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A New Gumbel Generated Family of Distributions: Properties, Bivariate Distribution and Application

Elebe E. Nwezza[1,*], Chinonyerem V. Ogbuehi[2], Uchenna U. Uwadi[1], C.O. Omekara[2]

[1]Department of Mathematics/Computer Science/Statistics/Informatics, Alex Ekwueme Federal University Ndufu alike, Ikwo, Nigeria
[2]Department of Statistics, Micheal Okpara University of Agriculture, Umudike, Nigeria
*Corresponding author: emma_nwezza@yahoo.com

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Abstract In this paper, we propose a new class of Gumbel generated distributions called Gumbel-Marshall-Olkin family of distributions. The new family of distributions is represented as linear mixture of exponentiated-G distribution. Some of the sub-models are presented. We derived some characterizations such as the quantile, moments, moment generating function, entropy and order statistics of the proposed family of distributions. The estimation of the unknown parameters of the new class of distribution is through the maximum likelihood. The consistency of the MLEs of the sub-model is assessed by means of simulation. Furthermore, we derive the bivariate density function of the new class of distributions. Two real life data sets are used to illustrate the potential usefulness of the sub-models of the proposed class of distributions. The results of the applications clearly indicate that the sub-models of the proposed class of distribution provided better fit among the other competing models.

Keywords: *Gumbel distribution, Marshall-Olkin distribution, Bivariate distribution, Moment, Maximum Likelihood*

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1. Introduction

In recent years, there have been growing interests in developing families of statistical distributions by extending already existing distributions through the addition of one or more parameters. The primary focus is to generate more flexible distributions that will provide better fits to many practical situations where ordinarily the classical distributions would not have provided adequate fits. [1] proposed a method of generating new family of distributions for any baseline distribution with cumulative density function (cdf) $F(x;\xi)$ and define the corresponding cdf $G(x;p,\xi)$ as

$$G(x;p,\xi)=\frac{F(x;\xi)}{1-\overline{p}\overline{F}(x;\xi)};0<p<\infty \tag{1}$$

where $\overline{p}=1-p$ and $\overline{F}(x;\xi)=1-F(x;\xi)$ is the survival function of the baseline distribution with vector of parameters ξ. For $p=1$, $G(x;p,\xi)=F(x;\xi)$.

In the literatures, there are many other families of distributions such are exponentaited-G by [2], beta–G by [3], transmuted-G by [4], gamma-G [5], Kumaraswamy-G by Cordeiro and de Castro (2011) [6], McDonald-G by [7].

Furthermore, [8] proposed a method of generating families of continuous distribution called the transformed-transformer (T-X) with the cdf of a class of continuous distributions for any given baseline distribution $F(x;\xi)$ defined as

$$G(x;\theta)=\int_a^{W(F(x;\xi))} r(t)\,dt = R\left(W\left(F(x;\xi)\right)\right)$$

and the corresponding probability density function (pdf) is given by

$$g(x;\theta)=r\left(W\left(F(x;\xi)\right)\right)\left(\frac{\partial}{\partial x}W\left(F(x;\xi)\right)\right)$$

where $W(F(x;\xi))$ satisfies the following conditions: $W(F(x;\xi))\in[a,b]$, $W(F(x;\xi))$ is differentiable and monotonically non-decreasing, $W(F(x;\xi))\to a$ as $x\to-\infty$ and $W(F(x;\xi))\to b$ as $x\to\infty$.

[9] proposed Gumbel-G family of distributions following [8]. However, in this paper, we propose a new wider class of continuous distributions which generalizes the Gumbel-G family of distributions by taking

$$W\left(F(x;\xi)\right)=\log\left(\left(\left(G_{MO}(x;\xi)\right)^{-1}-1\right)^{-1}\right)$$

and

$$r(t)=\frac{1}{\sigma}\exp\left(-\left(\frac{t-\mu}{\sigma}\right)\right)\exp\left(-\exp\left(-\left(\frac{t-\mu}{\sigma}\right)\right)\right)$$

Here, $G_{MO}(x;\xi)$ is the cdf of eq. (1).

The new wider class of continuous distributions is called Gumbel Marshall Olkin family of distributions (GMO-G) having cdf and pdf are respectively given by

$$G(x;\theta)=\exp\left(-Bp^{1/\sigma}\left(\frac{F(x;\xi)}{\overline{F}(x;\xi)}\right)\right) \quad (2)$$

and

$$g(x;\theta)=\frac{Bp^{1/\sigma}f(x;\xi)}{\sigma F(x;\xi)\overline{F}(x;\xi)}\left(\frac{F(x;\xi)}{\overline{F}(x;\xi)}\right)^{-\frac{1}{\sigma}} \times\exp\left(-Bp^{1/\sigma}\left(\frac{F(x;\xi)}{\overline{F}(x;\xi)}\right)^{-\frac{1}{\sigma}}\right) \quad (3)$$

where $B=\exp\left(\frac{\mu}{\sigma}\right)$, $\theta=(p,\mu,\sigma,\xi)$ is the vector of parameters of the GMO-G. For $p=1$ in eqs. (2) and (3), GMO-G reduces to the cdf and pdf of Gumbel-G of [9], which is a special case of the newly proposed family of distributions..

The hazard rate function (hrf) and survival function (s.f) of GMO-G are respectively given by

$$hrf=\frac{Bp^{1/\sigma}f(x)}{\sigma F(x)(1-F(x))}\left(\frac{F(x)}{\overline{F}(x)}\right)^{-\frac{1}{\sigma}} \times\left(\exp\left(Bp^{1/\sigma}\left(\frac{F(x)}{\overline{F}(x)}\right)^{-\frac{1}{\sigma}}\right)-1\right)^{-1}$$

and

$$S.f=1-\exp\left(-Bp^{1/\sigma}\left(\frac{F(x;\xi)}{\overline{F}(x;\xi)}\right)\right).$$

The main motivations behind this paper are to generate more flexible distributions having bimodal, bathtub, symmetric, monotone increasing, increasing-decreasing-increasing, J and reverse-J shapes; hazard rates of constant, J, reverse-J, monotone increasing, increasing-decreasing-increasing shapes as shown in Figure 2, Figure 3, Figure 4 and Figure 5; construct heavy-tailed distributions which are not longer-tailed for modeling real-life data as shown in figure1; and generate models that will provide better fit even when compared with models having the same baseline distribution.

The remaining part of the paper is organized as follows. The linear representation of the distribution and density functions of the new class of distributions are presented in section 2. In section 3, we presented some special models of the proposed new family of distributions. The quantile function is presented in section 4. In section 5, the shapes of density and hazard rate functions are discussed. We derived the moments including the ordinary, incomplete and generating function in section 6. The Entropy and the distribution of the order statistics are presented in sections 7 and 8. In section 9, we presented the method of estimation of the unknown parameters of the new family of distributions. Simulation studies on the consistency of the MLEs are presented in section 10. Bivariate extension of the proposed family of distribution is presented in section 11. Finally in section 12, we provided the concluding remarks.

