American Journal of Applied Mathematics and Statistics, 2019, Vol. 7, No. 3, 93-100
Available online at http://pubs.sciepub.com/ajams/7/3/2
Published by Science and Education Publishing
DOI:10.12691/ajams-7-3-2

Iterative Method for Approximating a Common Fixed Point for Family of Multivalued Nonself Mappings in Uniformly Convex Hyperbolic Spaces

Mollalgn Haile Takele[1,*], B. Krishna Reddy[2]

[1]Department of Mathematics, College of Science, Bahir Dar University, Ethiopia
[2]Department of Mathematics, University College of Science, Osmania University, India
*Corresponding author: mollalgnh@bdu.edu.et

Received February 11, 2019; Revised March 20, 2019; Accepted April 10, 2019

Abstract In this paper, authors constructed Mann type of iterative method for the finite family of multi valued, nonself and nonexpansive mappings in a uniformly convex hyperbolic space. Authors proved strong convergence theorems of the iterative method, which approximates a common fixed point for the family single valued and multi valued nonexpansive mappings in a complete uniformly convex hyperbolic space which is more general than a complete CAT(0) space and a uniformly convex Banach space. The results in this work extended many results in the literature.

Keywords: *fixed point, nonself mapping, nonexpansive mapping, multi valued mapping, mann type iterative method, uniformly convex metric space, hyperbolic space*

Cite This Article: Mollalgn Haile Takele, and B. Krishna Reddy, "Iterative Method for Approximating a Common Fixed Point for Family of Multivalued Nonself Mappings in Uniformly Convex Hyperbolic Spaces." *American Journal of Applied Mathematics and Statistics*, vol. 7, no. 3 (2019): 93-100. doi: 10.12691/ajams-7-3-2.

1. Introduction

Many nonlinear problems are naturally formulated as a fixed point problem for single valued or multi valued mapping. When a fixed point of nonexpansive mapping or contractive mapping exists, approximation technique is required. Following Picard's iterative method which fails to converge in general for mappings which are not strictly contraction, other approximation techniques were introduced to approximate a fixed point. In the last forty years, numerous researchers have been attracted by this direction, and they developed iterative methods to approximate fixed point for not only nonexpansive mappings but also for some general class of nonexpansive mappings in linear Banach spaces and nonlinear domains too. Fixed point theory and hence approximation techniques have been extended to metric spaces(see, for example, [1-10] and their references).

Let K be a non-empty subset of a metric space (E,d) with metric d. Then we denote the set of non-empty, closed and bounded subsets of E by $CB(E)$. We say K is proximal if for every $x \in E$ there exists $y \in K$ such that $d(x,y) = \inf\{d(x,z), \forall z \in K\}$.

We denote the set of non-empty, proximal and bounded subsets of K by $Prox(K)$. We see that in CAT (0) space or uniformly convex Banach space E every non-empty, closed and convex subset of E is proximal [11]. For $A, B \in CB(E)$, we define the Hausdorff distance between the two sets A and B by

$$D(A,B) = Max\left\{\sup_{x\in B} d(x,A), \sup_{x\in A} d(x,B)\right\},$$

where $d(x,A) = \inf\{d(x,a), \forall a \in A\}$. Furthermore, as Kuratowski in [12] presented that $(CB(E),D)$ is metric space if (E,d) is metric space and $(CB(E),D)$ is complete if (E,d) is complete.

Definition 1.1. Let $T: K \to 2^E$ be nonself multi valued mapping. Then the set of fixed points of T is defined by

$$F = F(T) = \{x \in K : x \in Tx\}.$$

For a single valued mapping $T: K \to E$, the set of fixed points is defined by

$$F = F(T) = \{x \in K : x = Tx\}.$$

In particular, investigations have been made on nonlinear hyperbolic spaces.

