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The Fuzzy Minimum Cost Flow Problem with the Fuzzy Time-Windows

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Abstract The Minimum Cost Flow Problem (MCFP) is a well-known combinatorial optimization and a logical distribution problem. The MCFP is an NP-hard problem with many applications in logistic networks and computer networks. The Fuzzy Minimum Cost Flow Problem with Fuzzy Time-Windows (FMCFPFTW) is an extension of the MCFP. The goal of the problem is to find the minimum amount of the fuzzy flow from the source to the sink that satisfies all constraints of the fuzzy shortest dynamic f-augmenting path with the fuzzy dynamic residual network. We consider a generalized fuzzy version of the MCFP of the fuzzy network. We propose the mathematical model of the FMCFPFTW. Finally, a new algorithm of the FMCFPFTW is presented.

Mathematics Subject Classification: 05C35, 90C27, 65G30, 68R10

Keywords: *combinatorial optimization, minimum cost flow, time-windows, fuzzy time-windows*

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1. Introduction

The Minimum Cost Flow Problem (MCFP) is a basic problem in the network flow theory, which is one of the classical combinatorial optimizations and an NP-hard problem with several applications. Over the past 40 years, the MCFP has been an area of research that has attracted many researchers. The MCFP has been studied extensively, see, [1,2,8,9,18,23,24]. We consider $\tilde{G} = (N, A, \tilde{l}, \tilde{b}, \tilde{c}_r, \tilde{c}_w, [\tilde{a}_{v_i}, \tilde{b}_{v_i}])$ be a fuzzy network without parallel arcs and loops, where N is a set of nodes and A is a set of arcs. For each node $v_i \in N$, $i = 1, \dots, n$ there is an associated of a three integer parameters, a waiting fuzzy cost $\tilde{c}_w(v_i, \tilde{t}_{v_i})$, a node fuzzy capacity $\tilde{l}(v_i, \tilde{t}_{v_i})$ and a fuzzy time-windows $[\tilde{a}_{v_i}, \tilde{b}_{v_i}]$ where, $\tilde{t}_{v_i} \in [\tilde{a}_{v_i}, \tilde{b}_{v_i}]$, $\tilde{t}_{v_i} \in \tilde{T}$ is a non-negative fuzzy time service, see, [3,16,19,20,24].

For each arc $(v_i, v_{i+1}) \in A, \forall i = 1, \dots, n$; there is an associated of a three integer parameters, a positive fuzzy transit time $\tilde{b}(a, \tilde{t}_{v_i})$, a fuzzy transit cost $\tilde{c}_r(a, \tilde{t}_{v_i})$, and a positive fuzzy capacity limit $\tilde{l}(a, \tilde{t}_{v_i})$. All these parameters are functions of the fuzzy time service $\tilde{t}_{v_i}$. The source node s and a sink node ρ has a fuzzy time-windows $[\tilde{a}_s, \tilde{b}_s]$, $[\tilde{a}_\rho, \tilde{b}_\rho]$ respectively, see [13,14,21,22]. The fuzzy flow must arrive at node $v_i \in N$ before a fuzzy time $\tilde{b}_{v_i}$, if it arrives before $\tilde{a}_{v_i}$, it just a fuzzy waiting time before treating the fuzzy flow at the node $v_i \in N$ (a fuzzy waiting time $\widetilde{w}(v_i) \succ 0$), see [4,5,6,7,18].

The problem is to determine how given the amount of the fuzzy flow can be sent from one node s to another node ρ. The fuzzy minimum cost on a fuzzy shortest dynamic f-augmenting path of the fuzzy dynamic residual network, subject to the fuzzy capacity limits on the arcs, see [10,11,15,17].

Traditionally, this problem is considered as a static one, where it is assumed that it takes a zero time to traverse the arc. All attributes of the fuzzy network, including a fuzzy cost to send a fuzzy flow on the arc, the fuzzy capacity of the arc, are a fuzzy time-invariant. A more realistic model is to consider the fuzzy time needed to traverse an arc. For each arc there is a fuzzy transit time $\tilde{b}(v_i, v_{i+1})$, $(v_i, v_{i+1}) \in A, i = 1, \dots, n$. For each node $v_i \in N$ there is a fuzzy time-windows $[\tilde{a}_{v_i}, \tilde{b}_{v_i}]$, subject to the condition that the schedule remains optimal for any fuzzy service time $\tilde{t}_{v_i} \leq \tilde{T}$. By the shortest dynamic f-augmenting path with a fuzzy dynamic residual network. The new version of the problem becomes so-called the Fuzzy Minimum Cost Flow Problem with Fuzzy Time-Windows (FMCFPFTW).