2. Linear Representation

Considering some useful series expansion

$$\exp(-z)=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}z^i$$

and

$$(1-z)^n=\sum_{i=0}^{\infty}(-1)^i\binom{n}{i}z^i.$$

Using the series expansion above, equation (2) can re-written as mixture of exponentiated-G (exp-G) cumulative function given by

$$G(x;\theta)=\sum_{m=0}^{\infty}t_m H(x)^m \quad (4)$$

where

$$t_m=(-1)^m\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}}{i!}B^i p^{i/\sigma}\binom{\frac{i}{\sigma}}{j}\binom{j-\frac{i}{\sigma}}{k}\binom{k}{m}.$$

Here $H_\gamma(x)=F(x)^\gamma$ is the cdf of Exp-G of power parameter γ . See [2,10,11,12].

Some mathematical properties of Gumbel Marshall Olkin –G family of distributions can be derived from those properties of the EXP-G distribution. See [11,12,13].

Differentiating eq. (4), eq. (3) can be re-written as

$$g(x)=\sum_{m=0}^{\infty}(m+1)t_m h_{m+1}(x). \quad (5)$$

Here, $h_\gamma(x)=\gamma f(x)F(x)^{\gamma-1}$ is the density function of Exp-G with power parameter γ.

3. Special Models

In this section, we consider two of the special models of GMO-G. However, equation (3) will be most tractable when the cdf and pdf of the baseline distribution have simply analytic expression.

3.1. GMO-Normal (GMO-N) Distribution

Suppose $\Phi(z)$ and $\phi(z)$ denotes the cdf and pdf of standard normal distribution where $z=\frac{x-\mu_1}{\sigma_1}$; $x,\mu_1\in\mathbb{R}$, $\sigma_1>0$, then the pdf of GMO-N distribution is given by

$$g(x;\theta)=\frac{Bp^{1/\sigma}\phi(z)}{\sigma\Phi(z)(1-\Phi(z))}\left(\frac{\Phi(z)}{(1-\Phi(z))}\right)^{-\frac{1}{\sigma}}\times\exp\left(-Bp^{1/\sigma}\left(\frac{\Phi(z)}{(1-\Phi(z))}\right)^{-\frac{1}{\sigma}}\right).$$

Here, $\theta=(p,\mu,\sigma,\mu_1,\sigma_1)$ is the parameter vector of GMO-N distribution. Figure 2 shows some possible shapes which are not limited to bathtub, symmetric, J-shape and monotone increasing shapes for some selected parameter values. Figure 3 also shows some possible shapes of the hazard rate function for some selected parameter values. These shapes indicate the flexibility of GMO-N and its potential to model real-life data.

3.2. GMO-Weibull (GMO-W) Distribution

A random variable with

$$\text{cdf}F(x)=1-\exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)$$

and

$$\text{pdf}f(x)=\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right);$$

where $\alpha>0$ and $\beta>0$ are the shape and scale parameters is said to follow Weibull distribution. By substituting $F(x)$ and $f(x)$ in equation (3), the pdf of GMO-W is defined as

$$g(x;\theta)=\frac{Bp^{1/\sigma}\alpha x^{\alpha-1}\exp\left(\left(\frac{x}{\beta}\right)^{\alpha}\right)}{\sigma\beta^{\alpha}\left(\exp\left(\left(\frac{x}{\beta}\right)^{\alpha}\right)-1\right)^{\left(\frac{1}{\sigma}+1\right)}}\times\exp\left(-Bp^{1/\sigma}\left(\exp\left(\left(\frac{x}{\beta}\right)^{\alpha}\right)-1\right)^{-\frac{1}{\sigma}}\right).$$

The plots of the pdf and hrf of GMO-W distribution for selected parameter values are shown in Figure 4 and Figure 5 respectively. Shapes such as bimodal, j and reverse-J, symmetric and left-skewed are for the density function while constant, S-shape, monotone increasing, increasing-decreasing-increasing shapes are for hazard rate function.

4. Quantile Function

The quantile function (qf) of GMO-G family is obtained by inverting equation (1.2) and it is given by

$$Q(u)=F^{-1}(u)=Q_G\left(\left(1-B^{-\sigma}p^{-1}(\ln(u))^{\sigma}\right)^{-1}\right). \quad (6)$$

Here, $u\in(0,1)$ and $Q_G(u)$ is the qf of the baseline distribution.

The effect of the parameters GMO-G on skewness and kurtosis are determined by its quantile measures using the Bowley's skewness and Moor's kurtosis measures. These measures are respectively given by

$$skewness=\frac{Q(1/4)+Q(3/4)-2Q(1/2)}{Q(3/4)-Q(1/4)}$$

and

$$kurtosis=\frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)}.$$

However, these measures are less sensitive to outliers and they do exist for distribution with moments. We considered the effect of parameters p and σ_1 on the skewness and kurtosis of GMO-Normal. The plots are presented in Figure 1. Both measures equal zero for the normal distribution.

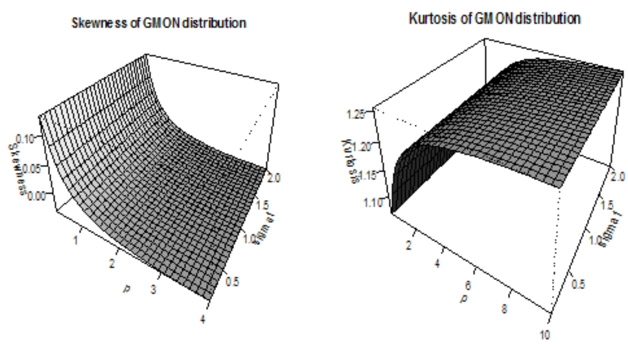

Figure 1. Plot of Skewness and Kurtosis of GMO-Normal distribution

Proposition 1: If Y follows Gumbel distribution, then for any baseline distribution with cdf $F(x)$ the qf can be expressed as

$$Q(u)=F^{-1}\left(V^{-1}\left(\left(1+\exp\left(-R^{-1}(u)\right)\right)^{-1}\right)\right) \quad (7)$$

where $V^{-1}(.)$ is the qf of Marshall Olkin and $R^{-1}(u)$ the qf of the transformed distribution.

PROOF

$V^{-1}(.)$ and $R^{-1}(u)$ are obtained by inverting (1) and cdf of $r(u)$ and are respectively given by

$$V^{-1}(t)=V^{-1}\left(\frac{t-t\overline{p}}{1-t\overline{p}}\right)$$

and

$$R^{-1}(u)=R^{-1}\left(-\ln\left(\left(\ln\left(u^{-\frac{1}{B}}\right)\right)^{\sigma}\right)\right).$$

Substituting $V^{-1}(.)$ and $R^{-1}(u)$ in (7) and simplifying, (6) is obtained.

