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Exponentiated Gumbel Exponential Distribution: Properties and Applications

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Abstract In this article, we proposed a new distribution called exponentiated Gumbel exponential (EGuE) distribution. The new distribution is a member of the T-X family obtained through the logit transformation of the exponential random variable, using the exponentiated Gumbel distribution as the generator. The mathematical properties of the proposed distribution were studied. The maximum likelihood estimates of the parameters of the EGuE distribution were derived. The applicability of the new distribution is shown using a real data set.

Keywords: *exponentiated Gumbel distribution, T-X family, maximum likelihood estimation, logit function, exponential distribution*

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1. Introduction

The development of flexible distributions for modeling lifetime data has received widespread attention in recent times. This is largely due to the fact that some classical distributions are not flexible enough in modeling lifetime processes. Many methods have been proposed in the literature by statisticians for developing new distributions. Lee et al, [1] termed the recent methods of developing flexible distribution "method of Combination". This is because these methods attempt to combine existing distributions to form a new one or adding parameters to existing distribution. Some examples of this method include but not limited to the beta-generated family of distributions Eugene et al, [2], Kumaraswamy family of distributions Jones [3], Cordeiro and de Castro [4], exponentiated-G family Gupta et al. [5], transmuted-G family Shaw and Buckley [6] and generalized transmuted family Alizadeh et al. [7].

Alzatreeh et al. [8] introduced the Transformed-Transformer $(T-X)$ family which addressed one of the limitations of the beta class by allowing for any continuous distribution to be used as a generator. Let T be a random variable in the interval $[c,d]$, $-\infty < c < d < \infty$ with probability density function (pdf) and cumulative density function (cdf), $r(t)$ and $R(t)$ respectively. Let the function of the cdf $F(x)$ of a random variable X be defined as $V(F(x))$ such that $V(F(x))$ meets the conditions stated below:

I. $V(F(x)) \in [c,d]$.

II. $V(F(x))$ is differentiable and monotonically non-decreasing.

III. $V(F(x)) \to c$ as $x \to -\infty$ and $V(F(x)) \to d$ as $x \to \infty$

The cdf of the $T-X$ family is given by

$$G(x) = \int_c^{V(F(x))} r(t)\,dt = R(V(F(x))) \tag{1}$$

The pdf corresponding to (1) above is

$$g(x) = r(V(F(x)))\frac{d}{dx}(V(F(x))) \tag{2}$$

Let $V(F(x))$ be the logit of $F(x)$ i.e $V(F(x)) = \ln(F(x)/1-F(x))$

Hence (1) and (2) can be written in terms of the logit of $F(x)$ as

$$G(x) = \int_{-\infty}^{\ln(F(x)/1-F(x))} r(t)\,dt = R(\ln(F(x)/1-F(x))) \tag{3}$$

and

$$g(x) = r(\ln(F(x)/1-F(x))) \times \frac{f(x)}{F(x)(1-F(x))} \tag{4}$$

respectively.

In this paper, we introduced the exponentiated Gumbel exponential $(EGuE)$ distribution using the $T-X$ approach introduced by Alzatreeh et al. [8]. The generator used is exponentiated Gumbel distribution while the baseline distribution $F(x)$ is the exponential distribution. The mathematical properties of the new distribution are extensively explored. Maximum likelihood estimates of the new distribution are derived and the potentiality of the proposed model is illustrated using a lifetime data set.

The rest of this paper is organized as follows. Section 2 introduces the exponentiated Gumbel exponential distribution. Several mathematical properties of the proposed model such as the quantile function, shapes of the pdf and hrf, moments, entropy, order statistics and inequality measures are discussed in Section 3. Estimation of the parameters of the model and application of the new distribution to real data set is done in Sections 4 and 5 respectively. Section 6 concludes the paper.

2. Exponentiated Gumbel Exponential Distribution

Suppose that T is distributed as exponentiated Gumbel distribution with cdf and pdf respectively given by Nadarajah [9]

$$R(t)=1-\left[1-\exp\left\{-\exp\left(-\frac{t-\mu}{\sigma}\right)\right\}\right]^{\alpha}$$

and

$$r(t)=\frac{\alpha}{\sigma}\left[1-\exp\left\{-\exp\left(-\frac{t-\mu}{\sigma}\right)\right\}\right]^{\alpha-1}$$
$$\times\exp\left\{-\exp\left(-\frac{t-\mu}{\sigma}\right)\right\}\exp\left(-\frac{t-\mu}{\sigma}\right)$$
$$-\infty<x<\infty,\ -\infty<\mu<\infty,\ \sigma>0,\ \alpha>0$$

Let the random variable X be exponentially distributed with cdf and pdf given respectively by

$$F(x)=1-e^{-\theta x}$$

$$f(x)=\theta e^{-\theta x},x>0,\theta>0$$

The logit of $F(x)$ is

$\ln\left(F(x)/1-F(x)\right)=\ln\left(e^{\theta x}-1\right)$. Thus using (3), the cdf $EGuE$ is given by

$$G(x)=1-\left[1-exp\left\{-B\left(e^{\theta x}-1\right)^{-1/\sigma}\right\}\right]^{\alpha} \tag{5}$$

where $B=\exp\left(\frac{\mu}{\sigma}\right)$

The pdf associated with (5) is obtained by taking the derivative of (5) with respect to the random variable x. Hence the pdf of $EGuE$ distribution is

$$g(x)=\frac{\alpha\theta B}{\sigma}\left(1-e^{-\theta x}\right)^{-1/\sigma-1}exp\left[-B\left(e^{\theta x}-1\right)^{-1/\sigma}\right]$$
$$\times e^{-\left(\frac{\theta}{\sigma}\right)x}\left[1-exp\left\{-B\left(e^{\theta x}-1\right)^{-1/\sigma}\right\}\right]^{\alpha-1} \tag{6}$$