Definition 1.2. [13] A hyperbolic space is a triple (E,d,W), where the pair (E,d) a metric space and $W: E \times E \times [0,1] \to E$ is a mapping satisfying the following

a) $d(z,W(x,y,\lambda)) \le (1-\lambda)d(z,x) + \lambda d(z,y)$;
b) $d(W(x,y,\lambda),W(x,y,\gamma)) = |\lambda-\gamma| d(x,y)$;
c) $W(x,y,\lambda) = W(y,x,1-\lambda)$;

d) $d(W(x,y,\lambda),W(z,w,\lambda)) \le (1-\lambda)d(x,z) + \lambda d(y,w),\ \forall x,y,z,w \in E, \lambda,\gamma \in [0,1]$.

Every normed linear space, R-trees, the Hilbert balls with the hyperbolic metric, the Cartesian products of Hilbert balls, Hadamard manifolds and hence CAT(0) spaces are examples of hyperbolic spaces and the detailed concepts and examples can be found in [13,14,15,16,17].

The following is found in [7].

A metric space E is said to be convex if it satisfies part a) of definition 1.2, hence, even in convex metric space E, for all $x,y \in E, \lambda \in [0,1]$ the following hold;

a) $d(x,W(x,y,\lambda)) = \lambda d(x,y)$ & $d(y,W(x,y,\lambda)) = (1-\lambda)d(x,y)$.

b) $W(x,y,0) = x,\ W(x,y,1) = y$ & $W(x,x,\lambda) = W(x,x,1-\lambda) = x$.

Definition 1.3 [13] A hyperbolic space (E,d,W) is uniformly convex if for every $r > 0$ and $\varepsilon \in (0,2]$ there exists a $\delta > 0$ such that for all $x,y,u \in E$,

$$d(x,u) \le r, d(y,u) \le r \ \&\ d(x,y) \ge \varepsilon r$$
$$\Rightarrow d(W(x,y,\tfrac{1}{2}),u) \le (1-\delta)r.$$

The modulus of uniformly convexity of the hyperbolic space (E,d,W) is the mapping

$$\phi : (0,\infty) \times (0,2] \to (0,1],$$

which gives $\delta = \phi(r,\varepsilon)$ for any $r > 0$ & $\varepsilon \in (0,2]$. and we say that ϕ is monotone if it is decreasing with respect to r.

Authors in [9] proved that CAT(0) spaces are uniformly convex hyperbolic spaces. Thus, uniformly convex hyperbolic spaces are generalizations of both uniformly convex Banach spaces and CAT(0) spaces.

Definition 1.4. [1,5,18] Let K be a non-empty subset of a metric space E. Then the mapping $T : K \to 2^E$ is said to

a) be L-Lipschitzian if $D(Tx,Ty) \le Ld(x,y)$ for some $L > 0$ and for all $x,y \in K$;

b) be nonexpansive if $D(Tx,Ty) \le d(x,y)$ for all $x,y \in K$, when $L = 1$;

c) be Quasi nonexpansive if $F(T) \ne \emptyset$ and $D(Tx,Tp) \le d(x,y)$ for all $p \in F(T), x \in K$;

d) satisfy condition(C) if

$$\frac{1}{2}d(x,Tx) \le d(x,y) \Rightarrow D(Tx,Ty) \le d(x,y).$$

A single valued mapping $T : K \to E$ is said to

1) satisfy condition(C) if

$$\frac{1}{2}d(x,Tx) \le d(x,y) \Rightarrow d(Tx,Ty) \le d(x,y);$$

2) be nonexpansive if

$d(Tx,Ty) \le d(x,y)$ for all $x,y \in K$, when $L = 1$;

3) be Quasi nonexpansive if $F(T) \ne \emptyset$ and

$d(Tx,Tp) \le d(x,p)$ for all $p \in F(T), x \in K$.

Thus, we see that every nonexpansive mapping satisfies condition(C), hence, the class of mappings satisfying condition(C) is an intermediate between the class of nonexpansive mappings and that of the class of quasi nonexpansive mappings.