5. Shape of GMO-G pdf and Hazard Rate Function

The shapes of the density and hazard rate function of GMO-G can be described analytically. The critical points of the GMO-G density function are the roots of the equation below

$$\frac{f'(x)}{f(x)}-\frac{f(x)}{\sigma pF(x)\overline{F}(x)}+\frac{Bp^{1/\sigma-1}f(x)}{\sigma F(x)\overline{F}(x)}\left(\frac{F(x)}{\overline{F}(x)}\right)^{-\frac{1}{\sigma}} -\frac{f(x)}{F(x)}+\frac{f(x)}{\overline{F}(x)}=0 \quad (8)$$

By equation (8), there may be more than one root. Suppose $x=x_0$ is a root of (8), then $\tau(x_0)<0$, $\tau(x_0)>0$, and $\tau(x_0)=0$ correspond to the local maximum, local minimum, and point of reflection respectively. Here $\tau(x)=\frac{\partial^2\log g(x)}{\partial x^2}$ is given by

$$\tau(x)=\frac{f(x)f''(x)-f'(x)^2}{f(x)^2}-\frac{F(x)\overline{F}(x)f'(x)}{\sigma pF(x)^2\overline{F}(x)^2}$$
$$-\frac{f(x)^2(1-2F(x))}{\sigma pF(x)^2\overline{F}(x)^2}-\frac{Bp^{1/\sigma-1}}{\sigma F(x)^2\overline{F}(x)^2}\left(\frac{F(x)}{\overline{F}(x)}\right)^{-\frac{1}{\sigma}}$$
$$\times\left(\frac{f(x)^2}{\sigma}-F(x)\overline{F}(x)f'(x)+f(x)^2(1-2F(x))\right)$$
$$-\frac{F(x)f'(x)-f(x)^2}{F(x)^2}+\frac{\overline{F}(x)f'(x)+f(x)^2}{\overline{F}(x)^2}.$$

Figure 2 and Figure 4 show the graphical pdf plots of two GMO-G sub-models (GMO-N and GMO-W) for selected parameter values indicating different shapes.

Furthermore, the shape of the GMO-G hazard rate function can be described analytically. The critical points of the GMO-G hazard rate function are the roots of the eq. (9).

$$\frac{f'(x)}{f(x)}-\frac{f(x)}{\sigma F(x)\overline{F}(x)}$$
$$+\frac{Bp^{1/\sigma}f(x)}{\sigma F(x)\overline{F}(x)\left(1-\exp\left(-Bp^{1/\sigma}\left(\frac{F(x)}{\overline{F}(x)}\right)^{-\frac{1}{\sigma}}\right)\right)} \quad (9)$$
$$\times\left(\frac{F(x)}{\overline{F}(x)}\right)^{-\frac{1}{\sigma}}-\frac{f(x)}{F(x)}+\frac{f(x)}{\overline{F}(x)}=0.$$

There may be more than one root of equation (9). Thus, if $x=x_0$ is a root of (9), then $\tau(x_0)<0$, $\Psi(x_0)>0$, and $\Psi(x_0)=0$ correspond to the local maximum, local minimum, and point of reflection respectively. Here $\Psi(x)=\frac{\partial^2\log g(x)}{\partial x^2}$ is given by

$$\Psi(x)=\frac{f(x)f''(x)-f'(x)^2}{f(x)^2}$$
$$-\frac{F(x)\overline{F}(x)f'(x)-f(x)^2}{pF(x)^2\overline{F}(x)^2}$$
$$-\frac{F(x)f'(x)-f(x)^2}{F(x)^2}+\frac{\overline{F}(x)f'(x)+f(x)^2}{\overline{F}(x)^2}$$
$$-\frac{Bp^{1/\sigma}}{\sigma F(x)^2\overline{F}(x)}\left(\frac{F(x)}{\overline{F}(x)}\right)^{-\frac{1}{\sigma}}$$
$$\times\left[\frac{f(x)^2}{\sigma\left(1-\exp\left(-Bp^{1/\sigma}\left(\frac{F(x)}{\overline{F}(x)}\right)^{-\frac{1}{\sigma}}\right)\right)}\right.$$
$$-\frac{F(x)\overline{F}(x)f'(x)-f(x)^2(1-2F(x))}{\left(1-\exp\left(-Bp^{1/\sigma}\left(\frac{F(x)}{\overline{F}(x)}\right)^{-\frac{1}{\sigma}}\right)\right)}$$
$$\left.+\frac{Bp^{1/\sigma}f(x)^2\exp\left(-Bp^{1/\sigma}\left(\frac{F(x)}{\overline{F}(x)}\right)^{-\frac{1}{\sigma}}\right)}{\sigma\left(1-\exp\left(-Bp^{1/\sigma}\left(\frac{F(x)}{\overline{F}(x)}\right)^{-\frac{1}{\sigma}}\right)\right)^2}\left(\frac{F(x)}{\overline{F}(x)}\right)^{-\frac{1}{\sigma}}\right].$$

