

Pursuit and Evasion Game under Uncertainty

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Abstract This paper examined a class of multidimensional differential games. In particular, it considered a situation in which the pursuer and evader are affected by uncertain disturbances. A necessary and sufficient condition for the existence of saddle point for this class of games was developed.

Keywords: uncertain disturbances, pursuer, evader, differential games

1. Introduction and System Description

Let IR^n, IR^m and IR^l be Euclidean spaces and $\|\cdot\|$ denote the Euclidean norm. We consider the uncertain dynamical system modelled by:

$$x'_p(t) = A_p x_p(t) + B_p u_p(t) + v_p(t) \quad (1)$$

$$x_p(t_0) = x_p(0) \in IR^n \text{ (given)} \quad (2)$$

$$x'_e(t) = A_e x_e(t) + B_e u_e(t) + v_e(t) \quad (3)$$

$$x_e(t) = x_e(0) \in IR^m \text{ (given)} \quad (4)$$

The subscripts p and e in equations (1)-(4) stand for pursuer and evader respectively. Also, $A_p, A_e \in IR^{n \times n}$ and $B_p, B_e \in IR^{n \times m}$. The vectors $v_p(t)$ and $v_e(t)$ are uncertain vectors.

2. The Problem

The pursuer uses control $u_p(t)$ to attempt to capture the evader, while the evader uses control $u_e(t)$ to avoid being captured. The problem of interest is the minimization and the maximization of the final miss by pursuer and evader respectively, in the presence of uncertainties $v_p(t)$ and $v_e(t)$.

3. The Game

The final miss is defined as a weighted quadratic form:

$$\|x_p(T) - x_e(T)\| \quad (5)$$

To make the game meaningful, we shall impose the following limitations:

$$\int_0^T \|u_p(t)\|_{R_p}^2 dt \leq \Delta_p \quad (6)$$

$$\int_0^T \|u_e(t)\|_{R_e}^2 dt \leq \Delta_e \quad (7)$$

Where R_p, R_e are positive definite matrices such that

$$R_p : IR^m \rightarrow IR^n,$$

$$R_e : IR^m \rightarrow IR^n$$

$$\Delta_p, \Delta_e > 0,$$

T is the final time. Joining (6) and (7) to (5), are the following pay-off functional defined as:

$$J(u_p, u_e) = \frac{1}{2} \left\{ \|x_p(T) - x_e(T)\|^2 + \int_0^T [u_p(t)^T R_p u_p(t) - (u_e(t)^T R_e u_e(t))] dt \right\} \quad (8)$$

the controller $u_p(t)$ is the minimizer of $J(u_p, u_e)$ and the controller $u_e(t)$ is the maximizer. In equations (1) and (2), the uncertainty $v_p(t)$ is acting against the wish of the controller $u_p(t)$ and also acting against the wish of the maximizer $u_e(t)$. in equations (3) and (4).

Subsequently, we make the following assumptions:

Assumption A1: Assuming that the uncertainties

$$v_p(t) : V_p \rightarrow IR^{l_p}$$

$$v_e(t) : V_e \rightarrow IR^{l_e}$$

where $V_p \cap V_e = \emptyset$ and $IR^{l_p}, IR^{l_e} \subset IR^l$ are Lebesgue measurable and ranged within a compact set, then there exist constants such that

$$\|v_p(t)\| \leq \theta_p, \theta_p \in (0, \infty), \theta_p > 0$$

$$\|v_e(t)\| \leq \theta_e, \theta_e \in (0, \infty), \theta_e > 0$$

Assumption A2: There exist non-singular matrices $\hat{P}$ and $\hat{E}$ of appropriate dimensions such that:

$$u_p(t) = -\hat{P}x_p(t) \quad (9)$$

$$u_e(t) = -\hat{E}x_e(t) \quad (10)$$

Also given any Q_1 and $Q_2 \in PD^n$ there exist a unique solution $\hat{P}, \hat{E} \in PD^n$ such that the following matrix Lyapunov equations are satisfied.

$$\left\{ \begin{array}{l} \hat{P}A_p + A_p^T \hat{P} + Q_1 = 0 \\ \hat{E}A_e + A_e^T \hat{E} + Q_2 = 0 \end{array} \right\} \quad (11)$$

