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On a Sum and Difference of Two Quasi Lindley Distributions: Theory and Applications

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Abstract In this paper two basic random variables constructed from Quasi Lindley distribution have been introduced. One of these variables is defined as the sum of two independent random variables following the Quasi-Lindley distribution with the same parameter (2SQLindley). The second one is defined as the difference of two independent random variables following the Quasi-Lindley distribution with also the same parameter (2DQLindley). For both cases, we provided some statistical properties such as moments, incomplete moments and characteristic function. The parameters are estimated by maximum likelihood method. From simulation studies, the performance of the maximum likelihood estimators has been assessed. The usefulness of the corresponding models is proved using goodness-of-fit tests based on different real datasets. The new models provide consistently better fit than some classical models used in this research.

Keywords: *Quasi Lindley distribution, mixed distribution, maximum likelihood estimation, incomplete moments, characteristic function, stochastic ordering and extreme order statistics*

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1. Introduction

The analysis and modeling of lifetime data are very important in applied sciences such as engineering, public health, actuarial science, biomedical studies, demography, industrial reliability and other applied sciences. For this reason, it became really urgent to find statistical distributions to handle these real models data. The one of these distributions, is Lindley distribution which was introduced by [1] to analyze failure time data especially in applications modeling stress-strength reliability. The importance of the Lindley distribution comes from its power to model failure time data with increasing, decreasing, unimodal and bathtub-shaped hazard rates. Lindley distribution can be written as a mix of exponential and gamma distributions. The properties and inferential procedure for the Lindley distribution were studied by [2]. It is shown that the Lindley distribution is better than the exponential distribution when the hazard rate is unimodal or bathtub shaped.

Generalizations and transformations of existing distribution attract the statistical research to develop new models to prove useful in exploring skewness and tail properties, and also to improve the goodness-of-fit of the extended family. Lindley distribution has been generalized, extended, modified and mixed with other discrete distributions by many authors in recent years. [3] referred to the three Parameters-Lindley distribution, [4] introduced generalized Poisson-Lindley distribution, generalized Lindley distribution has been proposed by [5]. Further, more investigations applied to Lindley distribution to improve its performance and have been introduced by some interested researchers. For example; Marshall-Olkin Lindley distribution was introduced by [6], power Lindley distribution was presented by [7], two-parameter Lindley distribution by [8], Quasi Lindley distribution by [9], transmuted Lindley distribution by [10], transmuted Lindley-geometric distribution and beta-Lindley distribution by [11]. A retrospective study on Lindley distribution and its generalizations have been studied extensively by [12] and discrete Harris extended Lindley distribution by [13]. A latest version of the Lindley distribution, called modified Lindley distribution, is introduced by [14].

Recently, [15] have been introduced Generalization of the Lindley distribution based on the formation of a distribution of the sum of two iid variables that have a Lindley distribution with the same parameters and are independent. They also presented a distribution of the difference between the same two variables, where the results applied on real data sets provided consistently better fit than some classical distributions.

In this paper we will present a distribution of the sum of two iid variables that have a Quasi-Lindley distribution with the same parameters and are independent, where the Quasi - Lindley distribution is considered one of the generalizations of the Lindley distribution. Quasi - Lindley distribution, Lindley distribution and the exponential distribution in order to demonstrate the quality of reconciliation on the real data set. Also, the distribution of the difference between the two variables will be compared with both Laplace distribution and the distribution of the sum of two independent variables each of which has the same Laplace distribution with the same parameters to demonstrate the good of fit on the real data set.

The cumulative distribution function (cdf) and the probability density function (pdf) of quasi-Lindley distribution (QLD) of [8] are given by;

$$G_*(y,\alpha,\theta) = 1 - \frac{1+\alpha+\theta y}{\alpha+1} e^{-\theta y}, y>0, \theta>0, \alpha>-1 \quad (1)$$

$$g_*(y,\alpha,\theta) = \frac{\theta(\alpha+\theta y)}{\alpha+1} e^{-\theta y}, y>0, \theta>0, \alpha>-1 \quad (2)$$

2. On the Sum of two Independent Quasi Lindley Distribution

This section is focused on definitions and some properties of the sum of two independent random variables following the Quasi Lindley distribution (2SQLindely).

2.1. Definition

We consider the pdf given by

$$f(x) = \frac{\theta^2}{(\alpha+1)^2} x \left(\alpha^2 + \theta\alpha x + \frac{\theta^2}{6} x^2 \right) e^{-\theta x}, \quad (3)$$
$$x>0, \theta>0, \alpha>-1$$

The feature of this distribution is the following: let Y_1 and Y_2 be two independent random variables following the Quasi Lindley distribution (QL) with parameters α and θ. Then, the random variable $X = Y_1 + Y_2$ has the pdf given by (3). This result is a particular case of [16]. The proof is as follow; Since Y_1 and Y_2 are independent, the pdf of X is given by the convolution product

$$f(x) = \int_{-\infty}^{+\infty} g_*(x-t) g_*(t) dt \quad for\ x>0$$

$$= \int_0^x \frac{\theta}{\alpha+1}(\alpha+\theta(x-t)) e^{-\theta(x-t)} \frac{\theta}{\alpha+1}(\alpha+\theta t) e^{-\theta t} dt$$

$$= \frac{\theta^2}{(\alpha+1)^2} x \left(\alpha^2 + \theta\alpha x + \frac{\theta^2}{6} x^2 \right) e^{-\theta x}$$

This distribution is considered entirely new as it has not been previously deduced and is considered a generalized distribution for the distribution of the sum of two independent variables for which the Lindley distribution provided by [15]. Whereas, through this research and in what part if $\alpha = \theta$ is placed, we obtain the same results as that of [15].

First of all, after some algebraic manipulations, the cdf of the 2SQLindely distribution is given by

$$F(x) = 1 - \frac{1}{6(\alpha+1)^2} \begin{bmatrix} 6(1+\alpha)^2(1+\theta x) \\ +(3+6\alpha)\theta^2 x^2 \\ +\theta^3 x^3 \end{bmatrix} e^{-\theta x}, x>0 \quad (4)$$

Moreover, the survival function is given as follows

$$S(x) = \frac{1}{6(1+\alpha)^2} \begin{bmatrix} 6(1+\alpha)^2(1+\theta x) \\ +(3+6\alpha)\theta^2 x^2 + \theta^3 x^3 \end{bmatrix} e^{-\theta x}, x>0 \quad (5)$$

The hazard rate function, hrf, is given by

$$h(x) = \frac{f(x)}{S(x)}$$
$$= \frac{6\theta^2 \left(\alpha^2 x + \alpha\theta x^2 + \frac{\theta^3}{6} x^3 \right)}{6(1+\alpha)^2(1+\theta x) + (3+6\alpha)\theta^2 x^2 + \theta^3 x^3}, x>0 \quad (6)$$

Also, the corresponding cumulative hazard rate function, hrf, is given by

$$\Omega(x) = -\ln(S(x))$$
$$= 6(1+\alpha)^2 + \theta x\ ln \begin{bmatrix} 6(1+\alpha)^2(1+\theta x) \\ +(3+6\alpha)\theta^2 x^2 + \theta^3 x^3 \end{bmatrix}, x>0 \quad (7)$$

he corresponding quantile function (qf), say $Q(u)$, can be obtained by solving the following equation: $F(Q(u)) = Q(F(u)), u \in (0,1)$. It cannot be presented analytically but can be determined numerically for a given α and θ.