The reminder of this paper consists of six sections including organized Introduction. Section 2 presents some basic concepts, definitions, a fuzzy time-varying, a fuzzy time-windows and the fuzzy shortest dynamic f-augmenting path with a fuzzy dynamic residual network. In Section 3, we presented the fuzzy dynamic residual network with the fuzzy time-varying and fuzzy time-windows. In Section 4, we propose, the mathematical formulation model of the FMCFPFTW. In Section 5, we presented an algorithm of the FMCFPFTW of the fuzzy network. Finally, the conclusion is given in Section 6.

2. Some Basic Concepts and Definitions

Let $\tilde{G} = \left(N, A, \tilde{l}, \tilde{b}, \tilde{c}_r, \tilde{c}_w, [\tilde{a}_{v_i}, \tilde{b}_{v_i}]\right)$ be a fuzzy network without parallel arcs and loops, where N is a set of nodes and A is a set of arcs. All parameters $\tilde{l}, \tilde{b}, \tilde{c}_r, \tilde{c}_w$ of the fuzzy network are functions of a fuzzy service time $\tilde{t}_{v_i} \le \tilde{T}$, where $\tilde{t}_{v_i} \in [0, \tilde{T}]$, $\tilde{t}_{v_i} \in \left[\tilde{a}_{v_i}, \tilde{b}_{v_i}\right]$, $i = 1, \ldots, n$. Each node $v_i \in N$ has a fuzzy time-windows $\left[\tilde{a}_{v_i}, \tilde{b}_{v_i}\right]$. More specifically, $\tilde{l}\left(v_i, v_{i+1}, \tilde{t}_{v_{i+1}}\right)$ is a fuzzy capacity of the arc $(v_i, v_{i+1}) \in A$, $\tilde{l}\left(v_i, \tilde{t}_{v_i}\right)$, $v_i \in N \backslash \{s, \rho\}$ is a fuzzy capacity of each node v_i during the period from $\tilde{t}_{v_i}$ to $\tilde{t}_{v_{i+1}}$. A fuzzy transit time $\tilde{b}\left(v_i, v_{i+1}, \tilde{t}_{v_{i+1}}\right)$ is needed to send a fuzzy flow through the arc (v_i, v_{i+1}) at a fuzzy time $\tilde{t}_{v_{i+1}}$.

The fuzzy cost $\tilde{c}_r(v_i, v_{i+1}, \tilde{t}_{v_{i+1}})$ of a transmitting one unit of a fuzzy flow through the arc (v_i, v_{i+1}) at a fuzzy time $\tilde{t}_{v_{i+1}}$. A unit of the fuzzy cost $\tilde{c}_w(v_i, \tilde{t}_{v_i})$ is keeping one unit of the fuzzy flow that a waiting at the node v_i for the closed interval duration $[\tilde{t}_{v_i}, \tilde{t}_{v_{i+1}}]$. We assume that $\tilde{b}, \tilde{l}, \tilde{c}_r$ and $\tilde{c}_w$ are the fuzzy integers and two particular nodes s and ρ are the source and the sink nodes of the fuzzy network with a fuzzy time-windows $[\tilde{a}_s, \tilde{b}_s]$, $[\tilde{a}_\rho, \tilde{b}_\rho]$ respectively.

We consider $n = |N|, m = |A|$, and $\tilde{f}(v_i, v_{i+1}, \tilde{t}_{v_{i+1}})$ be a fuzzy flow value of the arc (v_i, v_{i+1}) during the period $\left[\tilde{t}_{v_i}, \tilde{t}_{v_i} + \tilde{b}\left(v_i, v_{i+1}, \tilde{t}_{v_{i+1}}\right)\right]$. The fuzzy flow value $\tilde{f}(v_i, \tilde{t}_{v_i})$, $\forall v_i \in N \backslash \{s, \rho\}$ is a fuzzy waiting at the node v_i during the closed period $\left[\tilde{t}_{v_i}, \tilde{t}_{v_{i+1}}\right]$. The total fuzzy flow value $\tilde{f}(\alpha, \tilde{T})$ under a schedule α, which specifies, how to send a fuzzy flow from the source node s to the sink node ρ within the fuzzy time-windows and a fuzzy time limit $\tilde{T}$, then,

$$\tilde{f}\left(\alpha, \tilde{T}\right) = \sum_{(v_i, \rho) \in A, \tilde{t}_{v_i} + \tilde{b}\left(v_i, \rho, \tilde{t}_{v_i}\right) \le \tilde{T}} \tilde{f}\left(v_i, \rho, \tilde{t}_{v_i}\right). \quad (1)$$

The FMCFPFTW is to find a feasible schedule send to a given fuzzy flow $\tilde{v}_{\tilde{f}}$ from s to ρ within the fuzzy time limit $\tilde{T}$ and minimize the total fuzzy cost satisfy all constraints.