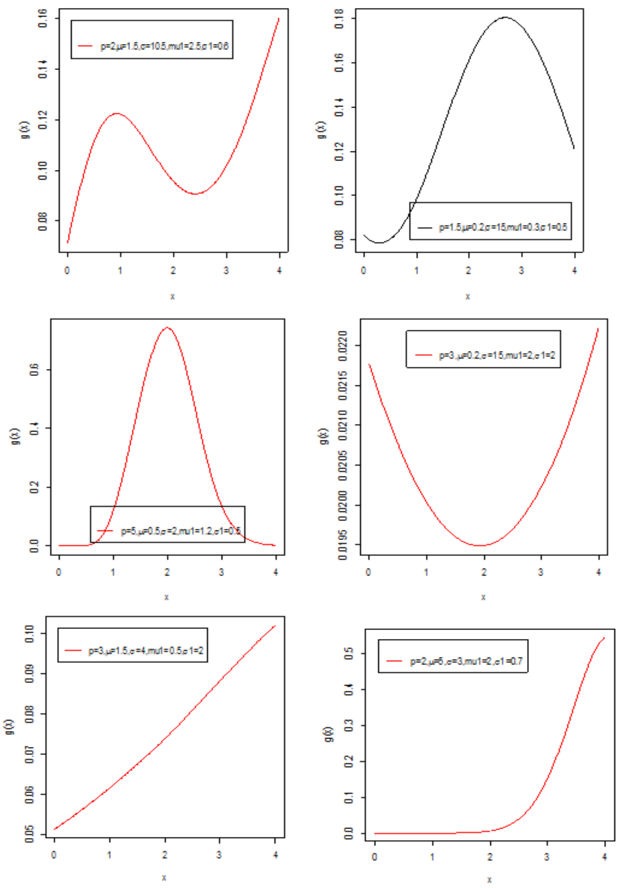

Figure 2. Plots of pdf of GMO-Normal for some selected parameter values

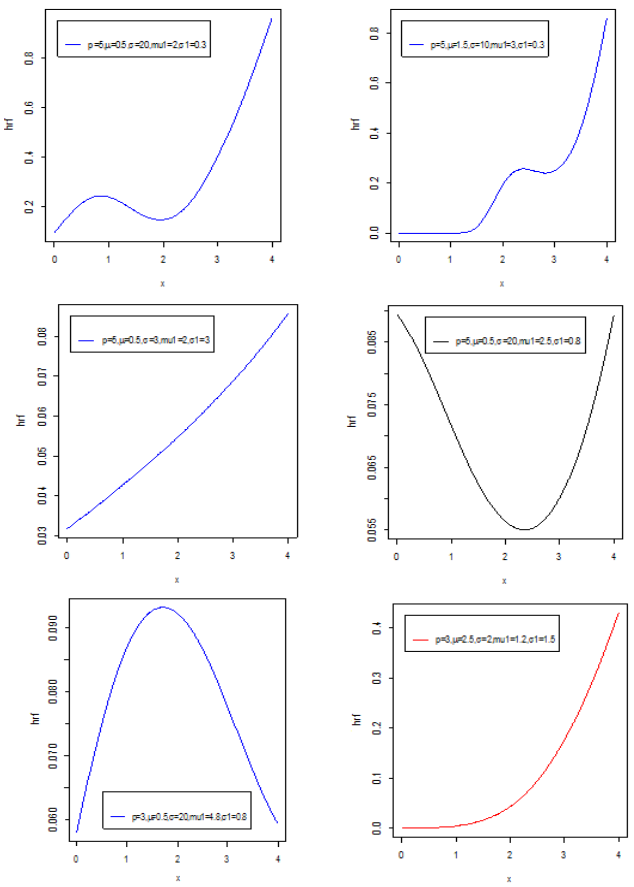

Figure 3. Plots of hazard rate function of GMO-Normal for some selected parameter values

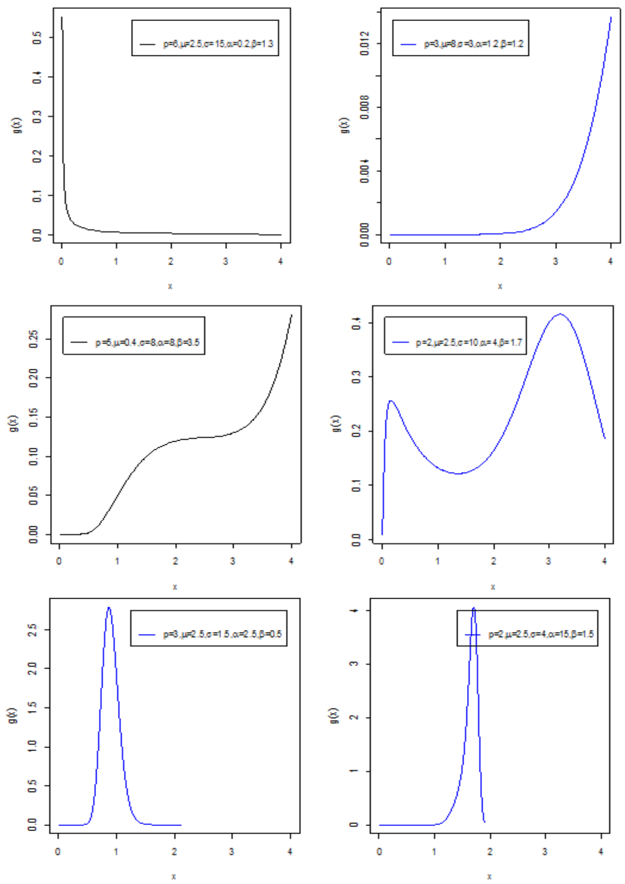

Figure 4. Plots of pdf of GMO-Weibull for some selected parameter values

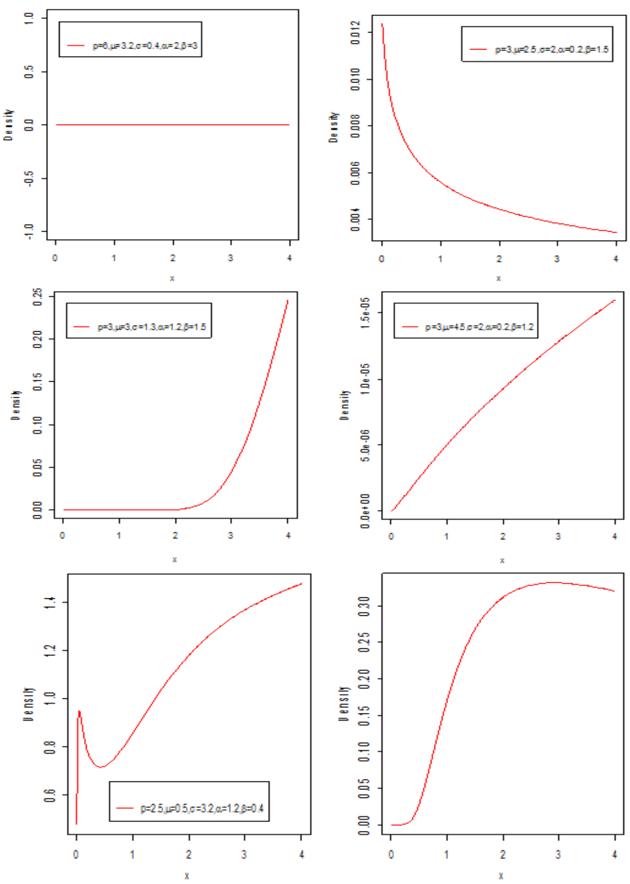

Figure 5. Plots of hazard rate function of GMO-Weibull for some selected parameter values.

Figure 3 and Figure 5 show the graphical hrf plots of two GMO-G sub-models (GMO-N and GMO-W) for selected parameter values indicating different shapes.

6. Moments

In this section, we obtained some moments associated with GMO-G family of distribution.

6.1. Ordinary Moment

Suppose X is a random variable from GMO-G distribution, then the n^{th} moment about the origin can be expressed by two formulas.

Firstly, let a random variable Y ~ $Exp^{m+1}-G$ having pdf $h_{m+1}(x)$ with power parameter $m+1$, then the n^{th} moment of X is given by

$$E\left(X^n\right)=\sum_{m=0}^{\infty}t_m E\left(Y^n\right).$$

Secondly,

$$E\left(X^n\right)=\sum_{m=0}^{\infty}(m+1)t_m\psi(n,\mathrm{m});$$

where

$$\psi(n,m)=\int_0^1\left(Q(u)\right)^n u^m du.$$

The incomplete moment plays important role in determining measures of inequality such as Lorenz, Bonferroni and Gini measures of inequality [14]. For a random variable having GMO-G density function, the w^{th} incomplete moment is given by

$$m_w(y)=\int_{-\infty}^{y}x^w g(x)dx$$

$$=\sum_{m=0}^{\infty}(m+1)t_m\int_0^{G(y)}\left(Q(u)\right)^w u^m du.$$

6.2. Moment Generating Function

The moment generating function (mgf) $M(t)=E\left(e^{tX}\right)$ of a random variable X having GMO-G defined as can be obtained firstly by

$$M(t)=\sum_{m=0}^{\infty}t_m M_{m+1}(t).$$

where $M_{m+1}(t)$ is the mgf of Y~Exp^{m+1}-G distribution.

Secondly, can be obtained by

$$M(t)=\sum_{m=0}^{\infty}(m+1)t_m\varphi(t,m).$$

where $\varphi(t,m)=\int_0^1\exp\left(tQ(u)\right)u^m du$, $Q(u)$ and u are as defined in equation (6).