The survival function of $EGuE$ is given by

$$S(x)=\left[1-exp\left\{-B\left(e^{\theta x}-1\right)^{-1/\sigma}\right\}\right]^{\alpha}$$

The hazard rate function $h(x)$, reversed - hazard rate function $R(x)$ and cumulative hazard rate function $H(x)$ of $EGuE$ distribution are respectively given by

$$h(x)=\frac{\left\{\begin{array}{l}\frac{\alpha\theta B}{\sigma}e^{-\left(\frac{\theta}{\sigma}\right)x}\left(1-e^{-\theta x}\right)^{-1/\sigma-1}exp\\ \left[\left[-\left(\frac{\theta}{\sigma}x+B\left(e^{\theta x}-1\right)^{-1/\sigma}\right)\right]\right]\end{array}\right\}}{1-exp\left\{-B\left(e^{\theta x}-1\right)^{-1/\sigma}\right\}}$$

$$R(x)=\frac{\left\{\begin{array}{l}\frac{\alpha\theta B}{\sigma}e^{-\left(\frac{\theta}{\sigma}\right)x}\left(1-e^{-\theta x}\right)^{-1/\sigma-1}exp\\ \left[\left[-B\left(e^{\theta x}-1\right)^{-1/\sigma}\right]\right]\end{array}\right\}}{1-\left[1-exp\left\{-B\left(e^{\theta x}-1\right)^{-1/\sigma}\right\}\right]^{\alpha}}$$
$$\times\left[1-exp\left\{-B\left(e^{\theta x}-1\right)^{-1/\sigma}\right\}\right]^{\alpha-1}$$

and

$$H(x)=-\alpha\ln\left(1-\left[1-exp\left\{-B\left(e^{\theta x}-1\right)^{-1/\sigma}\right\}\right]\right)$$

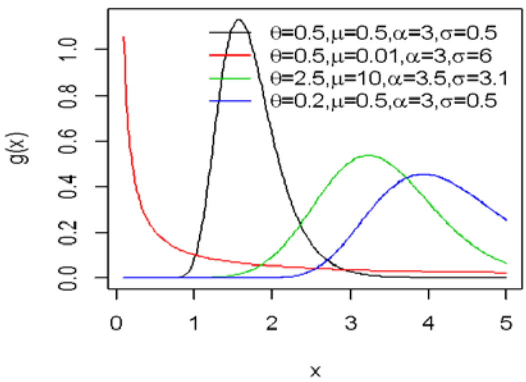

Figure 1. Plots of pdf of the EGuE distribution for selected parameter values

The plots of the pdf, cdf, and $h(x)$ of $EGuE$ for selected parameter values are displayed in Figure 1,

Figure 2 and Figure 3 respectively.

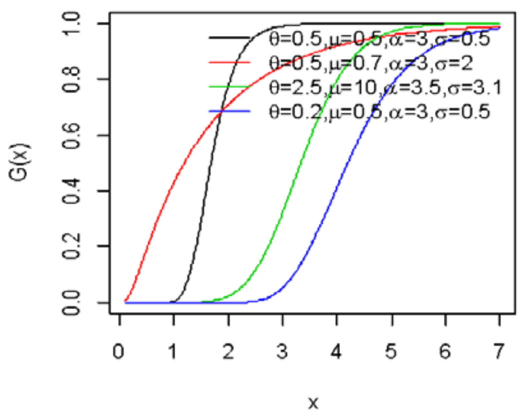

Figure 2. Plots of cdf of the EGuE distribution for selected parameter values

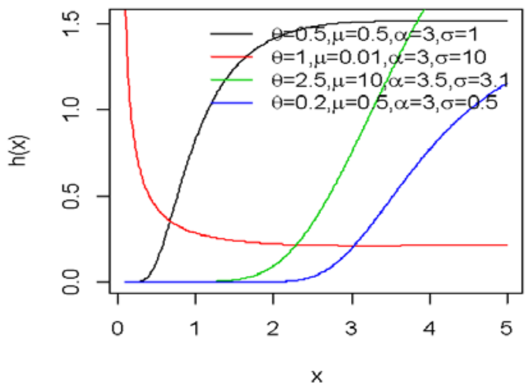

Figure 3. Plots of h(x) of the EGuE distribution for selected parameter values

3. Mathematical Properties of the (EGuE) Distribution

3.1. Quantile Function

The quantile function of X which follows the *EGuE* distribution is given by

$$Q(u)=\frac{1}{\theta}\log\left[\left\{-\frac{1}{B}\log\left[1-(1-u)^{\frac{1}{\alpha}}\right]\right\}^{-\sigma}+1\right] \qquad (7)$$

Proof:

The quantile function of *EGuE* is obtained by inverting (5). Equating the random variable u to (5) we have

$$u=1-\left[1-exp\left\{-B\left(e^{\theta x}-1\right)^{-1/\sigma}\right\}\right]^{\alpha}$$

$$\left[1-exp\left\{-B\left(e^{\theta x}-1\right)^{-1/\sigma}\right\}\right]^{\alpha}=(1-u)^{\frac{1}{\alpha}}$$

$$B\left(e^{\theta x}-1\right)^{-1/\sigma}=-\log\left(1-(1-u)^{\frac{1}{\alpha}}\right)$$

$$\left(e^{\theta x}-1\right)=\left\{-\frac{1}{B}\log\left(1-(1-u)^{\frac{1}{\alpha}}\right)\right\}^{-\sigma}$$

$$X=\frac{1}{\theta}\log\left[\left\{-\frac{1}{B}\log\left(1-(1-u)^{\frac{1}{\alpha}}\right)\right\}^{-\sigma}+1\right]=Q(u)$$

Substituting $u=\frac{1}{2}$ in (7) gives the median of *EGuE* distribution.

$$Q\left(\frac{1}{2}\right)=\frac{1}{\theta}\log\left[\left\{-\frac{1}{B}\log\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}\right)\right\}^{-\sigma}+1\right] \qquad (8)$$

The random variable u in (7) is uniformly distributed in the interval $(0,1)$. $Q(u)$ can be used to simulate a random sample of *EGuE* distribution. The skewness and Kurtosis of the proposed distribution can be studied using measures based on quantiles. The Galton [10] skewness (S) and Moor [11] Kurtosis (K) are usually used for this purpose.

These measures of skewness and kurtosis exist even when the moments of the distribution do not exist and they are not sensitive to outliers.Alizadeh [12]. These are some of the advantages of these measures over the ones based on moments. The expression for obtaining the Galton skewness (S) and Moor's kurtosis (K) respectively are given by

$$S=\frac{Q\left(3/4\right)+Q\left(1/4\right)-2Q\left(1/2\right)}{Q\left(3/4\right)-Q\left(1/4\right)} \qquad (9)$$

$$K=\frac{Q\left(7/8\right)-Q\left(5/8\right)+Q\left(3/8\right)-Q\left(1/8\right)}{Q\left(6/8\right)-Q\left(2/8\right)} \qquad (10)$$

3.2. Shapes of Pdf and Hrf

The shapes of the *pdf* and $h(x)$ can be described analytically. This can be done by taking their log, differentiating with respect to x and equating to zero. The shape of the *pdf* of *EGuE* can be described by

$$\left\{\begin{bmatrix}\dfrac{\theta Be^{-\theta x}}{\sigma}\left(e^{\theta x}-1\right)^{-1/\sigma-1} \\ \left[1-\dfrac{(\alpha-1)exp\left\{-B\left(e^{\theta x}-1\right)^{-1/\sigma}\right\}}{1-exp\left\{-B\left(e^{\theta x}-1\right)^{-1/\sigma}\right\}}\right]\end{bmatrix}\right\} \qquad (11)$$

$$-\theta\left(1+\frac{\left(\frac{1}{\sigma}+1\right)e^{-\theta x}}{\left(1-e^{-\theta x}\right)}+\left(\frac{1}{\sigma}-1\right)\right)=0$$

(11) may have more than one root. If $x=x_0$ is a root of (11), then it corresponds to a local maximum, local minimum or point of inflexion depending on

whether $\zeta(x_0)<0$, $\zeta(x_0)>0$ or $\zeta(x_0)=0$ where $\zeta(x_0)=\dfrac{d^2\log(g(x))}{dx^2}$.

The shape of the $h(x)$ of $EGuE$ distribution is described by

$$\left\{\begin{array}{l}\dfrac{\theta B e^{-\left(\frac{\theta}{\sigma}\right)x}\left(1-e^{-\theta x}\right)^{-1/\sigma-1}}{\sigma} \\ \left[1+\dfrac{exp\left\{-B\left(e^{\theta x}-1\right)^{-1/\sigma}\right\}}{1-exp\left\{-B\left(e^{\theta x}-1\right)^{-1/\sigma}\right\}}\right]\end{array}\right\} \tag{12}$$

$$-\theta\left(1+\left(\frac{1}{\sigma}-1\right)+\frac{\left(\frac{1}{\sigma}+1\right)e^{-\theta x}}{\left(1-e^{-\theta x}\right)}\right)=0$$

The roots of (12) may be more than one. If $x=x_0$ is a root of (12), then it corresponds to a local maximum, local minimum or point of inflexion depending on whether $\vartheta(x_0)<0$, $\vartheta(x_0)>0$ or $\vartheta(x_0)=0$ where $\vartheta(x_0)=\dfrac{d^2\log(h(x))}{dx^2}$.

3.3. Useful Expansions

A representation of the *pdf* and *cdf* of $EGuE$ distribution is made in this sub-section. The *pdf* of $EGuE$ can be written as

$$\left\{\begin{array}{l} g(x)=\dfrac{\alpha\theta B}{\sigma}\left(1-e^{-\theta x}\right)^{-1/\sigma-1} \\ \underbrace{exp\left[-B\left(\dfrac{1-e^{-\theta x}}{e^{-\theta x}}\right)^{-1/\sigma}\right]}_{C}\end{array}\right\} \tag{13}$$

$$\times e^{-\left(\frac{\theta}{\sigma}\right)x}\underbrace{\left[1-exp\left\{-B\left(\frac{1-e^{-\theta x}}{e^{-\theta x}}\right)^{-1/\sigma}\right\}\right]^{\alpha-1}}_{A}$$