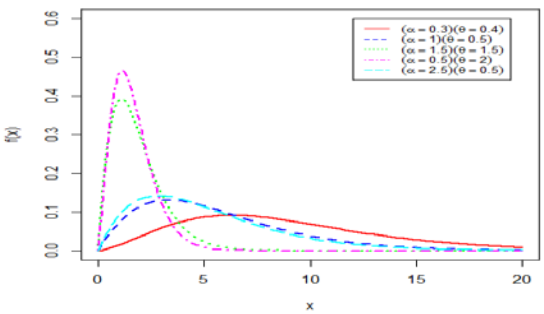

Figure 1. The pdf of the 2SQLindely distribution for some values of the parameters

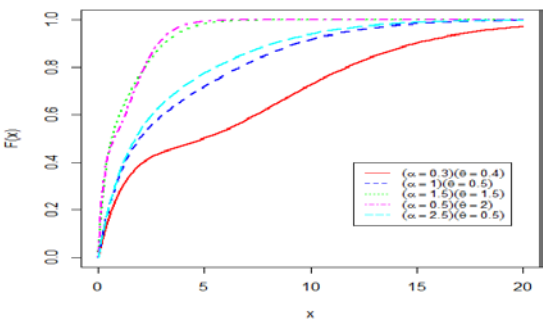

Figure 2. The cdf of the 2SQLindely distribution for some values of the parameters

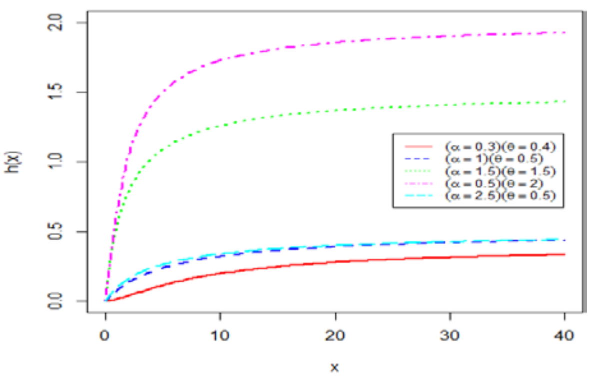

Figure 3. The hrf of the 2SQLindely distribution for some values of the parameters

Figure 1, Figure 2 and Figure 3 show the plots of the pdf, cdf and hrf for various values of the parameters α and , θ respectively.

We can derive the mode of the 2SQLindelydistribution by solving the following equation withrespect to x as follow;

$$-\frac{\theta^3}{6}x^3+\frac{\theta^2}{2}(1-2\alpha)x+\theta\left(2\alpha-\alpha^2\right)+\alpha^2=0 \qquad (8)$$

It is not possible to get an explicit solution of the equation (8) in the general case and therefore numerical methods should be used such as bisection method or fixed-point method to solve it.

2.2. Moments

The r^th moment about origin of the 2SQLindely has been obtained as

$$\mu_r^*=\frac{(r+1)!}{6(1+\alpha)^2\theta^r}\left[6\alpha^2+6\alpha(r+2)+r^2+5r+6\right] \qquad (9)$$

Proof: Let us introduce the gamma function defined by $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt, x>0$. By using the pdf of x given by (3), we have

$$\mu_r^*=E\left(x^r\right)=\int_0^x x^r f(x)dx$$

$$=\frac{\theta^2}{(1+\alpha)^2}\left[\begin{array}{l}\alpha^2\int_0^\infty x^{r+1}e^{-\theta x}dx+\theta\alpha\int_0^\infty x^{r+2}e^{-\theta x}dx\\+\frac{\theta^2}{6}\int_0^\infty x^{r+3}e^{-\theta x}dx\end{array}\right]$$

$$=\frac{\theta^2}{(1+\alpha)^2}\left[\alpha^2\frac{(r+2)}{\theta^{r+2}}+\theta\alpha\frac{(r+3)}{\theta^{r+3}}+\frac{\theta^2}{6}\frac{(r+4)}{\theta^{r+4}}\right]$$

$$=\frac{(r+1)!}{6(1+\alpha)^2\theta^r}\left[6\alpha^2+6\alpha(r+2)+r^2+5r+6\right]$$

The proof of the moments using the QL distribution as a baseline is given below. Let us recall that, for any $r\in\mathbb{N}$ and a random variable X following the QL distribution with parameters α and θ ,the r^th moment of X is given by

$$\bar{\mu}_r=\frac{(r+1)(\alpha+r+1)}{\theta^r(\alpha+1)}$$

Therefore, by $X=Y_1+Y_2$ and the binomial formula, the r^th moment of X is given by

$$\mu_r^*=E\left(y_1+y_2\right)^r=\sum_{k=0}^{r}\binom{r}{k}\dot{\mu}_{r-k}\dot{\mu}_k$$

$$=\sum_{k=0}^{r}\frac{(r-k+1)(\alpha+r-k+1)}{\theta^{r-k}(\alpha+1)}\times\frac{(k+1)(\alpha+k+1)}{\theta^k(\alpha+1)}$$

$$=\frac{r!}{(1+\alpha)^2\theta^r}\sum_{k=0}^{r}(\alpha+r-k+1)\times(\alpha+k+1)$$

$$=\frac{(r+1)!}{6(1+\alpha)^2\theta^r}\left[6\alpha^2+6\alpha(r+2)+r^2+5r+6\right].$$

So, the r^th moment will be

$$\mu_1^*=\frac{2(2+\alpha)}{(\alpha+1)\theta},\ \mu_2^*=\frac{6\alpha(\alpha+4)+20}{(1+\alpha)^2\theta^2}$$

$$\mu_3^*=\frac{24\left(\alpha^2+5\alpha+5\right)}{(1+\alpha)^2\theta^3}\ and\ \mu_4^*=\frac{120\left(\alpha^2+6\alpha+7\right)}{(1+\alpha)^2\theta^4}$$

Particularly, the mean of X is given by $\mu=\mu_1^*$ and the variance of X is given by

$$V(x)=\sigma^2=\mu_2^*-\mu_1^{*2}=\frac{2\left(\alpha^2+4\alpha+2\right)}{(1+\alpha)^2\theta^2}$$

Also, the skewness and kurtosis coefficients of X, respectively given by

$$\sqrt{\beta_1}=\frac{1}{\sigma^3}\sum_{k=0}^{3}\binom{3}{k}\mu_k^*(-\mu)^{3-k}$$

$$\beta_2=\frac{1}{\sigma^4}\sum_{k=0}^{4}\binom{4}{k}\mu_k^*(-\mu)^{4-k}$$

2.3. Incomplete Moments

Let r be a positive integer and X a random variable following the 2SQLindely distribution with parameters α and θ. Let us introduce the lower gamma function defined by

$$\gamma_{(s,y)}=\int_0^y t^{s-1}e^{-t}dt, s>0, y>0$$

Then,

$$\mu_r^*(t)=E\left(x^r|_{\{x\le t\}}\right)$$

$$=\frac{1}{(1+\alpha)^2\theta^r}\left[\begin{array}{l}\alpha^2\gamma(r+2,\theta t)\\+\alpha\gamma(r+3,\theta t)+\frac{1}{6}\gamma(r+r,\theta t)\end{array}\right] \qquad (10)$$