7. Entropy

The measure of uncertainty of a random variable is through entropy. The Renyi and Shannon entropies are the two popular entropies. However, the Renyi entropy is a generalization of Shannon entropy. For a random variable X with pdf $f(x)$, according to [15], the Renyi entropy is defined as

$$I_R(\gamma)=\frac{1}{1-\gamma}\ln\left(\int_{\forall x}f^{\gamma}(x)dx\right);\quad \gamma>0\ \ and\ \ \gamma\neq 1.$$

However, suppose X~GMO-G, the Renyi entropy is given by

$$I_R(\gamma)=\frac{\gamma}{1-\gamma}\ln\left(Bp^{1/\sigma}\right)-\frac{\gamma}{1-\gamma}\ln(\sigma)$$
$$+\frac{1}{1-\gamma}\ln\left(\sum_{i,j=0}^{\infty}w_{ij}I_{ij}(\sigma,\gamma)\right);$$

where $I_{ij}(\sigma,\gamma)=\int_{\forall x}f^{\gamma}(x)F^{-\gamma\left(\frac{1}{\sigma}+1\right)-\frac{i}{\sigma}+j}(x)dx$ and

$$w_{ij}=\frac{(-1)^{i+j}}{i!}\left(\gamma Bp^{1/\sigma}\right)^i\binom{\gamma\left(\frac{1}{\sigma}-1\right)+\frac{i}{\sigma}}{j}.$$

8. Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample from a population having pdf $f(x)$ and $X_{i:n}=X_{(1)}\leq X_{(2)}\leq ...\leq X_{(n)}$ the order statistics. The pdf $f(x_{i:n})$ of the i^{th} order statistics is given by

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}g(x)G(x)^{i+j-1}.$$

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}g(x)G(x)^{i-1}\left(1-G(x)\right)^{n-i}.$$

Following [16], the pdf of i^{th} ordered statistics of sample of size n from GMO-G population is given by

$$f(x_{i:n})=\sum_{j=0}^{n-i}v_j\sum_{q=0}^{\infty}(q+1)t_q h_{q+1}(x); \tag{10}$$

where

$$t_q=(-1)^q\sum_{k,m,n=0}^{\infty}\frac{(-1)^{k+m+n}}{k!(q+1)}p^{\frac{1}{\sigma}+\frac{k}{\sigma}}\binom{\frac{k}{\sigma}+\frac{1}{\sigma}-1}{m}$$
$$\times\binom{m-\frac{k}{\sigma}-\frac{1}{\sigma}-1}{n}\binom{n}{q}\left[B(1+j)\right]^k,$$

$v_j = \frac{n!}{(i-1)!(n-i)!}\frac{B}{\sigma}(-1)^j\binom{n-i}{j}$ and $h_{q+1}(x)$ is pdf of Exp-G with power parameter $q+1$. By eq. (10), we can obtain the ordinary and incomplete moments, and moment generating function of $X_{i:n}$ for any baseline distribution.

9. Estimation

We consider the maximum likelihood method for the estimation of unknown parameters of GMO-G distribution. For a random sample $X_1, X_2, ..., X_n$ of size n from eq. (2), the log-likelihood function is given by

$$\begin{aligned} L(\theta) &= n\log(B) - n\log(\sigma) \\ &+ \frac{n}{\sigma}\log(p) + \sum_{i=1}^{n}\log\left(f(x_i;\xi)\right) \\ &- \sum_{i=1}^{n}\log\left(F(x_i;\xi)\right) - \sum_{i=1}^{n}\log\left(1-F(x_i;\xi)\right) \\ &+ \frac{1}{\sigma}\sum_{i=1}^{n}\log\left(1-F(x_i;\xi)\right) - \frac{1}{\sigma}\sum_{i=1}^{n}\log\left(F(x_i;\xi)\right) \\ &- Bp^{1/\sigma}\sum_{i=1}^{n}\left(\frac{F(x_i;\xi)}{\overline{F}(x_i;\xi)}\right)^{-\frac{1}{\sigma}}. \end{aligned}$$

The score functions for the parameters of the distribution are given by

$$U_p(\Theta) = \frac{n}{\sigma p} - \frac{Bp^{1/\sigma - 1}}{\sigma}\sum_{i=1}^{n}\left(\frac{F(x_i;\xi)}{\overline{F}(x_i;\xi)}\right)^{-\frac{1}{\sigma}}.$$

$$U_\mu(\Theta) = \frac{n}{\sigma} - \frac{Bp^{1/\sigma}}{\sigma}\sum_{i=1}^{n}\left(\frac{F(x_i;\xi)}{\overline{F}(x_i;\xi)}\right)^{-\frac{1}{\sigma}}.$$

$$\begin{aligned} U_\sigma(\Theta) &= \frac{n\mu}{\sigma^2} - \frac{n}{\sigma} + \frac{n}{\sigma^2}\log(p) \\ &- \frac{1}{\sigma^2}\sum_{i=1}^{n}\log\left(1-F(x_i;\xi)\right) + \frac{1}{\sigma^2}\sum_{i=1}^{n}\log\left(F(x_i;\xi)\right) \\ &+ \frac{Bp^{1/\sigma}\left(\log(p)-1\right)}{\sigma^2}\sum_{i=1}^{n}\left(\frac{F(x_i;\xi)}{1-F(x_i;\xi)}\right). \end{aligned}$$

$$\begin{aligned} U_\xi(\Theta) &= \sum_{i=1}^{n}\frac{f(x_i;\xi)^*}{f(x_i;\xi)} - \left(1+\frac{1}{\sigma}\right)\sum_{i=1}^{n}\frac{F(x_i;\xi)^*}{F(x_i;\xi)} \\ &+ \left(1-\frac{1}{\sigma}\right)\sum_{i=1}^{n}\frac{F(x_i;\xi)^*}{1-F(x_i;\xi)} \\ &- \frac{Bp^{1/\sigma}}{\sigma}\sum_{i=1}^{n}\frac{F(x_i;\xi)^*}{\left(1-F(x_i;\xi)\right)}\left(\frac{F(x_i;\xi)}{1-F(x_i;\xi)}\right)^{-\left(\frac{1}{\sigma}+1\right)}. \end{aligned}$$

The MLEs say, $\hat{\Theta}$ of Θ can be obtained as the solution to the non-linear equations $U_p(\Theta)=0,\ U_\mu(\Theta)=0,\ U_\sigma(\Theta)=0,$ and $U_\xi(\Theta)=0.$ Iterative techniques such as a Newton-Raphson type algorithm can be employed to evaluate $\hat{\Theta}$. Here, the optim function in R statistical software is used to obtain the numerical solution. Taking the minus expectation of the partial derivative of the score function with respect to the parameters of the distribution, we obtain the Fisher's information matrix $I(\Theta)_{(3+v)\times(3+v)} = (a_{ij})$; $i, j = 1,2,3,v$, where v is the number of parameters of the baseline distribution. However, under general regularity conditions, $\sqrt{n}\left(\hat{\Theta}-\Theta\right)$ is asymptotically multivariate normal distributed $N_{v+3}\left(0, I\left(\hat{\Theta}\right)^{-1}\right)$. By $I\left(\hat{\Theta}\right)^{-1}$, the asymptotic confidence interval for the parameters of the distribution appropriately constructed.