Applying general binomial expansion (14) to A in (13)

$$(1-z)^{\beta-1}=\sum_{i=0}^{\infty}(-1)^i\binom{\beta-1}{i}z^i, \quad \beta>0 \ and \ |z|<1 \tag{14}$$

we have

$$A=\sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}exp\left\{-Bi\left(\frac{1-e^{-\theta x}}{e^{-\theta x}}\right)^{-1/\sigma}\right\}$$

and

$$AC=\sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}exp\left\{-B(i+1)\left(\frac{1-e^{-\theta x}}{e^{-\theta x}}\right)^{-1/\sigma}\right\}$$

Applying power series expansion for exponential functions to AC

$$AC=\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{j!}\binom{\alpha-1}{i}B^j(i+1)^j\left(\frac{1-e^{-\theta x}}{e^{-\theta x}}\right)^{-j/\sigma}$$

Substituting for AC in (13)

$$g(x)=\frac{\alpha\theta B}{\sigma}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{j!}\binom{\alpha-1}{i}B^j(i+1)^j \times\underbrace{\left(1-e^{-\theta x}\right)^{-\left(\frac{1}{\sigma}(j+1)+1\right)}}_{D}e^{-\frac{\theta}{\sigma}(j+1)x}$$

Applying the general binomial expansion for negative powers to D we have

$$g(x)=\frac{\alpha\theta B}{\sigma}\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j}}{j!}\binom{\alpha-1}{i}\binom{\frac{1}{\sigma}(j+1)+k}{k}\times B^j(i+1)^j e^{-\theta\left(\frac{1}{\sigma}(j+1)+k\right)x}$$

Thus the $EGuE$ density can be represented as infinite linear combination of exponential distribution. Therefore

$$g(x)=\sum_{i,j,k=0}^{\infty}W_{ijk}e^{-\theta\left(\frac{1}{\sigma}(j+1)+k\right)x} \tag{15}$$

where

$$W_{ijk}=\frac{\alpha\theta(-1)^{i+j}}{\sigma j!}\binom{\alpha-1}{i}\binom{\frac{1}{\sigma}(j+1)+k}{k}B^{j+1}(i+1)^j$$

The expansion of cumulative density $EGuE$ distribution is obtained as follows

$$\left(G(x)\right)^h=\left(1-\left[1-exp\left\{-B\left(\frac{1-e^{-\theta x}}{e^{-\theta x}}\right)^{-1/\sigma}\right\}\right]^{\alpha}\right)^h$$

where h is a positive integer.

$$\left(G(x)\right)^h=\sum_{p=0}^{h}(-1)^p\binom{h}{p}\left[1-exp\left\{-B\left(\frac{1-e^{-\theta x}}{e^{-\theta x}}\right)^{-1/\sigma}\right\}\right]^{\alpha p}$$

$$\left(G(x)\right)^h = \sum_{p=0}^{h}\sum_{q=0}^{\infty}(-1)^{p+q}\binom{h}{p}\binom{\alpha p}{q}\exp\left\{-Bq\left(\frac{1-e^{-\theta x}}{e^{-\theta x}}\right)^{-1/\sigma}\right\}$$

$$\left(G(x)\right)^h = \sum_{p=0}^{h}\sum_{q,m=0}^{\infty}\frac{(-1)^{p+q+m}}{m!}\binom{h}{p}\binom{\alpha p}{q} \times (Bq)^m \underbrace{\left(1-e^{-\theta x}\right)^{-m/\sigma}}_{E} e^{-\frac{m\theta x}{\sigma}} \quad (16)$$

Applying general binomial expansion for negative powers to E in (16) reduces to

$$\left(G(x)\right)^h = \sum_{p=0}^{h}\sum_{q,m,l=0}^{\infty}\frac{(-1)^{p+q+m}}{m!}\binom{h}{p}\binom{\alpha p}{q}\binom{\frac{m}{\sigma}+l-1}{l} \times (Bq)^m e^{-\theta\left(\frac{m}{\sigma}+l\right)x}$$

$$\left(G(x)\right)^h = \sum_{p=0}^{h}\sum_{q,m,l=0}^{\infty} V_{pqml}\exp\left(-\theta\left(\frac{m}{\sigma}+l\right)x\right) \quad (17)$$

where

$$V_{pqml} = \frac{(-1)^{p+q+m}}{m!}\binom{h}{p}\binom{\alpha p}{q}\binom{\frac{m}{\sigma}+l-1}{l}(Bq)^m$$

3.4. Moments

If X is distributed as (6), then its *rth* noncentral moment is obtained as follows

$$\mu_r = E\left(X^r\right) = \int_{-\infty}^{\infty} x^r g(x)\,dx \quad (18)$$

Substituting (15) in (18)

$$\mu_r = \sum_{i,j,k=0}^{\infty} W_{ijk}\int_0^{\infty} x^r e^{-\theta\left(\frac{1}{\sigma}(j+1)+k\right)x}dx$$

$$\mu_r = \sum_{i,j,k=0}^{\infty} W_{ijk}\frac{\theta^{-(r+1)}\Gamma(r+1)}{\left(\frac{1}{\sigma}(j+1)+k\right)^{r+1}} \quad (19)$$

where W_{ijk} is as defined in (15) and $\Gamma(.)$ is a gamma function.