A fuzzy time-varying and a fuzzy time-windows are an essential ingredient which we must consider in our model. Consequently, some important concepts and procedures will have to be re-defined and re-constructed, to consider the existence of the dynamic fuzzy time. Without ambiguity, in the following, we will assume that the length of the arc is equal to the fuzzy cost use the interchangeably of terminologies a fuzzy cost and a fuzzy length by the fuzzy shortest path, the fuzzy cheapest path. Further, we denote $\tilde{P}(s, v_i)$ be a directed fuzzy path from s to v_i connecting to the nodes $s = v_1, v_2, \ldots, v_n$; $v_i \in N$; $i = 1, \ldots, n$, where $\tilde{\beta}(v_i)$, $\tilde{w}(v_i)$ and $\tilde{\tau}(v_i)$ are a fuzzy arrival time, a fuzzy waiting time, and a fuzzy departure time respectively at the node v_i with a fuzzy time $\tilde{t}_{v_i}$. The fuzzy path has a fuzzy service time $\tilde{t}_{v_i}$ if the total fuzzy time required to traverse this path where, $\tilde{t}_{v_i} \le \tilde{T}$.

Definition 2.1 Let $G = (N, A, l, b, c_r, c_w, [a_{v_i}, b_{v_i}])$ be a classic network where N is a limited set of nodes and A is a set of arcs. Each arc $(v_i, v_{i+1}) \in A$ has an integral number l, b, c_r and c_w, where l is a capacity of each node, b is a transit time of each arc, c_r is a cost of the transmitting one unit of the flow and c_w is a cost of each node. A time-window is defined by, for each node $v_i, v_{i+1} \in N$ has time-windows $[a_{v_i}, b_{v_i}]$ and $[a_{v_{i+1}}, b_{v_{i+1}}]$ respectively, $i = 1, \ldots, n$ see, Figure 1.

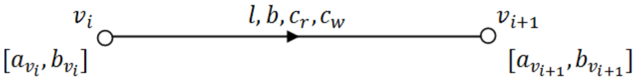

Figure 1. A representation of the classic time-windows of two nodes $v_i, v_{i+1} \in N$

Definition 2.2 [13] Let $X = \Re^n$ be a non-empty set, $\tilde{S} \subset X$. The fuzzy set $\tilde{S} = \{(x, \mu_{\tilde{S}}(x)) : x \in X\}$ is the set of ordered pairs where $\mu_{\tilde{S}} : X \to [0,1]$ is the membership function of the fuzzy set $\tilde{S}$. The fuzzy travel time between two places along with the most likely fuzzy time; For example, the fuzzy time $\tilde{T}$ to travel from node v_1 to node v_2 is between t_1 and t_3, but must possibly it is t_2. This sort of knowledge lets us construct 3-point fuzzy travel times.

Definition 2.3 [12] Every node $v_i \in N, i = 1, \ldots, n$ is assigned by the expert to one of two predetermined groups; a classical fuzzy time-windows and a fuzzy time-windows of a normal node. In an extreme case, a fuzzy time-window is tighter than the classical counterpart. The shown characteristics of the fuzzy time-windows are suggested to the shipper who can modify them. We consider the FMCFPFTW; the new optimization fuzzy model is presented by the fuzzy capacities calculating and the crisp equivalents of the fuzzy chance constraints.

Definition 2.4 The fuzzy path $\tilde{P}(s, v_i)$ is said to be a fuzzy dynamic f-augmenting path from s to v_i with a fuzzy time-varying and a fuzz time-windows with, $\tilde{t}_{v_i} \le \tilde{T}$, $\tilde{t}_{v_i} \in [\tilde{a}_{v_i}, \tilde{b}_{v_i}]$ if it satisfies:

$$\tilde{\tau}(v_{i-1}) + \tilde{b}(v_{i-1}, v_i, \tilde{\tau}(v_{i-1})) = \tilde{\beta}(v_i) \quad (2)$$

$$\tilde{\beta}(v_i) + \tilde{w}(v_i) = \tilde{\tau}(v_i) \quad (3)$$

$$\tilde{l}(v_{i-1}, v_i, \tilde{\tau}(v_{i-1})) \succ 0 \quad (4)$$

$$\tilde{l}(v_{i-1}, \tilde{t}_{v_{i-1}}) \succ 0 \quad (5)$$

for $i = 2, \ldots, n$ and $\tilde{\tau}(v_{i-1}) = \tilde{\beta}(v_{i-1})$, then there is no fuzzy waiting time $(\tilde{w}(v_{i-1}) = 0)$.