Proof: By using the pdf of X given by (3), we have

$$\mu_r^*(t) = \int_0^t x^r f(x)\,dx$$

$$= \frac{\theta^2}{(1+\alpha)^2}\left[\begin{array}{l} \alpha^2 \int_0^t x^{r+1} e^{-\theta x} + \theta\alpha \int_0^t x^{r+2} e^{-\theta x} dx \\ + \frac{\theta^2}{6} \int_0^t x^{r+3} e^{-\theta x} dx \end{array}\right]$$

$$= \frac{\theta^2}{(1+\alpha)^2}\left[\begin{array}{l} \alpha^2 \frac{\gamma(r+2,\theta t)}{\theta^{r+2}} + \theta\alpha \frac{\gamma(r+3,\theta t)}{\theta^{r+3}} \\ + \frac{\theta^2}{6} \frac{\gamma(r+4,\theta t)}{\theta^{r+4}} \end{array}\right]$$

$$= \frac{1}{(1+\alpha)^2 \theta^r}\left[\begin{array}{l} \alpha^2 \gamma(r+2,\theta t) + \alpha\gamma(r+3,\theta t) \\ + \frac{1}{6}\gamma(r+r,\theta t) \end{array}\right]$$

2.4. Moment Generating Function

The moment generating function $M_x(t)$ of a random variable $X \sim 2SQL(\alpha,\theta)$ is given by

$$M_x(t) = \frac{\theta^2\left[\alpha(\theta-t)+\theta\right]^2}{(1+\alpha)^2(\theta-t)^4} \qquad (11)$$

Proof. Let us recall that, for any $t \in \mathbb{R}$ and a random variable X following the 2SQLindely distribution with parameters α and θ .the moment generating function of X is given by

$$M_x(t) = \int_0^\infty e^{tx} f(x)\,dx$$

$$M_x(t) = \frac{\theta^2}{(1+\alpha)^2}\left[\begin{array}{l} \alpha^2 \int_0^\infty x e^{-(\theta-t)x} dx + \theta\alpha \int_0^\infty x^2 e^{-(\theta-t)x} dx \\ + \frac{\theta^2}{6} \int_0^\infty x^3 e^{-(\theta-t)x} dx \end{array}\right]$$

$$= \frac{\theta^2}{(1+\alpha)^2}\left[\frac{\alpha^2}{(\theta-t)^2} + \frac{2\theta\ \alpha}{(\theta-t)^3} + \frac{\theta^2}{(\theta-t)^3}\right]$$

$$= \frac{\theta^2\left[\alpha(\theta-t)+\theta\right]^2}{(1+\alpha)^2(\theta-t)^4}$$

2.5. Characteristic Function

The characteristic function of $2SQLindely$ distribution is given in the following.

Suppose that the random variable X have the $X \sim 2SQLindely(\alpha,\theta)$ then characteristic function, $\Phi_x(t)$, is

$$\Phi_x(t) = \frac{\theta^2\left[\alpha(\theta-it)+\theta\right]^2}{(1+\alpha)^2(\theta+t)^4} \qquad (12)$$

Where i $= \sqrt{-1}$.

2.6. Bonferroni and Lorenz curves

Bonferroni and Lorenz curve proposed by [17]. They are used to measure the inequality of the distribution of a random variable X. They are applied in many fields such as economics, reliability, demography, insurance, etc. The Bonferroni and Lorenz curves are defined as

$$B(p) = \frac{1}{p\mu}\int_0^q x f(x)\,dx$$

and

$$L(p) = \frac{1}{\mu}\int_0^q x f(x)\,dx$$

respectively, where q $=$ $F^{-1}(p)$.

If X has the pdf in (3), we can calculate Bonferroni and Lorenz curve of the $X \sim 2SQLindely$ (α, θ) distribution as

$$B(p) = \frac{1}{p\mu}\mu_1^*(q), L(p) = \frac{1}{\mu}\mu_1^*(q) \qquad (13)$$

where

$\mu_1^*(q) = E\left(x|_{\{x\le q\}}\right)$ defined in 10

2.7. Extreme Order Statistics

Let us consider a random sample $X_1 \ldots, X_n$ of size n from the 2SQLindely distribution with parameters α and θ. Let $X_{(1)} = \min(X_1 \ldots, X_n)$ be the sample minima and $X_{(n)} = \max(X_1 \ldots, X_n)$ be the sample maxima. Then, we have the following limit results

$$\lim_{t\to 0}\frac{F(xt)}{F(t)}$$

$$= \lim_{t\to 0}\frac{1-\frac{1}{6(\alpha+1)^2}\left[\begin{array}{l} 6(1+\alpha)^2(1+\theta xt) \\ +(3+6\alpha)\theta^2x^2t^2+\theta^3x^3t^3 \end{array}\right]e^{-\theta xt}}{1-\frac{1}{6(\alpha+1)^2}\left[\begin{array}{l} 6(1+\alpha)^2(1+\theta t) \\ +(3+6\alpha)\theta^2t^2+\theta^3t^3 \end{array}\right]e^{-\theta t}} \quad (14)$$

$$= x$$

In [18], they ensure the existence of a_n and b_n such that $\lim_{t\to\infty} P\left(a_n\left(X_{(1)} - b_n\right) \le x\right) = 1 - e^{-x}$.

We recognize the cdf of the exponential distribution with parameter 1, showing that can be $a_n\left(X_{(1)} - b_n\right)$ approximated by this distribution.

Moreover, we have

$$\lim_{t\to\infty}\frac{1-F(x+t)}{1-F(t)}$$

$$= \lim_{t\to\infty}\frac{\left[\begin{array}{l} 6(1+\alpha)^2(1+\theta(x+t)) \\ +(3+6\alpha)\theta^2(x+t)^2+\theta^3(x+t)^3 \end{array}\right]e^{-\theta x}}{\left[6(1+\alpha)^2(1+\theta t)+(3+6\alpha)\theta^2t^2+\theta^3t^3\right]} = e^{-\theta x}$$

To ensure the existence of a_n and b_n such that

$$\lim_{t\to\infty} P\left(a_n\left(X_{(n)} - b_n\right) \le x\right) = exp(-e^{-x})$$

We recognize the cdf of the Gumbel distribution with parameters 1 and $\frac{1}{\theta}$, showing that $a_n\left(X_{(n)} - b_n\right)$ can be approximated by this distribution.

2.8. Maximum Likelihood Estimation (MLE)

Assume $x_1, \dots, x_n$ be a random sample of size n from 2SQLindely(α, θ) then the likelihood function can be written as

$$L(\alpha,\theta)=\prod_{i=1}^{n}\left[\frac{\theta^2}{(\alpha+1)^2}x_i\left(\alpha^2+\theta\alpha x_i+\frac{\theta^2}{6}x_i^2\right)e^{-\theta x_i}\right]$$

(15)

By accumulation taking logarithm of equation (15), and the log- likelihood function can be written as

$$l(\alpha,\theta,\beta)=-2n\ln(1+\alpha)+2n\ln(\theta)-\theta\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\ln(x_i)+\sum_{i=1}^{n}\ln\left(\alpha^2+\theta\alpha x_i+\frac{\theta^2}{6}x_i^2\right) - (\alpha+1)\sum_{i=1}^{n}\ln(x_i) \quad (16)$$

Maximizing $l(\alpha, \theta)$with respect to $\alpha$and θ, we have the following system respectively with non-linear equations

$$\frac{dl}{d\alpha}=\frac{-2n}{(1+\alpha)}+\sum_{i=1}^{n}\frac{2\alpha+\theta x_i}{\alpha^2+\theta\alpha x_i+\frac{\theta^2}{6}x_i^2}=0 \quad (17)$$

$$\frac{dl}{d\theta}=\frac{2n}{\theta}-\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\frac{\alpha x_i+\frac{\theta}{3}x_i^2}{\alpha^2+\theta\alpha x_i+\frac{\theta^2}{6}x_i^2}=0 \quad (18)$$

The solution of this system has to be obtained numerically to get the estimates $\hat{\alpha}$and $\hat{\theta}$.