10. Simulation Studies

In this section, we assess the performance of the MLEs of GMO-Normal distribution, a sub model of GMO-G through simulation studies of different starting parameter values $p=2.5,\ \mu=2,\ \sigma=3,\ \mu_1=1.2,\ \sigma_1=0.8;$ $p=3,\ \mu=1.5,\ \sigma=3,\ \mu_1=0.5,\ \sigma_1=1.2$ and $p=3.5,$ $\mu=1.5,\ \sigma=3,\ \mu_1=1.5,\ \sigma_1=0.6.$ Simulated samples data of sizes n=50, 100, 150 generated by inverting the GMO-N distribution function given by

$$G(x) = \exp\left(-Bp^{1/\sigma}\left(\frac{\Phi(z)}{\overline{\Phi}(z)}\right)\right)$$

where $\Phi(z)$ is the distribution function of normal distribution with $z = \frac{x-\mu_1}{\sigma_1}$ and $\overline{\Phi}(z) = 1-\Phi(z)$. The MLEs, say $\hat{\Theta} = \hat{p}, \hat{\mu}, \hat{\sigma}, \hat{\mu}_1, \hat{\sigma}_1$ are determined and the process is repeated for N=1000 times. We obtain the average estimates (Mean value), average baises and the mean square error (MSE) given respectively by

$$\overline{\hat{\Theta}} = \frac{\sum_{i=1}^{N}\hat{\Theta}_i}{N},$$

$$bias_\Theta(n) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\Theta}_i - \Theta\right),$$

and

$$\widehat{MSE} = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\Theta}_i - \Theta\right)^2.$$

The results of the simulations for the three different starting parameter values are respectively shown in Table 1, Table 2, and Table 3. The values in the tables indicate that as the sample size increases the mean values tend to starting parameter values and the MSE decreases which conforms to the first order asymptotic theory.

Table 1. The Mean values, Average bias and MSEs of 1,000 simulations of the GMO-N distribution for the first set of starting parameter values

Parameter	Mean Value	Bias	MSE
n = 50			
$p = 2.5$	3.39587	0.8958703	29.42619
$\mu = 2$	2.324494	0.3244933	7.85408
$\sigma = 3$	4.099783	1.099783	5.035
$\mu_1 = 1.2$	1.201197	0.001197251	3.694475
$\sigma_1 = 0.8$	0.7983528	-0.001047231	0.4592548
n = 100			
$p = 2.5$	2.97454	0.4745403	5.70756
$\mu = 2$	2.239961	0.2399611	4.07647
$\sigma = 3$	3.737967	0.737967	2.61185
$\mu_1 = 1.2$	1.301246	0.1012455	0.7174885
$\sigma_1 = 0.8$	0.7562551	-0.04374486	0.1087711
n = 150			
$p = 2.5$	2.84734	0.3473398	4.555363
$\mu = 2$	2.230906	0.2309057	2.829908
$\sigma = 3$	3.512312	0.5123116	1.652731
$\mu_1 = 1.2$	1.264034	0.06403414	0.4262931
$\sigma_1 = 0.8$	0.769961	-0.03003901	0.07056685

Table 2. The Mean values, Average bias and MSEs of 1,000 simulations of the GMO-N distribution for the second set of starting parameter values

n = 50			
$p = 3$	3.685218	0.6852184	22.85455
$\mu = 1.5$	1.863394	0.3633938	7.211502
$\sigma = 3$	4.2039	1.2039	5.375771
$\mu_1 = 0.5$	0.6679074	0.1679074	4.131032
$\sigma_1 = 1.2$	1.118886	-0.08111376	0.5052979
n = 100			
$p = 3$	3.29551	0.2955102	7.265812
$\mu = 1.5$	1.807605	0.3076048	3.520821
$\sigma = 3$	3.745647	0.7456467	2.590324
$\mu_1 = 0.5$	0.6247453	0.1247453	1.282261
$\sigma_1 = 1.2$	1.133319	-0.06668094	0.235327
n = 150			
$p = 3$	3.217667	0.2176671	5.469239
$\mu = 1.5$	1.771239	0.2712389	2.546943
$\sigma = 3$	3.524388	0.5243885	1.660748
$\mu_1 = 0.5$	0.5839617	0.08396168	0.7318817
$\sigma_1 = 1.2$	1.151136	-0.04886168	0.1259547

Table 3. The Mean values, Average bias and MSEs of 1,000 simulations of the GMO-N distribution for the third set of starting parameter values

n =50			
$p = 3.5$	4.043453	0.5434532	21.17929
$\mu = 1.5$	1.823817	0.3238172	6.457807
$\sigma = 3$	4.135532	1.135532	5.306406
$\mu_1 = 1.5$	1.455288	-0.04471154	2.873199
$\sigma_1 = 0.6$	0.6183294	0.01832943	0.4969151
n = 100			
$p = 3.5$	3.802545	0.3025448	10.85041
$\mu = 1.5$	1.791826	0.2918256	3.61332
$\sigma = 3$	3.738561	0.7385607	2.58047
$\mu_1 = 1.5$	1.564739	0.0647388	0.4188419
$\sigma_1 = 0.6$	0.5682559	-0.0317441	0.06262047
n = 150			
$p = 3.5$	3.917156	0.4171556	9.941959
$\mu = 1.5$	1.73949	0.2394897	2.494105
$\sigma = 3$	3.517443	0.5174426	1.65395
$\mu_1 = 1.5$	1.538664	0.03866443	0.2520109
$\sigma_1 = 0.6$	0.5778611	-0.02213888	0.037728

11. Bivariate Extension of GMO-G

Let X and Y be two random variables from GMO-G family of distribution, the bivariate cdf is given by

$$H(x,y) = \exp\left(-Bp^{1/\sigma}\left(\frac{F(x,y)}{\overline{F}(x,y)}\right)^{-\frac{1}{\sigma}}\right). \tag{11}$$

By taking the partial derivative of (11) with respect to x and y, the bivariate pdf of GMO-G is given by

$$g(x,y) = Q_A(x,y) + Q_B(x,y)\left\{\frac{2\sigma\left(F(x,y)+1\right) + Bp^{1/\sigma}\left(\frac{F(x,y)}{\overline{F}(x,y)}\right)^{\frac{1}{\sigma}}}{\sigma F(x,y)}\right\};$$

where

$$Q_A(x,y) = \frac{Bp^{1/\sigma} f(x,y)}{\sigma\left(\overline{F}(x,y)\right)^3}\left(\frac{F(x,y)}{\overline{F}(x,y)}\right)^{-\frac{1}{\sigma}-1} \times \exp\left(-Bp^{1/\sigma}\left(\frac{F(x,y)}{\overline{F}(x,y)}\right)^{-\frac{1}{\sigma}}\right)$$

and

$$Q_B(x,y)=\frac{Bp^{1/\sigma}\frac{\partial F(x,y)}{\partial x}\frac{\partial F(x,y)}{\partial y}}{\sigma\left(\overline{F}(x,y)\right)^3}\left(\frac{F(x,y)}{\overline{F}(x,y)}\right)^{-\frac{1}{\sigma}-1}$$

$$\times\exp\left(-Bp^{1/\sigma}\left(\frac{F(x,y)}{\overline{F}(x,y)}\right)^{-\frac{1}{\sigma}}\right).$$