Definition 2.5 Let $\tilde{P}(s, v_i)$ be a fuzzy dynamic f-augmenting path from s to v_i, of the fuzzy time-varying and the fuzzy time-windows, with $\tilde{t}_{v_i} \le \tilde{T}$, $\tilde{t}_{v_i} \in \left[\tilde{a}_{v_i}, \tilde{b}_{v_i}\right]$, then the fuzzy capacity of the path $\tilde{P}(s, v_i)$ is defined as $\widetilde{Cap}(\tilde{P}(s, v_i)) = min\{\tilde{\delta}_1, \tilde{\delta}_2\}$ where;

$$\begin{aligned} \tilde{\delta}_1 &= \min_{2 \le i \le n} \tilde{l}\left(v_{i-1}, v_i, \tilde{\tau}\left(v_{i-1}\right)\right) \\ \text{and } \tilde{\delta}_2 &= \min_{2 \le i \le n} \min_{0 \le \tilde{t}_{v_i} \le \tilde{w}\left(v_{i-1}\right)} \tilde{l}\left(v_{i-1}, \tilde{\beta}\left(v_{i-1}\right)\right). \end{aligned} \quad (6)$$

In this work, the algorithm to be developed will search, successively, the fuzzy shortest paths from the source node s to the sink node ρ of the fuzzy dynamic residual network. The fuzzy transmit as much as possible of the fuzzy flow along the fuzzy paths, satisfy all constraints. In [20], we will need to the fuzzy network updating of the procedure to keep the needed information on the current fuzzy dynamic flow, which is to be introduced below. In our fuzzy network to updating of the procedure, we create, initially, a new fuzzy network to replace the original one.

Definition 2.6 For every arc $(v_i, v_{i+1}) \in A$, we create an artificial arc, denoted by $[v_{i+1}, v_i]$. It's a fuzzy transit time $\tilde{b}[v_{i+1}, v_i, \tilde{t}_{v_i}]$, a fuzzy transit cost $\tilde{c}_r[v_{i+1}, v_i, \tilde{t}_{v_i}]$, and the fuzzy capacity $\tilde{l}[v_{i+1}, v_i, \tilde{t}_{v_i}]$ where, $\tilde{t}_{v_i} \in [\tilde{a}_{v_i}, \tilde{b}_{v_i}]$ then;

For $v_i \in N$, $i = 1, \ldots, n$

$$\begin{aligned}&\tilde{b}\left[v_{i+1}, v_i, \tilde{t}_{v_i}\right]\\ &= -\tilde{b}\left(v_i, v_{i+1}, \tilde{u}\right)\\ &\quad if\ 0 \le \tilde{t}_{v_{i+1}} = \tilde{u} + \tilde{b}\left(v_i, v_{i+1}, \tilde{u}\right) \le \tilde{T}, 0 \le \tilde{u} \le \tilde{T}\\ &= \infty, otherwise,\end{aligned} \tag{7}$$

$$\begin{aligned}&\tilde{c}_r\left[v_{i+1}, v_i, \tilde{t}_{v_i}\right]\\ &= -\tilde{c}_r\left(v_i, v_{i+1}, \tilde{u}\right)\\ &\quad if\ 0 \le \tilde{t}_{v_{i+1}} = \tilde{u} + \tilde{b}\left(v_i, v_{i+1}, \tilde{u}\right) \le \tilde{T}, 0 \le \tilde{u} \le \tilde{T}\\ &= \infty, otherwise,\end{aligned} \tag{8}$$

$$\begin{aligned}&\tilde{l}\left[v_{i+1}, v_i, \tilde{t}_{v_i}\right] = 0,\\ &\forall\left(v_i, v_{i+1}\right) \in A; 0 \le \tilde{t}_{v_i} \le \tilde{T}; \tilde{t}_{v_i} \in \left[\tilde{a}_{v_i}, \tilde{b}_{v_i}\right].\end{aligned} \tag{9}$$

For every node $v_i \in N$, we define $\tilde{l}[v_i, \tilde{t}_{v_i}]$ as the fuzzy capacity within which a fuzzy flow can be waiting at v_i from a fuzzy time $\tilde{t}_{v_i}$ to $\tilde{t}_{v_{i-1}}$ and $\tilde{c}_w[v_i, \tilde{t}_{v_i}]$ as the fuzzy cost from a fuzzy flow to stay at v_i from a fuzzy time $\tilde{t}_{v_i}$ to $\tilde{t}_{v_{i-1}}$. Initially, let $\tilde{l}[v_i, \tilde{t}_{v_i}] = 0$ and,

$$\begin{aligned}&\tilde{c}_w\left[v_{i+1}, \tilde{t}_{v_{i+1}}\right] = -\tilde{c}_w\left(v_{i+1}, \tilde{t}_{v_{i+1}}\right);\\ &\forall i = 1, \ldots, n; \tilde{t}_{v_i} \in \left[0, \tilde{T}\right].\end{aligned} \tag{10}$$

No fuzzy flow can be sent along any artificial arcs in the fuzzy network $\tilde{G}$ as defined above since the fuzzy capacities of these arcs are a set to zero. Hence, this new fuzzy network is equivalence to the original one, and so we will still denote it by $\tilde{G}$.