4. Problem Formulation

Defining a new state variable

$$z(t) = x_p(t) - x_e(t) \quad (12)$$

then, from equations (1) – (4) we have

$$\begin{aligned} z'(t) = & A_p x_p(t) - A_e x_e(t) + B_p u_p(t) \\ & - B_e u_e(t) + v_p(t) - v_e(t) \end{aligned} \quad (13)$$

where

$$\left\{ \begin{array}{l} P = \hat{P}^{-1} \\ E = \hat{E}^{-1} \end{array} \right\} \quad (14)$$

We shall write equation (13) in a compact form as:

$$z'(t) = Fu_p(t) + Gu_e(t) + v(t) \quad (15)$$

where

$$\left\{ \begin{array}{l} F = B_p - A_p^T P \\ G = A_e^T E - B_e \\ v(t) = v_p(t) - v_e(t) \end{array} \right\} \quad (16)$$

We also impose the condition that

$$\|v(t)\| \leq \rho, \rho > 0.$$

On the basis of (15), the following problems arise:

Problem:1

$$\text{Min max } J(u_p, u_e) = \frac{1}{2} \{z(T)^T z(T) +$$

$$\int_{t_0}^T [(u_p(t)^T R_p u_p(t) - u_e(t)^T R_e u_e(t))] dt\}$$

Subject to

$$z'(t) = Fu_p(t) + Gu_e(t) + v(t), t \in [t_0, T]$$

This problem would be solved under the assumption that the pay-off functional defined by $J(u_p, u_e)$ can be separated to be of the form:

$$\begin{aligned} \min \max \{J(u_p, u_e)\} = & \mu(z(T)) \\ & + \int \{h_1(t, z, u_p) + h_2(t, z, u_e)\} dt \end{aligned} \quad (17)$$

Based on the aforementioned assumption and noting that

$$v(t) = v_p(t) - v_e(t) \quad (18)$$

we arrived at the formulation of the following two optimal control problems define by Problems (2) and (3)

Problem (2):

$$\text{Min } J(u_p) =$$

$$\frac{1}{2} \left[z(T)^T z(T) + \int_{t_0}^T \{u_p(t)^T R_p u_p(t)\} dt \right]$$

Subject to

$$z'(t) = Fu_p(t) + u_p(t) \quad (19)$$

$$z(t_0) = z_0$$

Problem (3):

$$\text{Max } J(u_e) = \frac{1}{2} \left[z(T)^T z(T) - \int_{t_0}^T \{u_e(t)^T R_e u_e(t)\} dt \right]$$

Subject to

$$z'(t) = Gu_e(t) - v_e(t) \quad (20)$$

$$z(t_0) = z_0$$

5. Solution

Necessary Condition for a Saddle Point

For problems (2) and (3) we introduce the following assumptions:

i. The matrix functions $R_p(\cdot), R_e(\cdot), F(\cdot)$ and $G(\cdot)$ are constant matrices of appropriate dimensions.

ii. Control functions $u_p(\cdot), u_e(\cdot)$ and disturbances $v_p(\cdot), v_e(\cdot)$ in problems (2) and (3) are generated by strategies.

Now consider problem (2) and define the Hamiltonian for the problem as

$$H(z, u_p, \lambda) = \frac{1}{2} (u_p(t)^T R_p u_p(t)) + \lambda^T v_p(t) \quad (21)$$

The adjoint equation satisfies

$$\left. \begin{array}{l} \lambda' = 0 \\ \lambda(T) = z(T) \end{array} \right\} \quad (22)$$

$$Hu_p = 0 \Rightarrow \bar{u}_p(t) = R_p^{-1}(t)F^T(t)\lambda(t)$$

Now,

$$\max_{\|v_p\| \leq \theta_p} H \Leftrightarrow \max_{\|v_p\| \leq \theta_p} \{\lambda^T v_p\}$$

Therefore

$$\bar{v}_p(t) = \frac{\theta_p \lambda(t)}{\|\lambda(t)\|} \quad (23)$$

6. Deduction

From (23) we deduce the following three cases, namely:

- (a) $\lambda(t) > 0 : \bar{v}_p(t) = \theta_p$,
- (b) $\lambda(t) < 0 : \bar{v}_p(t) = -\theta_p$
- (c) $\lambda_p(t) = 0$

then any admissible $v_p(t)$ is an optimal solution

We knock off case (b) since $\lambda(t)$ cannot be negative as an adjoint vector.