The moment generating function, $M_X(t)$ of *EGuE* distribution is given by

$$M_X(t) = E\left(e^{tX}\right) = E\left[\sum_{r=0}^{\infty}\frac{t^r X^r}{r!}\right] = \sum_{r=0}^{\infty}\frac{t^r}{r!}E\left(X^r\right) \quad (20)$$

Substituting (19) into (20) we have

$$M_X(t) = \sum_{r,i,j,k=0}^{\infty}\frac{t^r}{r!}W_{ijk}\frac{\theta^{-(r+1)}\Gamma(r+1)}{\left(\frac{1}{\sigma}(j+1)+k\right)^{r+1}} \quad (21)$$

3.5. Renyi and q-Entropies

The Renyi entropy of a random variable X is the measure of variation of uncertainty. The Renyi entropy is given by

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\left\{\int_0^{\infty} g^{\gamma}(x)\,dx\right\} \quad for\ \gamma > 0\ and\ \gamma \neq 1 \quad (22)$$

To obtain the Renyi entropy of the *EGuE* distribution, we substitute (6) in (22) and apply the general binomial expansion and power series expansion for exponential function. Hence,

$$g^{\gamma}(x) = \left(\frac{\alpha\theta B}{\sigma}\right)^{\gamma}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{j!}\binom{\gamma(\alpha-1)}{i}B^j(\gamma+1)^j \times \exp\left(-\frac{\theta}{\sigma}(\gamma+j)x\right)\left(1-e^{-\theta x}\right)^{-\gamma\left(1/\sigma+1\right)-\frac{j}{\sigma}}$$

Applying the general binomial expansion again

$$g^{\gamma}(x) = \left(\frac{\alpha\theta B}{\sigma}\right)^{\gamma}\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j}}{j!}B^j(\gamma+1)^j \times \binom{\gamma(\alpha-1)}{i}\binom{\frac{1}{\sigma}(\gamma+j)+(\gamma+k)-1}{k} \times \exp\left(-\theta\left(\frac{1}{\sigma}(\gamma+j)+k\right)x\right)$$

$$g^{\gamma}(x) = \sum_{i,j,k=0}^{\infty} V_{ijk}\exp\left(-\theta\left(\frac{1}{\sigma}(\gamma+j)+k\right)x\right)$$

where

$$V_{ijk} = \left(\frac{\alpha\theta B}{\sigma}\right)^{\gamma}\frac{(-1)^{i+j}}{j!}B^j(\gamma+1)^j\binom{\gamma(\alpha-1)}{i}\binom{\frac{1}{\sigma}(\gamma+j)+(\gamma+k)-1}{k}$$

$$\int_0^{\infty} g^{\gamma}(x) = \sum_{i,j,k=0}^{\infty} V_{ijk}\int_0^{\infty}\exp\left(-\theta\left(\frac{1}{\sigma}(\gamma+j)+k\right)x\right)dx$$

$$\int_0^{\infty} g^{\gamma}(x) = \sum_{i,j,k=0}^{\infty} V_{ijk}\left(\theta\left(\frac{1}{\sigma}(\gamma+j)+k\right)\right)^{-1}$$

Hence the Renyi entropy of *EGuE* distribution is given by

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\left\{\sum_{i,j,k=0}^{\infty} V_{ijk}\left(\theta\left(\frac{1}{\sigma}(\gamma+j)+k\right)\right)^{-1}\right\}$$

The q-entropy is given by

$$H_q(X) = \frac{1}{1-q}\log\left(1-\int_0^\infty g^q(x)\,dx\right), q>0 \quad and \quad q\neq 1$$

$$H_q(X) = \frac{1}{1-q}\log\left(1-\sum_{i,j,k=0}^{\infty} V_{ijk}\left(\theta\left(\frac{1}{\sigma}(q+j)+k\right)\right)^{-1}\right)$$

3.6. Order Statistics

Let $X_1, X_2 \cdots X_n$ be a random sample from the $EGuE$ distribution. Then the pdf of the rth order statistic can be expressed as

$$g_{r;n}(x) = \frac{\sum_{l_1=0}^{n-r}(-1)^{l_1}\binom{n-r}{l_1}g(x)G(x)^{l_1+r-1}}{B(r,n-r+1)} \quad (23)$$

where $B(.,.)$ is the beta function. Substituting (15) and (17) into (23) and replacing h with l_1+r-1 we have

$$g_{r;n}(x) = \frac{\sum_{l_1=0}^{n-r}\sum_{i,j,k,q,m,l=0}^{\infty}\sum_{p=0}^{l_1+r-1}C_{ijkqmll_1p}}{B(r,n-r+1)} \times \exp\left(-\theta x\left\{\frac{1}{\sigma}(j+m+1)+k+l\right\}\right)$$

where

$$C_{ijkqmll_1p} = \sum_{p=0}^{l_1+r-1}\frac{(-1)^{l_1+p+q+m}}{m!}\binom{n-r}{l_1}\binom{l_1+r-1}{p}\binom{\alpha p}{q} \times \binom{\frac{m}{\sigma}+l-1}{l}(Bq)^m W_{ijk}$$

Hence, the rth order statistic of $EGuE$ distribution may be expressed as a mixture of exponential density with parameter $\theta\left\{\frac{1}{\sigma}(j+m+1)+k+l\right\}$