3. The Fuzzy Dynamic Residual Network with Fuzzy Time-Varying and Fuzzy Time-Windows

Let $\tilde{P}(s, \rho)$ be a fuzzy dynamic f-augmenting path from s to ρ, and $\tilde{f}_{\tilde{P}}$ be a fuzzy flow value send along $\tilde{P}(s, \rho)$ which satisfies $\tilde{f}_{\tilde{P}} \in [0, \widetilde{Cap}\left(\tilde{P}(s, \rho)\right)]$. For $i = 1, \ldots, n-1$; $\tilde{t}_{v_i} \in [\tilde{a}_{v_i}, \tilde{b}_{v_i}]$, $\tilde{t}_{v_i} \in [0, T]$ and for each node $v_i \in N \backslash \{s, \rho\}$, then the update of the fuzzy capacity of the arc is given by:

- If (v_i, v_{i+1}) is not an artificial arc. Let,

$$\tilde{l}(v_i, v_{i+1}, \tilde{\tau}(v_i)) + \tilde{f}_{\tilde{P}} = \tilde{l}(v_i, v_{i+1}, \tilde{\tau}(v_i)) \tag{11}$$

$$\tilde{l}[v_{i+1}, v_i, \tilde{\beta}(v_{i+1})] - \tilde{f}_{\tilde{P}} = \tilde{l}[v_{i+1}, v_i, \tilde{\beta}(v_{i+1})] \tag{12}$$

- If (v_i, v_{i+1}) is an artificial arc. Let

$$\tilde{l}[v_i, v_{i+1}, \tilde{\tau}(v_i)] + \tilde{f}_{\tilde{P}} = \tilde{l}(v_i, v_{i+1}, \tilde{\tau}(v_i)) \tag{13}$$

$$\tilde{l}(v_{i+1}, v_i, \tilde{\beta}(v_{i+1})) - \tilde{f}_{\tilde{P}} = \tilde{l}(v_{i+1}, v_i, \tilde{\beta}(v_{i+1})) \tag{14}$$

For $i = 1, \ldots, n-1$, the update of the fuzzy capacity node is given by:

- If the fuzzy waiting time at node v_i is positive $(\widetilde{w}(v_i) \succ 0)$. Let

$$\tilde{l}(v_i, \tilde{t}_{v_i}) + \tilde{f}_p = \tilde{l}(v_i, \tilde{t}_{v_i}),\ \tilde{t}_{v_i} = \tilde{\tau}(v_i) \tag{15}$$

$$\tilde{l}[v_i, \tilde{t}_{v_i}] - \tilde{f}_p = \tilde{l}[v_i, \tilde{t}_{v_i}],\ \tilde{t}_{v_i} = \tilde{\beta}(v_i). \tag{16}$$

The fuzzy network $\tilde{G} = (N, A, \tilde{l}, \tilde{b}, \tilde{c}_r, \tilde{c}_w, [\tilde{a}_{v_i}, \tilde{b}_{v_i}])$ generated by the fuzzy network updating the procedure above is said to be a fuzzy dynamic residual network of the fuzzy time-varying and the fuzzy time-windows. The optimization problem in the original fuzzy network of the fuzzy dynamic residual network with a fuzzy time-varying and a fuzzy time-windows is equivalent to the sense that there is a one-to-one correspondence between their feasible solutions.

In the original fuzzy network, we assume that all fuzzy transit times $\tilde{b}(v_i, v_{i+1}, \tilde{t}_{v_{i+1}}) \succ 0$. Thus, the first fuzzy dynamic f-augmenting path will only contain the arcs with positive fuzzy transit times $\tilde{b}(v_i, v_{i+1}, \tilde{t}_{v_{i+1}}) \succ 0$ and a positive fuzzy waiting time $\widetilde{w}(v_i, \tilde{t}_{v_i}) \succ 0$. But in a fuzzy dynamic residual network of the fuzzy time-varying and fuzzy time-windows, a fuzzy transit time associated with an artificial arc is negative number, and a fuzzy flow can be stored at the node for a negative waiting time. Therefore, a fuzzy dynamic f-augmenting path can be found in the fuzzy dynamic residual network with a fuzzy time-varying and a fuzzy time-window may contain some arcs of the negative transit times and negative waiting time $(w(v_i) \prec 0)$.