2.9. Simulation Study

In this section, we estimate the bias (Bias(.)) and the mean square error (MSE(.)) for the parameters (α, θ) = (1.7,1.5),(0.7,0.5) using simulation study. The population parameter is generated using software "Mathcad 14" package program. The sampling distributions are obtained for different sample sizes n = [20,40, 100, 300, 500] from N= 1000 replications. The simulations results are reported in Table 1 and Table 2.

Table 1. Simulation results for the 2SQLindely distribution at $(\boldsymbol{\alpha}, \boldsymbol{\theta})$ = (1.7,1.5)

n	Bais (α)	MSE(α)	Bais(θ)	MSE(θ)
20	0.0791	0.0732	0.0341	0.0574
40	0.0683	0.0515	0.0262	0.0427
100	0.0424	0.0387	0.0124	0.0363
300	0.0223	0.0173	0.0095	0.0292
500	0.0196	0.0158	0.0085	0.0276

Table 2. Simulation results for the 2SQLindely distribution at $(\boldsymbol{\alpha}, \boldsymbol{\theta})$ = (0.7,0.5)

n	Bais (α)	MSE(α)	Bais(θ)	MSE(θ)
20	0.0931	0.0811	0.0814	0.0909
40	0.0814	0.0796	0.0802	0.0751
100	0.0762	0.0653	0.0516	0.0718
300	0.0468	0.0569	0.0411	0.0647
500	0.0442	0.0549	0.0387	0.0635

2.10. Applications

Here, we use two data sets to illustrate the power of the proposed 2SQLindely distribution. We compare the proposed distribution with the 2SQLindely, QL, Lindley and exponential distributions.

The first real data set is data of survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by [19].

1, 33, 44, 56, 59, 72, 74, 77, 92, 93, 96, 100, 100, 102, 105,107, 107, 108, 108, 108, 109, 112, 113, 115, 116, 120, 121, 122, 122, 124,130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 195, 196, 197, 202, 213, 215, 216, 222, 230, 231, 240, 245, 251, 253,254, 254, 278, 293, 327, 342,347, 361, 402, 432, 458, 555.

The second real data set represents the waiting times (in minutes) before service of 100 Bank customers and examined and analyzed by [2].

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1,3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9 ,4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1,,6.2, 6.2, 6.2, 6.3, 6.7 ,6.9 ,7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6,9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4,12.5, 12.9, 13.0, 13.1, 13.3 , 13.6 , 13.7, 13.9, 14.1, 15.4, 15.4, 17.3,17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0,27.0, 31.6, 33.1, 38.5

For comparing the goodness of fit of the models, we found the unknown parameters by the maximum likelihood method, -2log likelihood ($-2l(.)$), AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), Corrected Akaike Information Criterion (CAIC) , Kolmogorov-Smirnov(K-S) and Cramérvon Mises W_n^2 statistic, given by

$$AIC = -2l(.) + 2k,$$

$$CAIC = AIC + \frac{2k(k+1)}{n-k-1},$$

$$BIC = -2l(.) + k\ln(n),$$

$$K-S = \max_{i}\left(\left[\frac{i}{n}-F(x_i)\right],\left[\frac{i-1}{n}-F(x_i)\right]\right)$$

and

$$W_n^2 = \frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{2i-1}{2n}-F(x_i)\right]^2$$

Table 3. MLEs and the measures $-2l(.)$, AIC, BIC, CAIA,K-S and W_n^2 for First data set

Distribution	Estimates	$-2l(.)$	AIC	$CAIC$	BIC	K-S	W_n^2
2SQLindely	$\hat{\alpha} = 99.455$ $\hat{\theta} = 0.011$	**871.587**	**875.587**	**875.761**	**880.141**	**0.109**	**0.29**
QLindley	$\hat{\alpha} = 46.762$ $\hat{\theta} = 0.0054$	899.876	903.876	904.05	908.429	0.232	1.319
Lindley	$\hat{\theta} = 0.013$	897.228	881.228	881.285	883.505	0.224	1.404
Exp	$\hat{\theta} = 0.0052$	899.897	901.955	901.955	904.174	0.246	1.32

Table 4. MLEs and the measures $-2l(.)$, AIC, BIC, CAIA,K-S and W_n^2 for second data set

Distribution	Estimates	$-2l(.)$	AIC	$CAIC$	BIC	K-S	W_n^2
2SQLindely	$\hat{\alpha} = 66.813$ $\hat{\theta} = 0.205$	**634.603**	**638.603**	**638.727**	**643.813**	**0.042**	**0.028**
QLindley	$\hat{\alpha} = 156.811$ $\hat{\theta} = 0.102$	658.04	662.04	662.164	667.25	0.173	0.715
Lindley	$\hat{\theta} = 0.187$	638.076	640.076	640.117	642.681	0.068	0.058
Exp	$\hat{\theta} = 0.101$	658.042	660.042	660.083	662.647	0.173	0.715

Table 3 and Table 4 summarize the results of the Fitted 2SQLindely, QL, Lindley and exponential distributions for the two considered data sets.

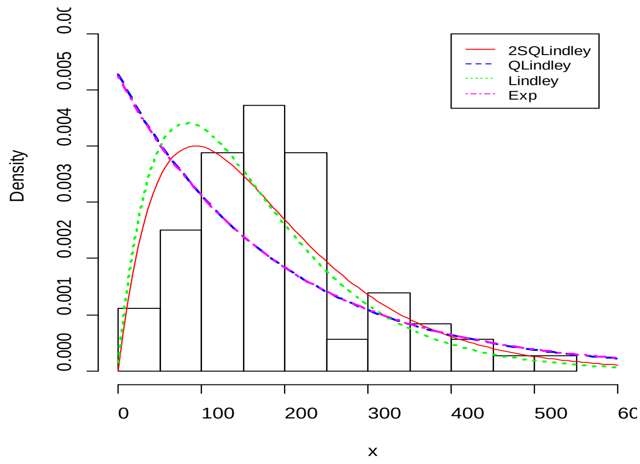

Figure 4. Estimated pdfs for the first data set

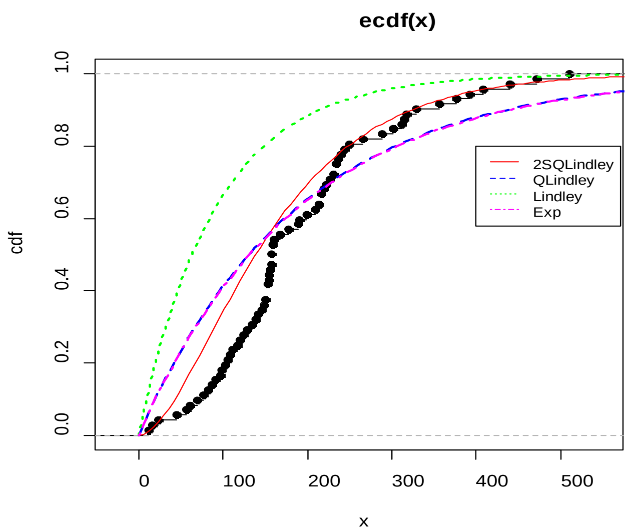

Figure 5. Estimated cdfs for the first data set.