Example 1.1. [18] Let $T : [0,3] \to \Re$ be defined by

$$T(x) = \begin{cases} 0, x \ne 3 \\ 1, x = 3 \end{cases}.$$

Then the map T satisfies condition(C) but is not nonexpansive mapping.

We may have a more general class of mappings: the class of strictlypseudocontractive mappings and their generalizations.

Definition 1.5. [8] Let K be non-empty subset of a hyperbolic space E and let $T : K \to 2^E$ be a multi valued mapping. Then T is said to be

a) inward mapping if for any $x \in K$,

$$Tx \subseteq \left\{ \begin{array}{l} w : w = x \ \ or \\ y = W\left(x, w, 1-\dfrac{1}{c}\right), y \in K, c \ge 1 \end{array} \right\}.$$

b) k-strictly pseudocontractive mapping if for all $x,y \in K$, there exists $k \in [0,1)$ such that

$$D^2(Tx,Ty) \le d^2(x,y) + 4kd^2\left(W(x,v,\tfrac{1}{2}), W(u,y,\tfrac{1}{2})\right),\ .$$

where $u \in Tx, v \in Ty$.

Thus, in particular, if $k = 0$, then T is nonexpansive mapping. Moreover, if T is single valued mapping we have $u = Tx$ and $v = Ty$.

Fixed point and common fixed point iterative methods are applicable in many areas such as convex optimization, control theory, differential inclusions, economics and physics. Consequently, the existence as well as methods of approximating fixed point and common fixed point for single valued and multi valued, self (nonself), contractive and nonexpansive type of mappings in Banach Spaces and generalizations to general metric spaces have been extensively studied by numerous authors of the field. In particular, fixed point results in a CAT(0) space and generalizations to hyperbolic spaces, which can be applied to graph theory, Biology and computer science have been extensively investigated by several authors.

Lim [19] was the first to introduce the delta convergence which is analogous to weak convergence in Banach spaces.

Definition 1.6. [9] Let E be a metric space and $\{x_n\}$ a bound sequence. Then for any point $x \in E$, if we define r by

$$r(x,x_n) = \lim_{n\to\infty} Sup\, d(x,x_n).$$

Then the asymptotic radius $r(\{x_n\})$ of the sequence $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{r(x,x_n), x \in E\}$ and the asymptotic centre $A(\{x_n\})$ of $\{x_n\}$ is given by

$$A(\{x_n\}) = \{x \in E : r(x,\{x_n\}\} = r(\{x_n\}).$$

Moreover, a sequence $\{x_n\}$ in a metric space E is said to be Δ convergent to the point $x \in E$ if $A(\{x_{n_k}\}) = \{x\}$ for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

Let K be a non-empty subset of a metric space E. Then the infimum of $r(x, x_n)$ over K is the asymptotic radius of the sequence $\{x_n\}$ with respect to K and is denoted by

$$A(K,\{x_n\}) = \inf\{r(x,\{x_n\}), x \in K\}.$$

The set of asymptotic centre of $\{x_n\}$ with respect to K is given by

$$AK(\{x_n\}) = \{x \in E : r(x,\{x_n\}) \le r(y,\{x_n\}), \forall\, y \in K\}.$$

If the point x in the hyperbolic space E is the unique asymptotic centre of every subsequence of a bounded sequences $\{x_n\}$, then the sequence $\{x_n\}$ Δ converges to x and we write it as $\Delta - \lim_{n\to 0} x_n = x$ or $x_n \underset{\to}{\Delta} x$.

Consequently, fixed point iterative methods for the finite family of single valued and multi valued mappings in uniformly convex Banach spaces as well as in CAT(0) spaces have been studied by various authors (see, [20,21,22] and their references). Results have also been extended to uniformly convex hyperbolic space which is more general than uniformly convex Banach space and CAT(0) space (see, [2] and references).

In particular, approximation techniques for common fixed point of nonself mappings via metric projection have been constructed by numerous researchers of the field [10]. However Colao and Marino in [23] presented that the computation for metric projection is costly, and they introduced iterative method by using inward condition without metric projection calculation. Consequently, authors in [8,24-30] constructed iterative methods for approximating a common fixed point for family of nonself and inward mappings for single valued and multi valued mappings in Hilbert spaces, Banach spaces and CAT(0) spaces as well.