The kth raw moment of the order statistic of the $EGuE$ distribution is

$$E\left(X_{r;n}^k\right) = \int_{-\infty}^{\infty} x^k g_{r;n}(x)\,dx$$

$$E\left(X_{r;n}^k\right) = \frac{\sum_{l_1=0}^{n-r}\sum_{i,j,k,q,m,l=0}^{\infty}\sum_{p=0}^{l_1+r-1}C_{ijkqmll_1p}}{B(r,n-r+1)} \times \int_0^{\infty} x^k \exp\left(-\theta x\left\{\frac{1}{\sigma}(j+m+1)+k+l\right\}\right)dx$$

$$E\left(X_{r;n}^k\right) = \frac{\sum_{l_1=0}^{n-r}\sum_{i,j,k,q,m,l=0}^{\infty}\sum_{p=0}^{l_1+r-1}C_{ijkqmll_1p}\Gamma(k+1)}{B(r,n-r+1)\left(\theta\left\{\frac{1}{\sigma}(j+m+1)+k+l\right\}\right)^{k+1}}$$

3.7. Incomplete Moments

The rth incomplete moment $M_r(z)$ is given by

$$M_r(z) = \int_0^z x^r g(x)\,dx \quad (24)$$

Substituting (15) in (24), the rth incomplete moment of $EGuE$ distribution becomes

$$M_r(z) = \sum_{i,j,k=0}^{\infty} W_{ijk}\int_0^z x^r e^{-\theta\left(\frac{1}{\sigma}(j+1)+k\right)x}dx$$

$$M_r(z) = \frac{\sum_{i,j,k=0}^{\infty} W_{ijk}\nu\left((r+1),\left(\frac{1}{\sigma}(j+1)+k\right)\theta z\right)}{\left(\theta\left(\frac{1}{\sigma}(j+1)+k\right)\right)^{r+1}} \quad (25)$$

where $\nu(r,z) = \int_0^z x^{r-1}e^{-x}dx$ is the lower incomplete gamma function.

3.8. Probability Weighted Moments (PWM)

The following relation may be used to obtain the probability weighted moments of a random variable.

$$\tau_{r,s} = E\left[X^r G^h(x)\right] = \int_{-\infty}^{\infty} x^r g(x) G^h(x)\,dx \quad (26)$$

Substituting (15) and (17) in (26) and replacing h with s we have the PWM of $EGuE$ as

$$\tau_{r,s} = \frac{\sum_{i,j,k=0}^{\infty}\sum_{p=0}^{s}\sum_{qml=0}^{\infty} W_{ijk}V_{pqml}}{\theta^{r+1}\left[\frac{1}{\sigma}(j+m+1)+(m+l)\right]^{r+1}}\Gamma(r+1)$$

3.9. Mean Deviations

The spread from the center of a population can be measured using the deviation from mean or deviation from the median. Letting the mean deviation from the mean, and mean deviation from the median be D_μ and D_M respectively. The mean deviation about mean is given by

$$D(\mu) = E(|X-\mu|) = \int_0^\infty |X-\mu|\, g(x)dx = 2\mu G(\mu) - 2\int_0^\mu x g(x)\,dx \quad (27)$$

Using the result of the lower incomplete moment in (25) we have

$$D(\mu) = 2\mu G(\mu) - \frac{2\sum_{i,j,k=0}^{\infty} W_{ijk}\nu\left(2,\left(\frac{1}{\sigma}(j+1)+k\right)\theta\mu\right)}{\left(\theta\left(\frac{1}{\sigma}(j+1)+k\right)\right)^2}$$

Letting $\int_0^{\mu} xg(x)dx = J(\mu)$. Hence (27) can be written as

$$D(\mu) = 2\mu G(\mu) - 2J(\mu)$$

where

$$J(\mu) = \frac{\sum_{i,j,k=0}^{\infty} W_{ijk} \nu\left(2, \left(\frac{1}{\sigma}(j+1)+k\right)\theta\mu\right)}{\left(\theta\left(\frac{1}{\sigma}(j+1)+k\right)\right)^2}$$

For the mean deviation from the median, we have

$$D(M) = E(|X-M|) = \int_0^{\infty} |X-M| g(x) dx$$

$$= \mu - 2\int_0^{M} xg(x)dx$$

$$D(M) = \mu - 2J(M)$$

where

$$J(M) = \frac{\sum_{i,j,k=0}^{\infty} W_{ijk} \nu\left(2, \left(\frac{1}{\sigma}(j+1)+k\right)\theta M\right)}{\left(\theta\left(\frac{1}{\sigma}(j+1)+k\right)\right)^2}$$

3.10. Moment Residual Life Function

The nth moment of residual life of X is given by

$$m_n(t) = \frac{1}{S(t)} \int_t^{\infty} (x-t)^n g(x) dx \tag{28}$$

where $S(t)$ is the survival function Substituting (15) in (28) we have

$$m_n(t) = \frac{\sum_{i,j,k=0}^{\infty} \sum_{r=0}^{n} (-t)^{n-r} W_{ijk}}{S(t)}$$

$$\times \int_t^{\infty} x^r \exp\left(-\left(\frac{1}{\sigma}(j+1)+k\right)\theta x\right) dx$$

Thus the moment of the residual life function is given by

$$m_n(t) = \frac{\sum_{i,j,k=0}^{\infty} \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} t^{n-r} W_{ijk}}{S(t)\left(\left(\frac{1}{\sigma}(j+1)+k\right)\theta\right)^{r+1}}$$

$$\times \Gamma\left((r+1), \left(\frac{1}{\sigma}(j+1)+k\right)\theta t\right)$$

where $\Gamma(r,z) = \int_z^{\infty} x^{r-1} e^{-x} dx$ is the upper incomplete gamma function. The mean residual life function is obtained from (28) by substituting $n = 1$.