The marginal distributions are respectively given by

$$g(y)=\frac{Bp^{1/\sigma}f(y)}{\sigma F(y)\overline{F}(y)}\left(\frac{F(y)}{\overline{F}(y)}\right)^{-\frac{1}{\sigma}}$$

$$\times\exp\left(-Bp^{1/\sigma}\left(\frac{F(y)}{\overline{F}(y)}\right)^{-\frac{1}{\sigma}}\right)$$

and

$$g(x)=\frac{Bp^{1/\sigma}f(x)}{\sigma F(x)\overline{F}(x)}\left(\frac{F(x)}{\overline{F}(x)}\right)^{-\frac{1}{\sigma}}\exp\left(-Bp^{1/\sigma}\left(\frac{F(x)}{\overline{F}(x)}\right)^{-\frac{1}{\sigma}}\right).$$

The conditional density functions are respectively given by

$$g(x|y)=\left(Q_a(x,y)+Q_b(x,y)\left\{\frac{\left[\begin{array}{l}2\sigma\left(F(x,y)+1\right)\\+Bp^{1/\sigma}\left(\frac{F(x,y)}{\overline{F}(x,y)}\right)^{-\frac{1}{\sigma}}\end{array}\right]}{\sigma F(x,y)}\right\}\right)$$

$$\times\frac{F(y)\overline{F}(y)}{f(y)}\left(\frac{F(y)}{\overline{F}(y)}\right)^{\frac{1}{\sigma}}\exp\left(Bp^{1/\sigma}\left(\frac{F(y)}{\overline{F}(y)}\right)^{-\frac{1}{\sigma}}\right)$$

and

$$g(y|x)=\left(Q_a(x,y)+Q_b(x,y)\left\{\frac{\left[\begin{array}{l}2\sigma\left(F(x,y)+1\right)\\+Bp^{1/\sigma}\left(\frac{F(x,y)}{\overline{F}(x,y)}\right)^{-\frac{1}{\sigma}}\end{array}\right]}{\sigma F(x,y)}\right\}\right)$$

$$\times\frac{F(x)\overline{F}(x)}{f(x)}\left(\frac{F(x)}{\overline{F}(x)}\right)^{\frac{1}{\sigma}}\exp\left(Bp^{1/\sigma}\left(\frac{F(x)}{\overline{F}(x)}\right)^{-\frac{1}{\sigma}}\right);$$

where

$$Q_a(x,y)=\frac{f(x,y)}{\left(\overline{F}(x,y)\right)^3}\left(\frac{F(x,y)}{\overline{F}(x,y)}\right)^{-\frac{1}{\sigma}-1}$$

$$\times\exp\left(-Bp^{1/\sigma}\left(\frac{F(x,y)}{\overline{F}(x,y)}\right)^{-\frac{1}{\sigma}}\right)$$

and

$$Q_b(x,y)=\frac{\frac{\partial F(x,y)}{\partial x}\frac{\partial F(x,y)}{\partial y}}{\left(\overline{F}(x,y)\right)^3}\left(\frac{F(x,y)}{\overline{F}(x,y)}\right)^{-\frac{1}{\sigma}-1}$$

$$\times\exp\left(-Bp^{1/\sigma}\left(\frac{F(x,y)}{\overline{F}(x,y)}\right)^{-\frac{1}{\sigma}}\right).$$

12. Applications

In this section, we considered the applications of two sub models of GMO-G (GMO-Weibull and GMO-Lomax) to real life data sets to show the potentials of the new class of distributions. Comparison with other models having the same baseline distribution is made based on goodness-of-statistics Cramer Von Mises (W^*), Anderson Darling, Koromogrov-Smirnov, Akaike Information Criterion, and Bayesian Information Criterion. The model with the least value of goodness-of-fit statistics provides the best fit [17].

First data set: The first real data set is on the observed survival times (weeks) for AG positive reported by [18]. The data set is as follow: 65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65. The GMO-Weibull, Exponentiated-Weibull(ExpW) due to [2], Beta-Weibull (BW) due to [19], Gumbel-Weibull(GuW) due to [9], logistic-Weibull(LW) and Weibull distribution are fitted to the data set.

Second data set: The second real data set refers to 30 devices failure times reported by Meeker and Escobar [20] and Tahir et al. [21]. The data set are as follow: 275,13, 147, 23, 181, 30, 65, 10, 300, 173, 106, 300, 300, 212, 300, 300, 300, 2, 261, 293, 88,247, 28, 143, 300, 23, 300, 80, 245, 266. The GMO-Lomax, Beta-Lomax (BL) due to [22], Gumbel-Lomax(GuL) due to [23], Exponentiated-Lomax(ExpL) due to [24], logistic-Lomax (LL) due to [25] and Lomax distribution are fitted to the data set.

The results of the two applications to the real data sets are shown in Table 4, Table 5 and Table 6, Table 7 for first and second data set respectively. The parameter estimates with standard errors in parenthesis are shown in Table 4 and Table 6 while the goodness-of-fit statistics are contained in Table 5 and Table 7 for the first and second data set respectively. The plots of the estimated pdfs and empirical cdf with cdfs of some of the competing models in the first and second application are shown respectively in Figure 6 and Figure 7. The plots in Figure 6 and Figure 7 indicate that the sub models of the newly proposed class of distributions provide better fits among the competing models in agreement with the Table 5 and Table 7.

Table 4. The parameter estimates and standard errors of (in parentheses) of the first data set

Model	p	μ	σ	a	b	α	θ
GMOW	0.1962 (0.6680)	-1.5159 (11.9814)	9.3802 (10.8786)	-	-	3.2793 (4.7433)	56.6393 (63.3664)
GuW	1	2.1061 (1.3339)	0.3903 (0.3838)	-	-	0.0944 (0.0622)	0.0028 (0.0015)
BW	-	-	-	0.7967 (0.4953)	0.0638 (0.0161)	0.8714 (0.0024)	2.5268 (0.0024)
ExW	-	-	-	1.6149 (1.7986)	-	0.6012 (0.4083)	31.1669 (49.3803)
LW				1.2009 (0.5772)		1.6124 (0.9316)	92.6622 (21.1288)
W	-	-	-	-	-	0.8841 (0.1831)	59.1654 (16.9361)

Table 5. The goodness of fit statistics for the first data set

Model	W	A	AIC	BIC	K-S	-log
GMOW	0.0235	0.1745	177.2849	181.451	0.1092	83.6425
GuW	0.2025	1.2495	190.7061	194.0389	0.2002	91.3530
BW	0.0659	0.4791	181.2613	184.5942	0.1515	86.6317
ExW	0.0890	0.6165	181.5105	184.0101	0.1698	87.7553
LW	6.2042	34.4041	179.1987	181.6984	0.9382	86.5994
W	0.0704	0.5076	178.2193	179.8857	0.1491	87.1096