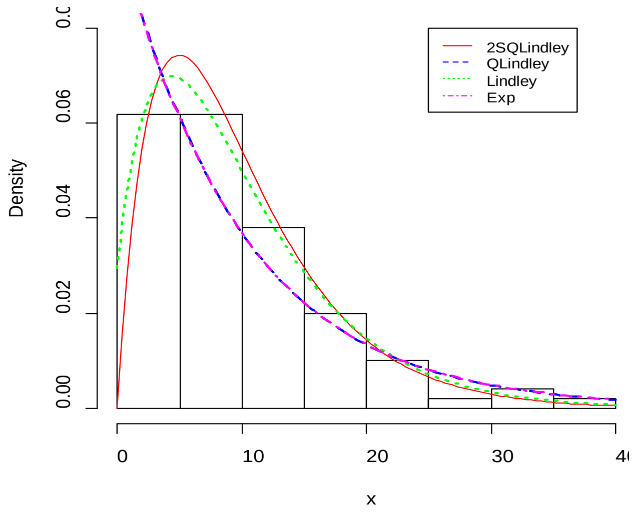

Figure 6. Estimated pdfs for the second data set

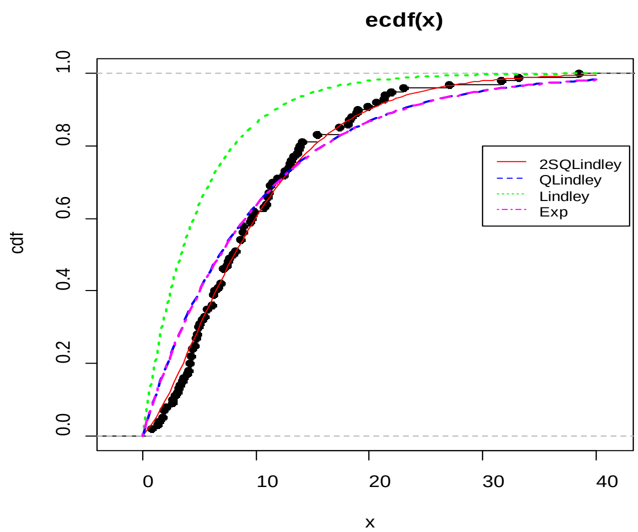

Figure 7. Estimated cdfs for the second data set.

From Table 3 and Table 4, it is obvious that the smallest $-2l(.)$, AIC, BIC, CAIA,K-S and W_n^2statistic are acquired for the 2SQLindely distribution. In summary, we can conclude that the 2SQLindely model can be adequate for modeling these data.

3. On the Difference of Two Independent Quasi Lindley Distribution

This section is devoted to the difference of two independent random variables following the QL distribution with pdf given by equation (2), 2DQLindely distribution.

3.1. Definition

Let Y_1 and Y_2 be two independent random variables following the QL distribution with parameters α and θ. Then, the random variable $X = Y_1 - Y_2$ has the pdf given by :

$$f(x) = \frac{\theta e^{-\theta|x|}}{4(1+\alpha)^2}\left(\theta(2\alpha+1)|x| + 2\alpha^2 + 2\alpha + 1\right), \quad (19)$$
$$x \in \mathbb{R}, \theta > 0, \alpha > -1$$

Proof: Since the support of Y_1 and Y_2 is $(0, +\infty)$ and the support of X is $\mathbb{R}$, let us notice that the cdf and pdf of $-Y_2$ are, respectively, given by

$$G_{**}(y,\alpha,\theta) = \frac{1+\alpha-\theta y}{\alpha+1} e^{\theta y}, y < 0, \theta > 0, \alpha > -1$$

$$g_{**}(y,\alpha,\theta) = \frac{\theta(\alpha-\theta y)}{\alpha+1} e^{\theta y}, y < 0, \theta > 0, \alpha > -1$$

Since Y_1 and - Y_2 are independent, the pdf of X is given by the convolution product

$$f(x) = (f_* f_{**})(x) = \int_{-\infty}^{\infty} f_*(x-t)\, f_{**}(t)\, dt$$
$$= \int_{-\infty}^{\inf(x,0)} \frac{\theta^2}{(1+\alpha)^2}\left[\alpha + \theta(x-t)\right] e^{-\theta(x-t)} (\alpha - \theta t) e^{\theta t} dt$$
$$= \frac{\theta^2}{(1+\alpha)^2} e^{-\theta x} \int_{-\infty}^{\inf(x,0)} \left[\alpha + \theta(x-t)\right](\alpha - \theta t)\, dt$$

When $x \geq 0$, we have $inf(x;\ 0) = 0$ implying that

$$f(x) = \frac{\theta}{4(1+\alpha)^2} e^{-\theta x}\left[\theta(2\alpha+1)x + 2\alpha^2 + 2\alpha + 1\right]$$

When $x < 0$, we have $inf(x;\ 0) = 0$ implying that

$$f(x) = \frac{\theta}{4(1+\alpha)^2} e^{\theta x}\left(-\theta(2\alpha+1)x + 2\alpha^2 + 2\alpha + 1\right)$$

By putting the above results together, we obtain the desired result.

Like the distribution of the sum of two independent variables, this distribution is considered entirely new as it has not been previously deduced and is considered a generalized distribution for the distribution of the difference between two independent variables for which the Lindley distribution provided by Chesneau, et al (2020). Whereas, through this research and in what part if $\alpha = \theta$ placed, we obtain the same results as that of Chesneau, et al (2020).

The cdf of the 2DQLindley distribution is given by

$$F(x) = \begin{cases} \frac{1}{4(1+\alpha)^2}\left[2(\alpha+1)^2 - \theta(2\alpha+1)x\right] e^{\theta x}, x < 0 \\ 1 - \frac{1}{4(1+\alpha)^2}\left[\theta(2\alpha+1)x + 2(1+\alpha)^2\right] e^{-\theta x}, x \geq 0 \end{cases} \quad (20)$$

Proof:$x < 0$ by using (19), we have

$$F(x) = P(t \leq x) = \int_{-\infty}^{x} f(t)\, dt$$
$$= \frac{\theta}{4(1+\alpha)^2}\left[\begin{array}{c} -\theta(2\alpha+1)\int_{-\infty}^{x} t e^{\theta t} dt \\ +\left(2\alpha^2 + 2\alpha + 1\right)\int_{-\infty}^{x} e^{\theta t} dt \end{array}\right]$$
$$= \frac{1}{4(1+\alpha)^2}\left[2(\alpha+1)^2 - \theta(2\alpha+1)x\right] e^{\theta x}$$

Since the distribution of X is symmetric around 0, for $x \geq\ 0$ we have

$$F(x) = 1 - F(-x)$$
$$= 1 - \frac{1}{4(1+\alpha)^2}\left[\theta(2\alpha+1)x + 2(1+\alpha)^2\right] e^{-\theta x}$$

We obtain the desired result by putting the above equalities together.

By using $F(x)$, the corresponding survival function is given by

$$S(x) = \begin{cases} 1 - \frac{1}{4(1+\alpha)^2}\left[\begin{array}{c} 2(\alpha+1)^2 \\ -\theta(2\alpha+1)x \end{array}\right] e^{\theta x}, x < 0 \\ \frac{1}{4(1+\alpha)^2}\left[\begin{array}{c} \theta(2\alpha+1)x \\ +2(1+\alpha)^2 \end{array}\right] e^{-\theta x}, x \geq 0 \end{cases} \quad (21)$$

The hazard rate function, hrf, is given by

$$h(x) = \begin{cases} \dfrac{\theta\left[\begin{array}{c} -\theta(2\alpha+1)x \\ +2\alpha^2 + 2\alpha + 1 \end{array}\right]}{\left[\theta(2\alpha+1)x + 2(1+\alpha)^2\right]}, x < 0 \\ \dfrac{\theta\left[\begin{array}{c} \theta(2\alpha+1)x \\ +2\alpha^2 + 2\alpha + 1 \end{array}\right]}{4(1+\alpha)^2 e^{\theta x} - \left[\begin{array}{c} \theta(2\alpha+1)x \\ +2(1+\alpha)^2 \end{array}\right]}, x \geq 0 \end{cases} \quad (22)$$

Also, the corresponding cumulative hazard rate function, hrf, is given by

$$\Omega(x) = -\ln\left(S(x)\right)$$

$$= \begin{cases} -\ln\left(1 - \dfrac{1}{4(1+\alpha)^2}\begin{bmatrix} 2(\alpha+1)^2 \\ -\theta(2\alpha+1)x \end{bmatrix} e^{\theta x}\right), x < 0 \\ \ln(4) + 2\ln(1+\alpha) + \theta x + \ln\begin{bmatrix} \theta(2\alpha+1)x \\ +2(1+\alpha)^2 \end{bmatrix}, x \geq 0 \end{cases} \quad (23)$$

The corresponding quantile function, say $Q(u)$, can be obtained by solving the following equation: $F(Q(u)) = Q(F(u)), u \in (0,1)$. It cannot be presented analytically but can be determined numerically for a given α and θ.

