.25, 0.5)	6.5798E_002	5.5213E_004
(0.5, 0.5)	7.3478E_002	7.8084E_004
(0.75, 0.5)	1.8764E_002	5.5213E_004
(0.25, 0.1)	3.2385E_001	1.2245E_001
(0.5, 0.1)	5.7492E_001	1.7317E_001
(0.75. 0.1)	4.2954E_001	1.2245E_001

6. Conclusion

Many excellent properties of wavelet, such as ''locality'' and vanishing moments, causes the wavelet basis to be a better choice than others, in function approximation.. In other way it is seen that decomposition method can be an alternative way for the solution of partial differential equations that have no exact solutions. These two methods have been applied for Lane-Emden equation, successfully. As the results of illustrative examples show the solutions are in a good agreement with the exact ones, even up to 9 digits significant. This agreement of the solutions are so high that one can clams that these methods are a powerful numerical tools for fast and accurate solution, for such kind of differential equations.

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