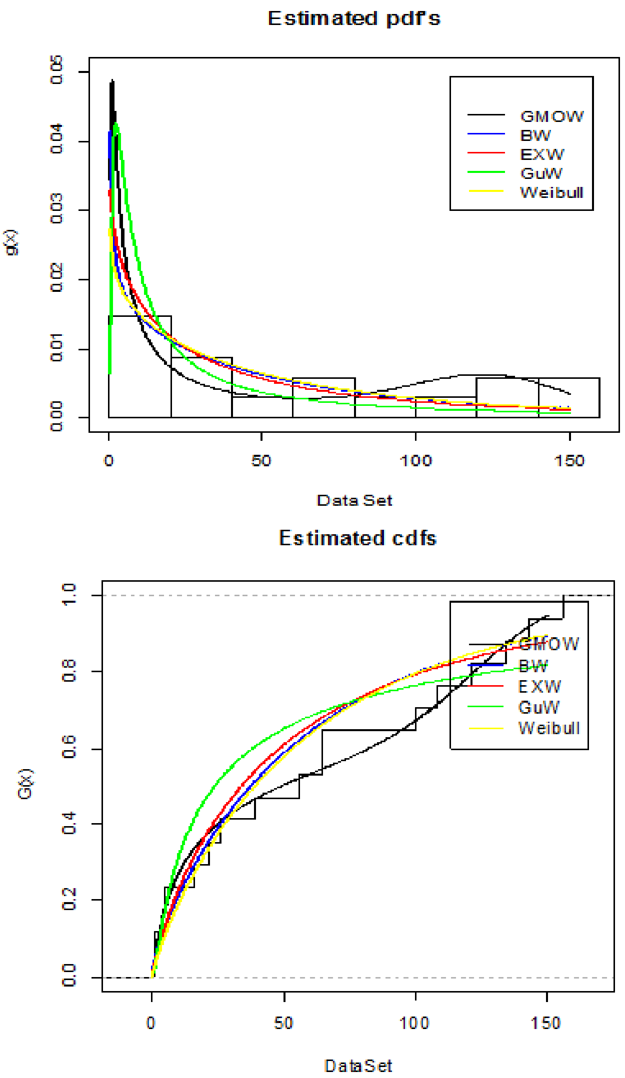

Figure 6. Plots of estimated pdfs and cdfs with empirical cdf of the first data set

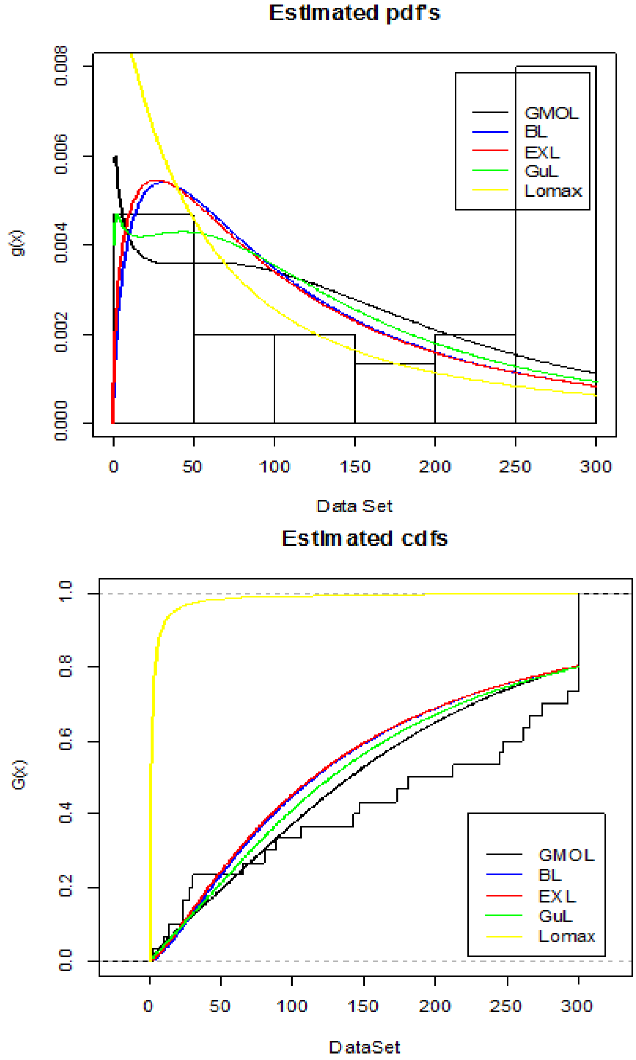

Figure 7. Plots of estimated pdfs and cdfs with empirical cdf of the second data set

Table 6. The parameter estimates and standard errors of (in parentheses) of the second data set

Model	p	μ	σ	a	b	α	λ
GMOL	7.3083 (180.5080)	2.6518 (24.8572)	4.5207 (1.9515)	-	-	17.8241 (11.4496)	330.455 (194.4526)
GuL	1	5.0809 (2.9066)	4.3659 (1.8972)	-	-	9.3140 (5.7825)	120.9559 (68.7018)
BL	-	-	-	1.6054 (0.4314)	16.8237 (26.7395)	0.1328 (0.2079)	151.9141 (94.0866)
ExL	-	-	-	1.4554 (0.3814)	-	2.1243 (0.8548)	197.4543 (111.6175)
LL	-	-	-	13.7426 (2.4047)	-	0.0991 (0.0078)	0.0054 (0.0041)
lomax						1.0702 (0.3307)	103.3856 (44.6811)

Table 7. The goodness of fit statistics of the second data set

Model	W	A	AIC	BIC	K-S	-log
GMOL	0.3129	1.8752	380.6972	387.7032	0.1977	185.3486
GuL	0.3601	2.1019	383.5192	389.1240	0.2062	187.7596
BL	0.4044	2.2962	387.06	392.6648	0.2155	189.53
ExL	0.3968	2.2618	384.355	388.5586	0.2156	189.1775
LL	0.4487	2.4989	390.6905	394.8941	0.2498	192.3453
lomax	0.4992	2.7417	389.5037	392.3061	0.8846	192.7519

13. Conclusion

We proposed a new class of Gumbel generated family of distributions called Gumbel-Marshall Olkin-G family of distribution which has Gumbel-G as a special model. The cdf and pdf of the new class of distributions are represented as linear combination of exponentiated-G family of distribution. Some sub models of the new class of distributions are presented and several continuous distributions can be obtained for any baseline distribution. Furthermore, some statistical properties such as the quantile function. Ordinary and incomplete moments, generating function, entropy, and density function of the order statistics are derived. The unknown parameters of the new family of distributions are estimated through maximum likelihood method. The consistency of the MLEs is considered using GMO-N by means of simulations studies. We further derived the bivariate extension of the new class of distributions. Finally, the potentials of the new class of the distributions are illustrated by means of comparing the GMO-W and GMO-Lomax distributions with other competing distributions in two real life data sets. The goodness-of-statistics indicate that the two sub models of the new class of distributions provide the best fit among other competing models.

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