Figure 8, Figure 9 and Figure 10 show the plots of the pdf, cdf and hrf for various values of the parameters α and , θ respectively.

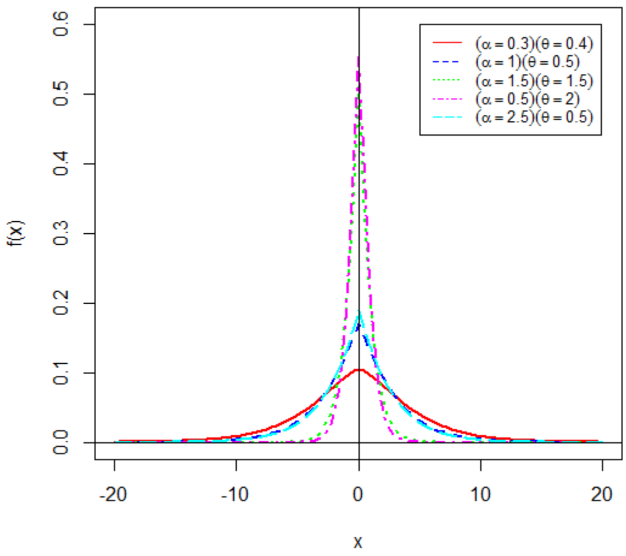

Figure 8. The pdf of the 2DQLindley distribution for some values of the parameters

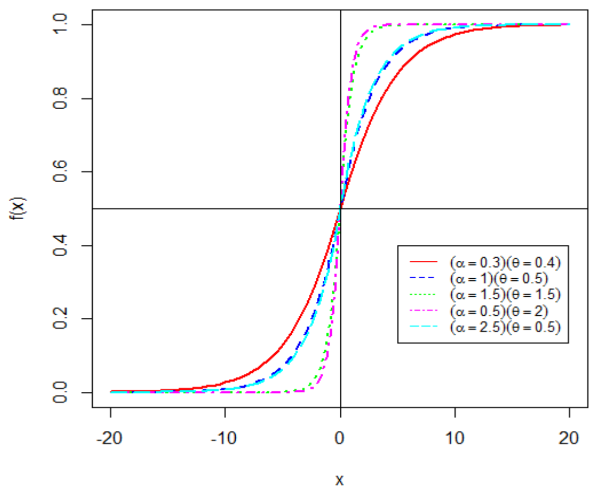

Figure 9. The cdf of the 2DQLindley distribution for some values of the parameters

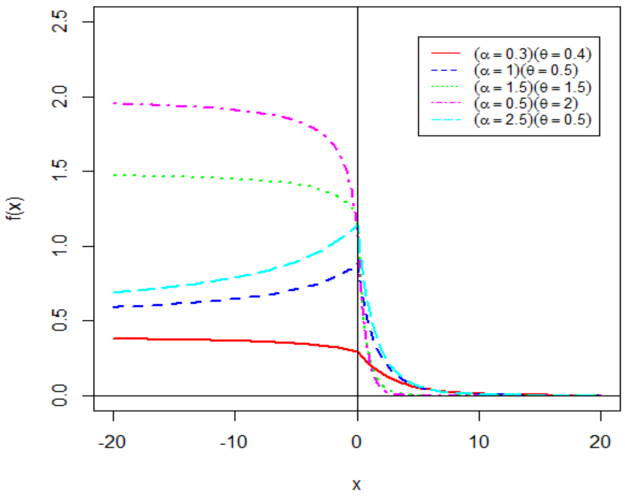

Figure 10. The hrf of the 2DQLindley distribution for some values of the parameters.

3.2. Mixture

Let $Z_1, Z_2 and\ Z_3$ be three random variables following the Laplace distribution with parameter θ and B a random variable following the Bernoulli distribution with parameter $\dfrac{\alpha^2}{(1+\alpha)^2}$, all these random variables are independent. Let X be a random variable following the 2DQLindley distribution with parameters α and θ. Then, we have the following stochastic representation:

$$X \overset{(d)}{=} Z_1 B + (1-B)(Z_2 + Z_3)$$

Proof: It is enough to remark that we can write $f(x)$ given by (19) as

$$f(x) = \frac{\alpha^2}{(1+\alpha)^2}\left[\frac{\theta}{2}e^{-\theta|x|}\right]$$
$$+\frac{2\alpha+1}{(1+\alpha)^2}\left[\frac{\theta}{4}(1+\theta|x|)e^{-\theta|x|}\right]$$
$$= pf_1(x) + (1-p)f_2(x)$$

where

$$p = \frac{\alpha^2}{(1+\alpha)^2},\ f_1(x) = \frac{\theta}{2}e^{-\theta|x|}$$
$$and\ \ f_2(x) = \frac{\theta}{4}(1+\theta|x|)e^{-\theta|x|}.$$

One can notice that $f_1(x)$ is the pdf of the Laplace distribution with parameter θ r and $f_2(x)$is the pdf of the sum of two independent random variables both following the Laplace distribution with parameter θ as common distribution, see, [Kotz, et al (2001) , Section 2.3].

3.3. Moments

The r[th] moment about origin of the 2DQLindley has been obtained as

$$\mu_r^* = \mu_r^* = \left[1+(-1)^r\right]\frac{(r+1)}{2\theta^r}\left[1+\frac{r(2\alpha+1)}{2(1+\alpha)^2}\right] \quad (24)$$

Proof: Since the distribution of X is symmetric around 0 and the integral is well defined, for any $s \in \mathbb{N}$, we have $\mu_{2S+1}^* = 0$. By the use of the gamma function, for any $s \in \mathbb{N}$, we have

$$\mu_{2S}^* = E\left(x^{2S}\right)$$

$$= \frac{\alpha^2}{(1+\alpha)^2}\int_{-\infty}^{\infty} x^{2S}\left[\frac{\theta}{2}e^{-\theta|x|}\right]dx$$

$$+\frac{2\alpha+1}{(1+\alpha)^2}\int_{-\infty}^{\infty} x^{2S}\frac{\theta}{4}\left(1+\theta|x|\right)e^{-\theta|x|}dx$$

$$= \frac{\alpha^2}{(1+\alpha)^2}\frac{(2s+1)}{\theta^{2S}} + \frac{2\alpha+1}{(1+\alpha)^2}\left[\frac{(2s+1)}{2\theta^{2S}}+\frac{(2s+2)}{2\theta^{2S}}\right]$$

$$= \frac{(2s+1)}{\theta^{2S}}\left[1+\frac{S(2\alpha+1)}{(1+\alpha)^2}\right].$$

By distinguishing the odd and even integer, we can get the proof.