3.11. Inequality Measures

The Lorenz and Bonferroni curves are very important inequality measures in income and wealth distribution. The Lorenz curve for $EGuE$ distribution is given by

$$L_F(t) = \frac{\int_0^t xg(x)dt}{E(X)} \tag{29}$$

substituting (19) and (25) for $r = 1$ in (29) we have

$$L_F(t) = \frac{\theta^2 \sum_{i,j,k}^{\infty} W_{ijk} \nu\left(2, \left(\frac{1}{\sigma}(j+1)+k\right)\theta t\right)}{\sum_{i,j,k}^{\infty} W_{ijk}}$$

while and Bonferroni curve is given by

$$B_F(t) = \frac{L_F(t)}{G(t)} \tag{30}$$

$$B_F(t) = \frac{\theta^2 \sum_{i,j,k}^{\infty} W_{ijk} \nu\left(2, \left(\frac{1}{\sigma}(j+1)+k\right)\theta t\right)}{\sum_{i,j,k}^{\infty} W_{ijk} \left(1 - \left[1 - exp\left\{-B\left(e^{\theta t}-1\right)^{-1/\sigma}\right\}\right]^{\alpha}\right)}$$

4. Estimation

Given a random sample $X_1, X_2 \cdots X_n$ of size n, the log-likelihood function of $EGuE$ distribution is given by

$$\begin{aligned} l = {} & n\left(\log\alpha + \log\theta - \log\sigma + \frac{\mu}{\sigma}\right) - \frac{\theta}{\sigma}\sum_{i=1}^{n} x_i \\ & -\left(\frac{1}{\sigma}+1\right)\sum_{i=1}^{n}\log\left(1-e^{-\theta x}\right) \\ & - BS_i + (\alpha-1)\sum_{i=1}^{n}\log\left[1-exp\{-BS_i\}\right] \end{aligned} \tag{31}$$

where $S_i = \left(e^{\theta x}-1\right)^{-\frac{1}{\sigma}}$

Let $\Omega = (\alpha, \mu, \sigma, \theta)^T$ be the vector of unknown parameters, the score function associated with it is given by

$$Z(\Omega) = \left(\frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \mu}, \frac{\partial l}{\partial \sigma}, \frac{\partial l}{\partial \theta}\right)^T$$

where $\frac{\partial l}{\partial \alpha}$, $\frac{\partial l}{\partial \mu}$, $\frac{\partial l}{\partial \sigma}$, and $\frac{\partial l}{\partial \theta}$ are the partial derivatives of l with respect to α, μ, σ, and θ. The score function's elements are:

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(1-exp\{-BS_i\}\right)$$

$$\frac{\partial l}{\partial \mu}=\frac{n}{\sigma}-\frac{B}{\sigma}\sum_{i=1}^{n}S_i+\frac{B(\alpha-1)}{\sigma}\sum_{i=1}^{n}\left(\frac{S_i\exp(-BS_i)}{1-\exp(-BS_i)}\right)$$

$$\frac{\partial l}{\partial \sigma}=-n\left(\frac{\sigma+\mu}{\sigma^2}\right)+\frac{\theta}{\sigma^2}\sum_{i=1}^{n}x_i+\frac{1}{\sigma^2}\sum_{i=1}^{n}\log\left(1-e^{-\theta x}\right)$$
$$-\frac{B}{\sigma^2}\sum_{i=1}^{n}\left(S_i\left\{\log\left(e^{\theta x}-1\right)-\mu\right\}\right)+\frac{(\alpha-1)}{\sigma^2}$$
$$\times\sum_{i=1}^{n}\left(\frac{B\exp(-BS_i)\left(S_i\left(\log\left(e^{\theta x}-1\right)-\mu\right)\right)}{1-\exp(-BS_i)}\right)$$

$$\frac{\partial l}{\partial \theta}=\frac{n}{\theta}-\frac{1}{\sigma}\sum_{i=1}^{n}x_i-\left(\frac{1}{\sigma}+1\right)\sum_{i=1}^{n}\left(\frac{\theta e^{-\theta x}}{1-e^{\theta x}}\right)$$
$$+\frac{B\theta}{\sigma}\sum_{i=1}^{n}\left(\exp(\theta x)\left(e^{\theta x}-1\right)^{-\frac{1}{\sigma}-1}\right)$$
$$-\frac{(\alpha-1)B\theta}{\sigma}\sum_{i=1}^{n}\left(\frac{\exp(\theta x_i-BS_i)\left(e^{\theta x}-1\right)^{-\frac{1}{\sigma}-1}}{1-\exp(-BS_i)}\right)$$

The maximum likelihood estimates are now obtained by numerically solving $Z(\Omega)=0$ which is a system of non-linear equations. Numerical optimization methods are normally used in solving such systems of equations.

5. Application

In this section, we presented an application of *EGuE* distribution to a real data set. The fit of *EGuE* is compared to the fits of other competing models with the same baseline distribution; exponentiated Weibull exponential (EWE) Elgarhy et al. [13], Weibull exponential (WE) Oguntunde et al. [14], Kumaraswamy exponential (KE) Cordeiro and de Castro [4] and exponential (E) distributions.