Definition 3.1 Let $\tilde{\beta}(v_i), \tilde{\tau}(v_i)$ and $w(v_i)$, $i = 1, \ldots, n$, where a negative waiting time $(w(v_i) \prec 0)$, the fuzzy path $\tilde{P}(s, v_i)$ is said to be a fuzzy f-augmenting path from s to v_i. If for $i = 2, \ldots, n$, $\tilde{t}_{v_i} \in [\tilde{a}_{v_i}, \tilde{b}_{v_i}]$ then, for each node $v_i \in N \backslash \{s, \rho\}$ it satisfies:

$$\tilde{\tau}(v_{i-1}) + b[v_{i-1}, v_i, \tilde{\tau}(v_{i-1})] = \tilde{\beta}(v_i) \tag{17}$$

$$\tilde{\beta}(v_i) + w(v_i) = \tilde{\tau}(v_i) \tag{18}$$

$$\tilde{l}[v_{i-1}, v_i, \tilde{\tau}(v_{i-1})] \succ 0,\quad \text{if } (v_{i-1}, v_i) \text{ is an artificial arc} \tag{19}$$

$$\tilde{l}[v_{i-1}, \tilde{t}_{i-1}] \succ 0,\ \tilde{t}_{v_{i-1}} = \tilde{\beta}(v_{i-1}), \tag{20}$$

if $w(v_{i-1}) \prec 0$, where $0 \le \tilde{\beta}(v_i) \le \tilde{T}, 0 \le \tilde{\tau}(v_i) \le \tilde{T}$; for $i = 2, \ldots, n$.

For a node $v_i \in N \backslash \{s\}$ with a fuzzy time-varying and a fuzzy time-windows, $\tilde{t}_{v_i} \in [\tilde{a}_{v_i}, \tilde{b}_{v_i}]$, $\tilde{t}_{v_i} \le \tilde{T}$ the fuzzy cost of the fuzzy shortest dynamic f-incrementing path from s to v_i with the fuzzy time $\tilde{t}_{v_i}$ is said to be infinity if**;**

- there is no a fuzzy path from s to v_i, or
- all the fuzzy paths from s to v_i either have a fuzzy time greater than $\tilde{t}_{v_i}$ or a violate some other constraints and thus infeasible.

4. A Mathematical Model of the FMCFPFTW

- A mathematical model of the FMCFPFTW is given by the following:

$$\min \sum_{(v_i,v_{i+1})\in A}\sum_{\tilde{t}_{v_i}\le T;\tilde{a}_{v_i}\le \tilde{t}_{v_i}\le \tilde{b}_{v_i}} \tilde{c}_r(v_i,v_{i+1},\tilde{t}_{v_i})\tilde{f}(v_i,v_{i+1},\tilde{t}_{v_i}) + \sum_{v_i\in N}\sum_{\tilde{t}_{v_i}\le T;\tilde{a}_{v_i}\le \tilde{t}_{v_i}\le \tilde{b}_{v_i}} \tilde{c}_w(v_i,\tilde{t}_{v_i})\tilde{f}(v_i,\tilde{t}_{v_i}) \quad (21)$$

Subject to:

$$\sum_{(s,v_i)\in A}\sum_{\tilde{t}_{v_i}\le T}\tilde{f}(s,v_i,\tilde{t}_{v_i}) = \sum_{(v_i,\rho)\in A}\sum_{\{\tilde{t}_{v_i}:\tilde{t}_{v_i}+\tilde{b}(v_i,\rho,\tilde{t}_{v_i})\le T\}}\tilde{f}(v_i,\rho,\tilde{t}_{v_i});\forall v_i\in N,\ \tilde{t}_{v_i}\in[0,\tilde{T}] \quad (22)$$

$$\sum_{(v_i,v_{i+1})\in A;\tilde{t}'_{v_i}+\tilde{b}(v_i,v_{i+1},\tilde{t}'_{v_i})=\tilde{t}_{v_i}}\tilde{f}(v_i,v_{i+1},\tilde{t}'_{v_i}) + \tilde{f}(v_{i+1},\tilde{t}_{v_{i+1}}) = \sum_{(v_{i+1},v_i)\in A}\tilde{f}(v_{i+1},v_i,\tilde{t}_{v_{i+1}}) \quad (23)$$

$$0\le \tilde{f}(v_i,v_{i+1},\tilde{t}_{v_{i+1}})\le \tilde{l}(v_i,v_{i+1},\tilde{t}_{v_{i+1}});\forall v_i,v_{i+1}\in N\backslash\{s,\rho\},\tilde{t}_{v_{i+1}}\in[\tilde{a}_{v_{i+1}},\tilde{b}_{v_{i+1}}];i=1,\dots,n \quad (24)$$

$$0\le \tilde{f}(v_i,\tilde{t}_{v_i})\le \tilde{l}(v_i,\tilde{t}_{v_i}),\forall(v_i,v_{i+1})\in A,\tilde{t}_{v_i}\in[\tilde{a}_{v_i},\tilde{b}_{v_i}];i=1,\dots,n. \quad (25)$$

Equation (21) is the objective function that minimizes the total fuzzy cost. Equation (22) is a fuzzy flow value departure from the source node s and arrival to the sink node ρ with the traveling fuzzy time $\tilde{t}_{v_i}\le\tilde{T}$. Equation (23) is fuzzy flow conservation. Equations (24) and (25) are the fuzzy capacity constraints on the arcs and nodes respectively. All constraints are satisfying a fuzzy time-varying and a fuzzy time-windows constraint.