Also, owing to the r[th] moment, we have

$$\mu_1^* = 0,\ \mu_2^* = \frac{2\left(\alpha^2+4\alpha+2\right)}{\theta^2(1+\alpha)^2},$$

$$\mu_3^* = 0 \text{ and } \mu_4^* = \frac{24\left(\alpha^2+6\alpha+3\right)}{\theta^4(1+\alpha)^2}$$

In particular, the mean of X is given by $\mu = \mu_1^* = 0$ and the variance of X is given by

$$V(x) = \sigma^2 = \mu_2^* = \frac{2\left(\alpha^2+4\alpha+2\right)}{\theta^2(1+\alpha)^2}$$

We can see that the variance of the 2SQLindley and 2DQLindley distributions are the same

The skewness of X is equal to 0 and the kurtosis of X is given by

$$\beta_2 = \frac{1}{\sigma^4}E(X-\mu)^4 = \frac{6\left(\alpha^2+6\alpha+3\right)^2}{\left(\alpha^2+4\alpha+2\right)^2}$$

3.4. Moment Generating Function

The moment generating function $M_x(t)$ of a random variable X $\sim 2DQLindley(\alpha,\theta)$ is given by

$$M_x(t) = \frac{\theta^2\left[\theta^2(\alpha+1)^2-\alpha^2t^2\right]}{(1+\alpha)^2\left(\theta^2-t^2\right)^2} \quad (25)$$

Proof. Let us recall that, for any $t \in \mathbb{R}$ and a random variable X following the QL distribution with parameters α and θ .the moment generating function of X is given by

$$M_{y_i}(t) = \frac{\theta(\alpha\theta-\alpha t+\theta)}{(1+\alpha)(\theta-t)^2}.$$

Hence, using the representation $X = Y_1 - Y_2$ with Y_1 and $(-Y_2)$ independent and identically distributed, the characteristic function for X is given by

$$M_x(t) = M_{y_1}(t)M_{(-y_2)}(-t)$$

$$= \frac{\theta(\alpha\theta-\alpha t+\theta)}{(1+\alpha)(\theta-t)^2}\times\frac{\theta(\alpha\theta-\alpha t+\theta)}{(1+\alpha)(\theta+t)^2}$$

$$= \frac{\theta^2\left[\theta^2(\alpha+1)^2-\alpha^2t^2\right]}{(1+\alpha)^2\left(\theta^2-t^2\right)^2}$$

We can also write themoment generating function as follow,

$$M_x(t) = \frac{\alpha^2}{(1+\alpha)^2}\left[\frac{\theta^2}{\theta^2-t^2}\right]+\frac{2\alpha+1}{(1+\alpha)^2}\left[\frac{\theta^2}{\theta^2-t^2}\right]^2$$

which is exactly the moment generating function of $Z_1B + (1-B)(Z_2+Z_3)$, Which reflect the targeted result.

3.5. Characteristic Function

Suppose that the random variable X have the X $\sim 2D$QLindley(α, θ) then characteristic function, $\Phi_x(t)$, is

$$\Phi_x(t) = \frac{\theta^2\left[\theta^2(\alpha+1)^2+\alpha^2t^2\right]}{(1+\alpha)^2\left(\theta^2+t^2\right)^2} \quad (26)$$

Where i $= \sqrt{-1}$.

Proof:

The proof is simple.

3.6. Maximum Likelihood Estimation (MLE)

Assume $x_1, \ldots, x_n$ be a random sample of size n from 2DQLindley(α, θ) then the likelihood function can be written as

$$L(\alpha,\theta) = \prod_{i=1}^{n}\left[\frac{\theta}{4(1+\alpha)^2}\left[\begin{matrix}\theta(2\alpha+1)|x_i|\\+2\alpha^2+2\alpha+1\end{matrix}\right]e^{-\theta|x_i|}\right] \quad (27)$$

By accumulation taking logarithm of equation (27), and the log- likelihood function can be written as

$$l(\theta,\alpha) = n\ln\theta - n\ln 4 - 2n\ln(1+\alpha) - \theta\sum_{i=1}^{n}|x_i| + \sum_{i=1}^{n}\ln\left[\theta(2\alpha+1)|x_i|+2\alpha^2+2\alpha+1\right] \quad (28)$$

Maximizing $l(\alpha,\theta)$with respect to α and θ, we have the following system with non-linear equations

$$\frac{dl}{d\alpha}=\frac{n}{\theta}-\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\left[\frac{(2\alpha+1)|x_i|}{\theta(2\alpha+1)|x_i|+2\alpha^2+2\alpha+1}\right]=0$$
(29)

$$\frac{dl}{d\theta}=\frac{-2n}{(1+\alpha)}+\sum_{i=1}^{n}\left[\frac{4\alpha+2}{\theta(2\alpha+1)|x_i|+2\alpha^2+2\alpha+1}\right]=0 \quad (30)$$

This equation cannot be solved analytically. However, some numerical algorithm allows to approach the solution in a precise way.

3.7. Simulation Study

Table 5. Simulation results for the 2DQLindley distribution at (α,θ) = (1.3,1.7)

n	Bais (α)	MSE(α)	Bais(θ)	MSE(θ)
20	0.1237	0.1734	0.0994	0.1654
40	0.0978	0.1463	0.0971	0.1327
100	0.0813	0.0916	0.0823	0.0997
300	0.0778	0.0897	0.0678	0.0791
500	0.0751	0.0874	0.0654	0.0783

Table 6. Simulation results for the 2DQLindley distribution at (α,θ) = (0.4,0.9)

n	Bais (α)	MSE(α)	Bais(θ)	MSE(θ)
20	0.0998	0.1549	0.0934	0.1449
40	0.0962	0.1332	0.0887	0.1167
100	0.0842	0.1009	0.0841	0.0911
300	0.0766	0.0894	0.0699	0.0877
500	0.0691	0.0799	0.0687	0.0832

In this section, we estimate the bias (Bias(.)) and the mean square error (MSE(.)) for the parameters (α,θ) = (1.3,1.7),(0.4,0.9) using simulation study. The population parameter is generated using software "Mathcad 14" package program. The sampling distributions are obtained for different sample sizes n = [20,40, 100, 300, 500] from N= 1000 replications. The simulations results are reported in Table 5, Table 6.

3.8. Applications

In this section, we analyze two data sets in order to illustrate the good performance of the 2DQLindley distribution to compare with the Laplace and 2SLaplace distributions, both with parameters standardly denoted by μ and θ. Here, we consider an extended form of the 2DQLindley distribution by adding the location parameter μ in the pdf of the 2DQLindley distribution. Thus, the related pdf is given by

$$f(x)=\frac{\theta e^{-\theta|x-\mu|}}{4(1+\alpha)^2}\left[\theta(2\alpha+1)|x-\mu|+2\alpha^2+2\alpha+1\right]$$
$$x,\mu\in\mathbb{R},\theta>0,\alpha>-1.$$

The first data sets correspond to the tensile strength of craft paper, reported in [20]. The data set are given below:

6.3 11.1 20.0 24.0 26.1 30.0 33.8 34.0 38.1 39.9 42.0 46.1 53.1 52.0 52.5 48.0 42.8 27.8 21.9

The second data set representing lung cancer rates data for 44 US states is given by www.calvin.edu/stob/data/cigs.csv. The data are given below:

17.05 19.8, 15.98 22.07 22.83 24.55 27.27 23.57 13.58 22.8 20.3 16.59 16.84 17.71 25.45 20.94,26.48 22.04 22.72 14.2 15.6 20.98 19.5 16.7 23.03 25.95 14.59 25.02 12.12 21.89,19.45 12.11,23.68 17.45 14.11 17.6 20.74 12.01 21.22 20.34 20.55 15.53 15.92 25.88

Table 7 and Table 8 summarize the results of the Fitted 2DQLindley, Laplace and 2SLaplace distributions for the two considered data sets.