Case (c) is a trivial solution. Assume that the solution is not trivial, we consider case (a).

Let $\lambda(\cdot) : [t_0, T] \rightarrow \mathbb{R}^{n+1}$ be continuous and let $\lambda(t)$ be define as

$$\lambda(t) = M(t)\bar{z}(t) + N(t)\theta_p, \quad (24)$$

where $M(t)$ and $N(t)$ are square matrices and $\bar{z}(t) : [t_0, T] \rightarrow \mathbb{R}^n$ is a solution to (19). Differentiating (14) to get:

$$\lambda'(t) = M'(t)\bar{z}(t) + M(t)\bar{z}'(t) + N'(t)\theta_p \quad (25)$$

Hence,

$$M'(t)\bar{z}(t) + M(t)\bar{z}'(t) + N'(t)\theta_p = 0 \quad (26)$$

Therefore on the basis of (12) we have

$$\begin{aligned} M'(t)\bar{z}(t) + M(t)F(t)\{-R_p^{-1}F(t)\lambda(t)\} \\ + M(t)\theta_p + N'(t)\theta_p = 0 \end{aligned} \quad (27)$$

Substituting (14) in (17) we get

$$\begin{aligned} M'(t)\bar{z}(t) - M(t)^T F(t)R_p^{-1}(t)F(t)M(t)\bar{z}(t) \\ - M(t)^T F(t)R_p^{-1}(t)F(t)N(t)\theta_p \\ + (M(t) + N'(t))\theta_p = 0 \end{aligned} \quad (28)$$

For (18) to hold, it is necessary and sufficient that

$$\left. \begin{array}{l} M'(t) - M(t)F(t)R_p^{-1}(t)F(t)M(t) = 0 \\ M(T) = I \\ N'(t) + M(t) \\ -M(t)F(t)R_p^{-1}(t)F(t)N(t) = 0 \\ N(T) = I \end{array} \right\} \quad (29)$$

Let $\alpha(z(t), t)$ and $\beta(z(t), t)$ be strategies for $u_p(t)$ and $v_p(t)$ respectively, we then summarize the saddle point solution as follows:

$$\begin{aligned} \alpha(z(t), t) = \bar{u}_p(t) = -R_p^{-1}(t)F(t)^T \{M(t)\bar{z}(t) + N(t)\theta_p\} \\ \beta(\bar{z}(t), t) = \bar{v}_p(t) = \theta_p \end{aligned} \quad (30)$$

For problem (3) the Hamiltonian is defined as

$$\begin{aligned} H(u_e, v_e, \eta) = -\frac{1}{2}(u_e(t)^T R_e(t)u_e(t)) \\ + \eta(t)^T \{Gu_e(t) - v_e(t)\} \end{aligned} \quad (31)$$

Following the same procedure as explained in problem (2), the adjoint vector satisfies

$$\left. \begin{array}{l} \eta'(t) = 0 \\ \eta(T) = z(T) \end{array} \right\} \quad (32)$$

Now,

$$Hu_e = 0 \Rightarrow \bar{u}_e(t) = R_e^{-1}(t)G(t)\eta(t) \quad (33)$$

Since u_e is a maximizer, we choose $v_e(t)$ such that

$$\min_{\|v_e\| \leq \theta_e} H \Leftrightarrow \min_{\|v_e\| \leq \theta_e} \{-\eta(t)^T v_e\} \quad (34)$$

Therefore

$$\bar{v}_e(t) = -\frac{\theta_e \eta(t)}{\|\eta(t)\|} \quad (35)$$

The following three cases can be deduced from (25)

- (d) $\eta(t) > 0, \bar{v}_e(t) = -\theta_e$
- (e) $\eta(t) < 0, \bar{v}_e(t) = \theta_e$
- (f) $\eta(t) = 0$,

then any admissible $v_e(t)$ is an optimal solution.