5. An Algorithm of the FMCFPFTW

Let $\tilde{G}=(N,A,\tilde{l},\tilde{b},\tilde{c}_r,\tilde{c}_w,[\tilde{a}_{v_i},\tilde{b}_{v_i}])$ be a fuzzy network of the fuzzy time-varying and a fuzzy time-windows with nonzero fuzzy transit times, an arbitrary fuzzy costs and no negative cycles. We will develop an algorithm to find the fuzzy shortest dynamic f-augmenting path from s to ρ with the fuzzy time at most $\tilde{t}_{v_i}\le\tilde{T}$, where no fuzzy waiting time $(\tilde{w}(\mathrm{v_i})=0)$, $\forall v_i\in N\backslash\{s,\rho\}, i=1,\dots,n$ is permitted at any node. We will develop a procedure, called a fuzzy shortest dynamic f-augmenting path of the fuzzy time-varying and a fuzz time-window searching the process for the zero-fuzzy waiting time, which contains a two different searching operation:

- the first is a forward-searching,
- the second is a backward-searching.

Both operations are designed by using a fuzzy dynamic programming method, and the forward-searching is to deal with positive fuzzy transit times while the backward searching will deal a negative transit time.

Definition 5.1 Let $\tilde{P}(s,v_i)$ be a fuzzy dynamic f-augmenting path with the fuzzy time-varying and a fuzzy time-windows from s to a fuzzy service time $\tilde{t}_{v_i}\in[\tilde{a}_{v_i},\tilde{b}_{v_i}]$. A section $\tilde{P}(v_i,v_{i+1})$; $1\le i\le n$ is defined as a sub-path of $\tilde{P}(s,v_i)$ where all the fuzzy transit times have the same sign. A section of $\tilde{P}(s,v_i)$ is said to be positive if it's a fuzzy transit times are all positive.

A fuzzy dynamic f-augmenting path with a fuzzy time-varying and a fuzzy time-window consists of a several positive and a negative section alternatively. The number of these sections is said to be the alternating number of $\tilde{P}(s,v_i)$.

Definition 5.2 Let $\tilde{d}_z(v_i,\tilde{t}_{v_i})^k$ is the fuzzy length of the fuzzy shortest dynamic f-augmenting path with the fuzzy time-varying and a fuzzy time-windows from the source node s to the node v_i of a fuzzy time exactly $\tilde{t}_{v_i}\le\tilde{T}$, with the alternating number at most k.

Since there are no negative cycles in the fuzzy network $\tilde{G}$, a fuzzy shortest dynamic f-augmenting path $\tilde{P}$ cannot contain more than n nodes and each node cannot be visited more than once at any fuzzy time $\tilde{t}_{v_i}$. Therefore, $\tilde{P}$ cannot contain more than $n\tilde{T}$ sections. In other words, $\tilde{d}_z(v_i,\tilde{t}_{v_i})^k$ is the fuzzy length of the fuzzy shortest dynamic f-augmenting path with the fuzzy time-varying and fuzzy time-windows from s to v_i of the fuzzy time exactly $\tilde{t}_{v_i}$ when $k\ge n\tilde{T}$.

We now describe the procedure to solve the sub-problem, in which A^+ and A^- denote the set of all positive and negative arcs, respectively.

We will present an algorithm of the FMCFPFTW, our algorithm satisfies a fuzzy time-varying and a fuzzy time-windows constraint without a fuzzy waiting time of the fuzzy shortest dynamic f-augmenting path.

i) The first: An algorithm of the fuzzy shortest dynamic f-augmenting path of a fuzzy dynamic residual network:

Begin

Initialize: $\tilde{d}_z(s,0)^0:=0$, $\tilde{d}_z(s,\tilde{t}_{v_i})^0:=\infty$, $\tilde{t}_{v_i}\le,\tilde{T}$

$\tilde{d}_z(v_{i+1},\tilde{t}_{v_{i+1}})^0:=\infty$,

$\forall v_{i+1}\in N\backslash\{s\};\tilde{t}_{v_i}\in[\tilde{a}_{v_i},\tilde{b}_{v_i}],\tilde{t}_{v_i}\le\tilde{T},i=0,\dots,n;$