Table 7. MLEs and the measures $-2l(.)$, AIC, BIC, CAIA,K-S and W_n^2 for First data

Distribution	Estimates	$-2l(.)$	AIC	$CAIC$	BIC	K-S	W_n^2
2DQLindley	$\hat{\alpha}=0.961$ $\hat{\theta}=0.126$ $\hat{\mu}=34.00$	**155.049**	**161.049**	**162.649**	**163.882**	**0.323**	**1.028**
Laplace	$\hat{\theta}=0.015$ $\hat{\mu}=34.00$	172.105	176.105	176.855	177.993	0.996	6.333
2SLaplace	$\hat{\theta}=0.136$ $\hat{\mu}=34.805$	353.151	357.151	357.901	359.04	0.389	1.059

Table 8. MLEs and the measures $-2l(.)$, AIC, BIC, CAIA,K-S and W_n^2 for second data

Distribution	Estimates	$-2l(.)$	AIC	$CAIC$	BIC	K-S	W_n^2
2DQLindley	$\hat{\alpha}=1.345$ $\hat{\theta}=0.386$ $\hat{\mu}=20.3$	**257.793**	**263.793**	**264.293**	**269.146**	**0.268**	**0.632**
Laplace	$\hat{\theta}=0.025$ $\hat{\mu}=20.32$	351.565	355.565	355.858	359.134	0.998	14.665
2SLaplace	$\hat{\theta}=0.433$ $\hat{\mu}=19.862$	307.923	311.923	312.215	315.491	0.279	0.692

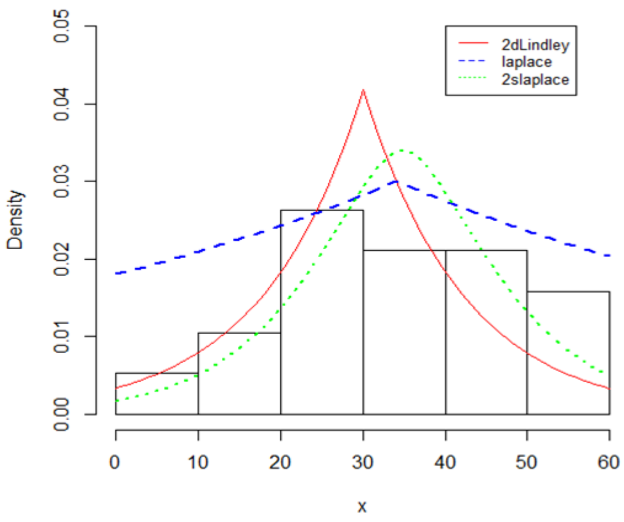

Figure 11. Estimated pdfs for the first data set

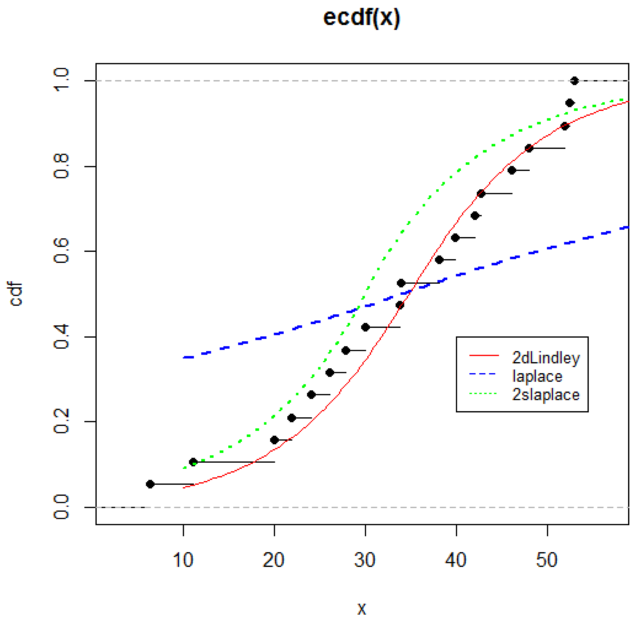

Figure 12. Estimated cdfs for the first data set

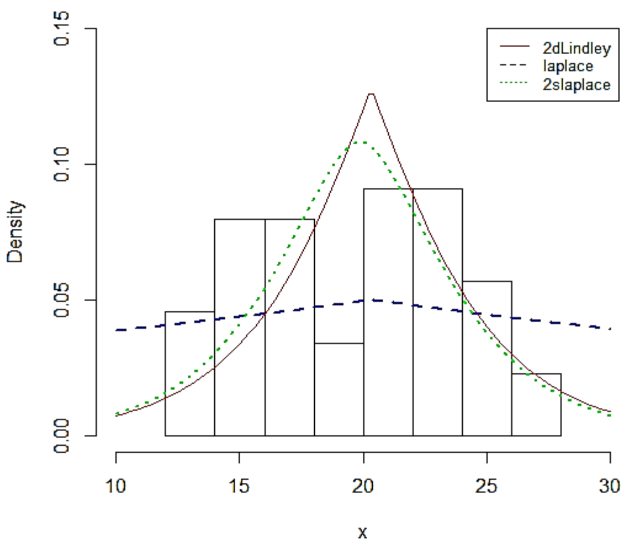

Figure 13. Estimated pdfs for the second data set

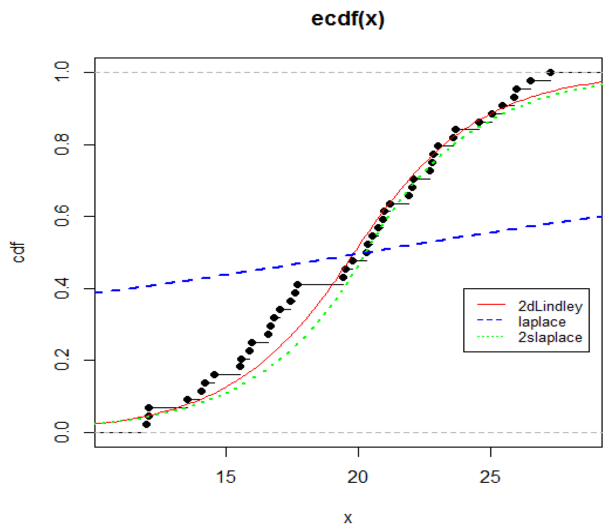

Figure 14. Estimated cdfs for the second data set

Figure 11, Figure 12, Figure 13 and Figure 14 shown that the smallest $-2l(.)$, AIC, BIC, CAIA,K-S and W_n^2statistic are acquired for the 2DQLindley distribution. In summary, we can conclude that the 2DQLindley model can be adequate for modeling these data.

4. Concluding Remarks

In this paper, we have derived single representations for the exact distribution of the sum and difference of independent QL random variables. We referred to the distributions of sum and difference of two independent QL random variables as the 2SQLindely and 2DQLindley distributions, respectively. Statistical properties such as moments, incomplete moments, characteristic function and extreme order statistics of the 2SQLindely distribution have been provided. At the same time, a comprehensive study of statistical properties of the 2DQLindley distribution also has been discussed. The model parameters are estimated by maximum likelihood method for both cases. From simulation studies, the performance of the maximum likelihood estimators has been assessed. The new models provide consistently better fit than some classical models available in the literature. In conclusion, proposed model with their attracting properties should have a promising future in distribution theory.

Finally, we can say this paper is a generalization of the results were obtained in Chesneau, et al (2020) when $\alpha = \theta$.

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