Since $v_e(t)$ is a minimizer and from (26) we shall knock off case (e) and we shall also assume a non-trivial solution, hence we shall let $\eta(\cdot) : [t_0, T] \rightarrow \mathbb{R}^{n+1}$ be continuous and be defined as

$$\eta(t) = D(t)\bar{z}(t) - E(t)\theta_e \quad (37)$$

Where $D(t)$ and $E(t)$ are square matrices and $\bar{z}(t) : [t_0, T] \rightarrow \mathbb{R}^n$ is a solution of (10). We observed that

$$\eta'(t) = D'(t)\bar{z}(t) + D(t)\bar{z}'(t) - E'(t)\theta_e \quad (38)$$

From (22)

$$D'(t)\bar{z}(t) + D(t)\bar{z}'(t) - E'(t)\theta_e = 0 \quad (39)$$

Therefore

$$\begin{aligned} D'(t)\bar{z}(t) + D(t)Q(t)\bar{u}_e(t) \\ - D(t)v_e(t) - E(t)\theta_e = 0 \end{aligned} \quad (40)$$

becomes

$$\begin{aligned} D'(t)\bar{z}(t) + D(t)R_e^{-1}G(t)^T \eta(t) \\ + D(t)\theta_e - E'(t)\theta_e = 0 \end{aligned} \quad (41)$$

On substituting (37) into (41) we have the following equation

$$\begin{aligned} D'(t)\bar{z}(t) + D(t)G(t)R_e^{-1}(t)G(t)^T D(t)\bar{z}(t) \\ - D(t)G(t)R_e^{-1}(t)G(t)E(t)\theta_e + D(t)\theta_e - E(t)\theta_e = 0 \end{aligned} \quad (42)$$

For (42) to hold

$$\left. \begin{aligned} D'(t) + D(t)G(t)R_e^{-1}(t)G(t)^T D(t) &= 0 \\ E'(t) - D(t) + D(t)G(t)R_e^{-1}(t)G(t)E(t) &= 0 \end{aligned} \right\} \quad (43)$$

Let $\bar{\alpha}(\bar{z}(t), t)$ and $\bar{\beta}(\bar{z}(t), t)$ be strategies for control $u_e(t)$ and disturbance respectively, then, the saddle point solution is summarized as follows:

$$\left\{ \begin{aligned} \bar{\alpha}(\bar{z}(t), t) = \bar{u}_e(t) &= R_e^{-1}G(t)[D(t)z(t) - E(t)\theta_e] \\ \bar{\beta}(\bar{z}(t), t) = \bar{v}_e(t) &= -\theta_e \end{aligned} \right. \quad (44)$$

7. Value of the Game

We now employ the results in (20) and (34) to compute the value of the objective functional defined in problems (2) and (3)

We recall that

$$z'(t) = Fu_p(t) + v_p(t) \quad (45)$$

$$\lambda'(t) = 0 \quad (46)$$

$$\lambda(T) = Z(T) \quad (47)$$

Multiply (45) by $\lambda(t)^T$ and equation (46) by $z(t)$ to get

$$\lambda(t)z'(t) = \lambda(t)^T Fu_p(t) + \lambda(t)^T v_p(t) \quad (48)$$

$$\lambda'(t)z(t) = 0 \quad (49)$$

adding equations (48) and (49) together to get

$$\begin{aligned} \lambda(t)^T z'(t) + \lambda'(t)^T z(t) \\ = \lambda(t)^T Fu_p(t) + \lambda(t)^T v_p(t) \\ = \frac{d}{dt}[\lambda(t)z(t)] \\ = \lambda(t)^T Fu_p(t) + \lambda(t)^T v_p(t) \end{aligned} \quad (50)$$

Integrating (50) we get the following:

$$\begin{aligned} \int_{t_0}^T \frac{d}{dt}[\lambda(t)^T z(t)] &= \int_{t_0}^T \{\lambda(t)^T Fu_p(t) + \lambda(t)^T v_p(t)\} dt \\ &= \int_{t_0}^T \{\lambda(t)^T Fu_p(t) + \lambda(t)^T v_p(t)\} dt \\ &= \lambda(T)^T z(T) - \lambda(t_0)^T z(t_0) \\ &= -\int_{t_0}^T (u_p(t)R_p u_p(t) + \lambda(t)^T \theta_p) dt \end{aligned}$$