Estimates of parameters of the *EGuE* distribution and other competing distributions were obtained using the method of maximum likelihood. The Kolmogorov-Smirnov (K-S), Cramer-von Mises (W^*), Anderson-Darling (A^*) statistics, Akaike information criterion (AIC) and Bayesian information criterion (BIC) are the goodness of fit criteria used to compare the fits of the distributions to the data set.

The real data set used for illustration is the survival times (in days) of 72 virulent tubercle bacilli infected guinea pigs. The data is obtained from Bjerkedal [15] and has been used in SElgarhy et al. [13]. It is right-skewed and unimodal data. The data is as shown below.

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, .08, 1.08,1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46,1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16,2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32,4.58, 5.55

The maximum likelihood estimates, standard errors of the estimates and log-likelihood values of *EGuE* and other competing models are shown in Table 1. The values of the K-S, W^*,A^* statistics, AIC and BIC are presented in Table 2.

We observe that the four-parameter *EGuE* distribution provides a better fit to the data set than the other distributions with the same baseline given that it has the lowest value in all the goodness of fit criteria considered.

The plot of the histogram and estimated *pdf* of *EGuE*, *EWE*, *WE*, *KE* and *E* distributions are displayed in Figure 4 while the empirical and estimated *cdfs* are shown in Figure 5. Both plots affirm the results of the goodness of fit criteria that *EGuE* distribution provides a better fit to the data set than the other competing models.

Table1. MLEs of parameters (Standard errors in parenthesis)

Distribution	Estimates				Loglik
EGuE	10.12 (8.54)	10.46 (8.57)	0.63 (0.27)	6.37 (4.96)	-100.8
EWE	0.04 (0.05)	23.96 (40.18)	1.04 (0.27)	2.34 (1.20)	-102.9
WE	0.03 (0.03)	60.90 (89.71)	1.54 (0.14)		-104.4
KE	0.545 (0.50)	2.31 (0.63)	2.06 (2.47)		-102.7
E	0.54 (0.06)				-115.8

Table 2. Goodness of fit statistics of the estimated models

Distribution	$K-S$	W^*	A^*	AIC	BIC
EGuE	0.078	0.067	0.421	209.62	218.72
EWE	0.103	0.107	0.715	213.92	223.03
WE	0.118	0.193	1.163	214.96	221.79
KE	0.101	0.100	0.674	211.45	218.28
E	0.269	1.137	5.840	233.55	235.82

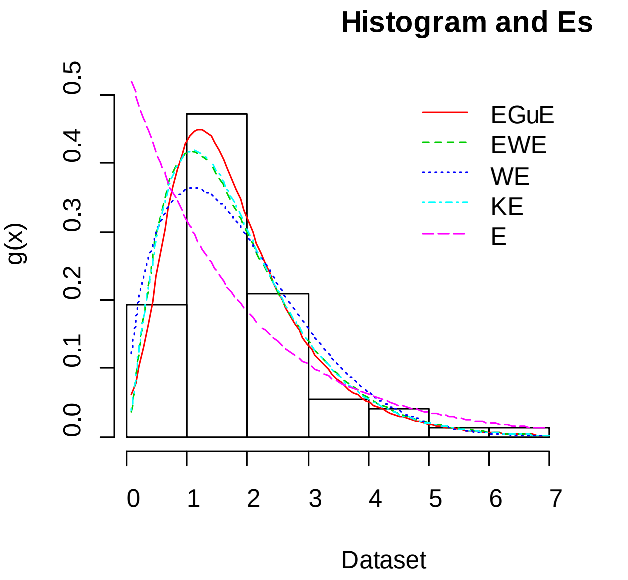

Figure 4. Plots of estimated pdf of EGuE distribution and other competing models based on the data set.

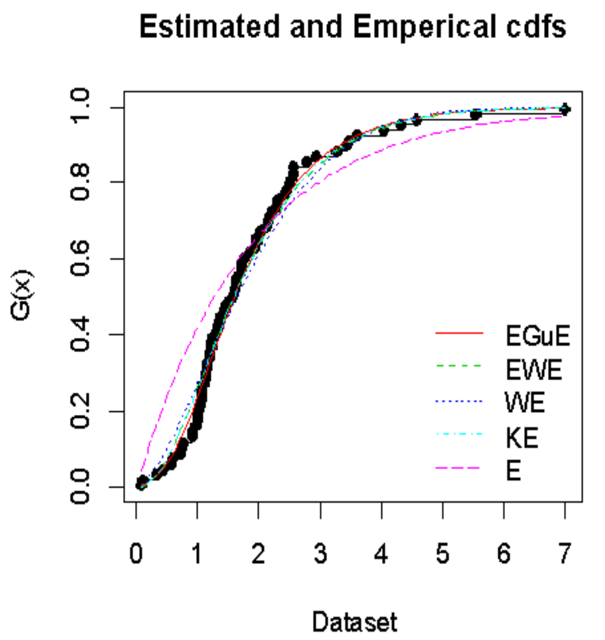

Figure 5. Plots of estimated cdf of EGuE distribution and other competing models based on the data sets.

6. Conclusion

In this paper, we introduced a four-parameter distribution called the exponentiated Gumbel exponential distribution using the transformed-transformer method introduced by Alzatreeh et al. [7]. We expressed the density of the new model as an infinite linear combination of exponential distribution**.** Several mathematical properties of the new model were derived. Estimates of the parameters of the proposed model were obtained using the method of maximum likelihood. The importance of the new model was demonstrated using a real data set.

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