Sort all values $\tilde{u}+\tilde{b}(v_i,v_{i+1},\tilde{u})$ for $\tilde{u}\in[1,\tilde{T}]$ and for all arcs $(v_i,v_{i+1})\in A^+$;

Sort all values $\tilde{u}+\tilde{b}(v_i,v_{i+1},\tilde{u})$ for $\tilde{u}\in[0,\tilde{T}-1]$ and for all arcs $[v_i,v_{i+1}]\in A^-$; $i:=0$;

Do

$i:=i+1;$

For all $v_{i+1}\in N,\tilde{t}_{v_i}\le\tilde{T}$ **do** $\tilde{d}_z(v_{i+1},\tilde{t}_{v_{i+1}})^i:=\tilde{d}_z(v_{i+1},\tilde{t}_{v_{i+1}})^{i-1}$

Case 1: i is an odd number:

For $1\le\tilde{t}_{v_i}\le\tilde{T}$ **do**

For every $v_{i+1}\in N\backslash\{s\}$ **do** the forward-searching operation:

$$\tilde{d}_z(v_{i+1},\tilde{t}_{v_{i+1}})^i := \min\{\tilde{d}_z(v_{i+1},\tilde{t}_{v_{i+1}})^i, \min_{\{v_i:(v_i,v_{i+1})\in A^+\}}\ \min_{\{\tilde{u}:\tilde{u}+\tilde{b}(v_i,v_{i+1},\tilde{u})=\tilde{t}_{v_i}\ ^\wedge\ \tilde{l}(v_i,v_{i+1},\tilde{u})>0\}} \tilde{d}_z(v_i,\tilde{u})^i+\tilde{c}_r(v_i,v_{i+1},\tilde{u})\};$$

Case 2: i is an even number:

For $\tilde{t}_{v_i}\in[\tilde{T}-1,0]$ **do**

For every $v_{i+1}\in N\backslash\{\rho\}$ **do** the backward-searching operation:

$$\tilde{d}_z(v_{i+1},\tilde{t}_{v_{i+1}})^i := \min\{\tilde{d}_z(v_{i+1},\tilde{t}_{v_{i+1}})^i, \min_{\{v_i:(v_i,v_{i+1})\in A^-\}}\min_{\{\tilde{u}:\tilde{u}+\tilde{b}([v_i,v_{i+1},\tilde{u}])=\tilde{t}_{v_i}\ ^\wedge\ \tilde{l}(v_i,v_{i+1},\tilde{u})>0\}} \tilde{d}_z(v_i,\tilde{u})^i+\tilde{c}_r([v_i,v_{i+1},\tilde{u}])\};$$

While there exists at least one $\tilde{d}_z(v_{i+1},\tilde{t}_{v_{i+1}})^i\ne\tilde{d}_z(v_{i+1},\tilde{t}_{v_{i+1}})^{i-1}$;

Let $\tilde{d}^*_z(\rho)=\min_{0\le\tilde{t}_{v_i}\le\tilde{T}}\tilde{d}_z(\rho,\tilde{t}_{v_i})^i$;

End

ii) The second: An algorithm of the FMCFPFTW:

Begin

$$\bar{v} := 0;$$

For $j = 1, \ldots, m$ **do**

Call the first algorithm;

If $\tilde{d}_z^*$ (ρ) $\prec \infty$ then call the algorithm UP-NET; (there is an f-augmenting path $\tilde{P}_j(s, \rho)$ with the fuzzy time-varying and the fuzzy time-windows of the fuzzy flow value $\tilde{f}_j = \widetilde{Cap}\left(\tilde{P}_j(s, \rho)\right)$; so update a fuzzy network)

Else stop; (no feasible solution to send all v units of the fuzzy flow from s to ρ within the fuzzy time $\tilde{t}_{v_i} \in \left[\tilde{a}_{v_i}, \tilde{b}_{v_i}\right], \tilde{t}_{v_i} \leq \tilde{T}, i = 0, \ldots, n$.

$\bar{v} := \bar{v} + \tilde{f}_j$;

If $\bar{v} \geq v$ then stop;

End

6. Conclusion

This paper presents a new version of the Minimum Cost Flow Problem (MCFP), a new version is the Fuzzy Minimum Cost Flow Problem with the Fuzzy Time-Windows (FMCFPFTW). We consider a generalized fuzzy version of the MCFP. The objective is to find an optimal schedule to send a fuzzy flow from the source to the sink satisfies all constraints of the fuzzy shortest dynamic f-augmenting path of the fuzzy dynamic residual network $\tilde{G}$. The fuzzy flow must arrive at the sink before a fuzzy nonnegative deadline $\tilde{T}$. Also, we propose a mathematical formulation model of the MFCFFPFTW. Finally, an algorithm of the FMCFPFTW is presented.

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