We know from (47) that $\lambda(T) = z(T)$, therefore,

$$\begin{aligned} z(T)^T z(T) + \int_{t_0}^T \{u_p(t)^T R_p u_p(t)\} dt \\ = \lambda(t_0)^T z(t_0) + \int_{t_0}^T \{\lambda(t)^T \theta_p\} dt \end{aligned}$$

Hence,

$$2J(\bar{u}_p(t)) = \lambda(t_0)^T z(t_0) + \int_{t_0}^T \{\lambda(t)^T \theta_p\} dt,$$

and from

$$\lambda(t) = M(t)z(t) + N(t)\theta_p,$$

therefore

$$\begin{aligned} 2J(\bar{u}_p(t)) &= \langle M(t_0)z(t_0), z(t_0) \rangle \\ &+ N(t_0)\theta_p^T z(t_0) \\ &+ \int_{t_0}^T \{(M(t)z(t) + N(t)\theta_p)^T \theta_p\} dt \end{aligned} \quad (51)$$

Similarly, we consider

$$z'(t) = Gu_e(t) - v_e(t) \quad (52)$$

$$\eta'(t) = 0 \quad (53)$$

$$\eta(T) = z(T) \quad (54)$$

From (52) and (53) we have

$$\eta(t)^T z'(t) = \eta(t)^T Gu_e(t) - \eta(t)^T v_e(t) \quad (55)$$

$$\eta(t)^T z(t) = 0 \quad (56)$$

Combining (55) and (56) we have

$$\eta(t)z'(t) + \eta'(t)z(t) = \eta(t)^T Gu_e(t) - \eta(t)^T v_e(t) \quad (57)$$

Therefore

$$\int_{t_0}^T \frac{d}{dt} \{\eta(t)z(t)\} dt = \quad (58)$$

$$\int_{t_0}^T \langle R_e u_e(t), u_e(t) \rangle dt + \int_{t_0}^T (\eta(t)^T \theta_e) dt$$

Now, given that $\eta(T) = z(T)$, then

$$\begin{aligned} z(T)^T z(T) - \int_{t_0}^T (R_e u_e(t), u_e(t)) dt \\ = \eta(t_0)^T z(t_0) + \int_{t_0}^T (\eta(t)^T \theta_e) dt \end{aligned} \quad (59)$$

From (37) substitute for $\eta(t)$ to get

$$\begin{aligned} & z(T)^T z(T) - \int_{t_0}^T (R_e u_e(t), u_e(t)) dt \\ &= z(t_0)^T D(t_0) z(t_0) - E(t_0) \theta_e^T z(t_0) \\ &+ \int_{t_0}^T \{ (D(t) \bar{z}(t) - E(t)^T \theta_e) \} dt \end{aligned} \quad (60)$$

Therefore

$$\begin{aligned} 2J(\bar{u}_e) &= z(t_0)^T D z(t_0) - E(t_0) \theta_e^T z(t_0) \\ &+ \int_{t_0}^T \{ (D(t) \bar{z}(t) - E(t) \theta_e)^T \theta_e \} dt \end{aligned} \quad (61)$$

Combining (51) and (61), the value of the game is given as

$$\begin{aligned} J(\bar{u}_p, \bar{u}_e) &= \frac{1}{2} \{ < (M(t_0) + D(t_0), z(t_0)) > \\ &+ (N(t_0) \theta_p - E(t_0) \theta_e) z(t_0) \\ &+ \int_{t_0}^T [(M(t) + D(t) \bar{z}(t) + (N(t) \theta_p)^T \theta_p \\ &- (E(t) \theta_e)] dt \} \end{aligned} \quad (62)$$

8. Sufficient Condition for a Saddle Point

We shall employ the sufficiency theorem given by [Gutman, S. (1975)] to show that the solution obtained for each of the cases is indeed a saddle point.

Let $V(z, t) : \mathbb{R}^n \times \mathbb{R}^1 \rightarrow \mathbb{R}^1$ be defined as

$$V(z, t) = z(t)^T M z(t) + N(t) \theta_p z(t) \quad (63)$$

We assume that $V(.,.)$ is continuous on $[t_0, T]$ and it is a C^1 - function.

Define

$$\begin{aligned} & L(z, t, u_p, v_p) \\ &= u_p(t)^T R_p(t) u_p(t) + \text{grad} V(z, t) F(z, t, u_p, v_p) \\ &= u_p(t)^T R_p(t) u_p(t) + 2z(t)^T M(t) z'(t) + N(t) \theta_p z'(t) \\ &+ z(t)^T M'(t) z(t) + N'(t) \theta_p z(t) \\ &= u_p(t)^T R_p(t) u_p(t) + 2z(t)^T [M(t)(F(t) u_p(t) + v_p(t))] \\ &+ N(t) \theta_p [F(t) u_p(t) + v_p(t)] \\ &+ z(t)^T M'(t) z(t) + N'(t) \theta_p z(t) \end{aligned} \quad (64)$$

By virtue of (23) and (24) we have

$$\begin{aligned} & L(z, t, u_p, v_p) \\ &= u_p(t)^T R_p(t) u_p(t) + z(t)^T M'(t) z(t) \\ &- 2z(t)^T M(t)^T F(t) R_p^{-1}(t) F(t)^T M(t) z(t) \\ &- 2z(t)^T M(t)^T F(t) R_p^{-1}(t) F(t) N(t) \theta_p + 2z(t)^T M(t) \theta_p \\ &- M(t)^T F(t) R_p^{-1}(t) F(t)^T N(t) \theta_p z(t) \\ &- N(t)^T F(t) R_p^{-1}(t) F(t) N(t) \theta_p^2 \\ &+ N(t) \theta_p + N'(t) \theta_p z(t) \end{aligned} \quad (65)$$

$$\begin{aligned} & L(z, t, u_p, v_p) \\ &= u_p(t)^T R_p(t) u_p(t) \\ &+ z(t)^T (M'(t) - M(t)^T F(t) R_p^{-1}(t) F(t) M(t)) z(t) \\ &- z(t)^T M(t)^T F(t) R_p^{-1}(t) F(t)^T M(t) + N'(t) + M(t) \\ &- M(t)^T F(t) R_p^{-1}(t) N(t) z(t) \theta_p \\ &- 2M(t)^T F(t) R_p^{-1}(t) F(t) N(t) \theta_p z(t) \\ &+ M(t) z(t) \theta_p - N(t) F(t) R_p^{-1}(t) F(t) N(t) \theta_p^2 \\ &+ N(t) \theta_p \end{aligned} \quad (66)$$

using (29), (66) is reduced to

$$\begin{aligned} & L(z, t, u_p, v_p) \\ &= u_p(t)^T R_p(t) u_p(t) + M(t) z(t) \theta_p \\ &- 2M(t)^T F(t) R_p^{-1}(t) F(t) N(t) \theta_p z(t) \\ &- N(t)^T F(t) R_p^{-1}(t) F(t) N(t) \theta_p^2 + N(t) \theta_p \\ &- z(t)^T M(t)^T F(t) R_p^{-1}(t) F(t)^T M(t) z(t) \end{aligned} \quad (67)$$

In (66) we substitute for $u_p(t) R_p(t) u_p(t)$, using (23) and (24) to get

$$L(z, t, u_p, v_p) = M(t) z(t) \theta_p + N(t) \theta_p \quad (68)$$

Now

$$\min_{u_p \in \mathbb{R}^m} L(z, t, \bar{u}_p, v_p) = \min_{u_p \in \mathbb{R}^m} [M(t) z(t) \theta_p + N(t) \theta_p] = 0 \quad (69)$$

and

$$\begin{aligned} & \max_{v_p \in \mathbb{R}^p} L(z, t, u_p, \bar{v}_p) \\ &= \max_{v_p \in \mathbb{R}^p} [M(t) z(t) \theta_p + N(t) \theta_p] > 0, \|v_p\| \leq \theta_p \end{aligned} \quad (70)$$

On the basis of (69) and (70), (30) is indeed a saddle point.

9. Conclusion

The idea of saddle point (min, max) controllers arises in engineering problems where extreme conditions are to be overcome (Gutman, 1975). A natural example is the "boosted period" of missiles when high thrust acts on the body so that every small deviation from the designed specifications cause unpredictable (input) disturbances in three nodes.

In this work we have considered situations where the disturbances affect the motions of the pursuer and the evader respectively. In our subsequent paper we hope to apply the results in this work to problems arising from pricing of general insurance policies, particularly in a competitive and non-cooperative market.

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