We raise an open question that, can we construct iterative methods which approximate common fixed point for the finite family nonself mappings in a uniformly convex Hyperbolic space which is more general than complete CAT(0) spaces and uniformly convex Banach spaces? Thus, it is our purpose in this paper to approximate a common fixed point for the finite family of nonself mappings with inward conditions in uniformly convex hyperbolic spaces, which is a positive answer to our question.

2. Preliminary Concepts

We use the following notations and definitions;

Definition 2.1. [31] A sequence $\{x_n\}$ in K is said to be Fejer monotone with respect to a subset F of K if

$$\forall x \in F, d(x_{n+1}, x) \le d(x_n, x), \forall n.$$

Lemma 2.1. [32] Let E be a metric space. Then, if $A, B \in CB(E)$) and $a \in A$, then for every $\gamma > 0$ there exists $b \in B$ such that $d(a,b) \le D(A,B) + \gamma$.

Lemma 2.2. [33] Let E be a metric space. Then if $A, B \in Prox(E)$ and $a \in A$, then there exists $b \in B$ such that $d(a,b) \le D(A,B)$.

Lemma 2.3. [34] Let E be a uniformly convex hyperbolic space with monotone modulus of uniformity convexity and $x \in E$, let $\{x_n\},\{y_n\}$ in E be two sequences, if there exists $r \ge 0$ satisfying the conditions $\limsup_{n\to\infty} d(x_n, x) \le r$,

$$\limsup_{n\to\infty} d(y_n, x) \le r$$

and

$$\lim_{n\to\infty} d(W(x_n, y_n, 1-\lambda_n), x) = r,$$

where $\{\lambda_n\} \subset [\varepsilon, 1-\varepsilon] \subset (0,1)$ and $\varepsilon \in (0,1)$. Then it holds that $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Definition 2.2. [24] Let F and K be two closed and convex non-empty sets in a metric space E and $F \subset K$. Then for any sequence $\{x_n\} \subset K$, if the sequence $\{x_n\}$ converges strongly to an element $x \in \partial K \setminus F$, where $x_n \ne x$ implies that $\{x_n\}$ is not Fejer-monotone with respect to the subset $F \subset K$, and we say the pair (F, K) satisfies condition(S).

Example 2.1. Let $F = \{0\} \subset K = [-1,1]$. Then, the pair (F, K) satisfies condition(S) with the metric induced by norm in $E = \Re$.

Definition 2.3 [35] The multi valued mapping $T : K \to 2^E$ with non-empty set of fixed points F is said to satisfy condition(I) if there exists a non decreasing non negative function $g : [0,\infty) \to [0,\infty)$ satisfying $g(0) = 0$, and $g(r) > 0$, $\forall r \in (0,\infty)$ such that

$$d(x_n, Tx_n) \ge g(d(x_n, F)), F \ne \varnothing \text{ holds.}$$

Definition 2.4. The mapping $T : K \to E$ is said to be semi compact if every bounded sequence $\{x_n\}$ in K satisfying

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0$$

has a convergent subsequence.

Furthermore, the multi valued mapping $T : K \to CB(E)$ is semi compact if every bounded sequence $\{x_n\}$ in K satisfying

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0$$

has a convergent subsequence.

3. Results and Discussion

Mann Type of iterative method

Let $T_1,T_2,...T_K : K \to \Pr ox(E)(CB(E))$ be a finite family of nonself and nonexpansive multi valued mappings on a non-empty, closed and convex subset K of a complete uniformly convex hyperbolic space E. Then it is our objective to construct Mann type of iterative method for approximating a common fixed point of the family and determine conditions for convergence of the iterative method. We use inward condition instead of the computation for metric projection which is costly, that is computationally expensive in many cases and we prove both delta and strong convergence results of the iterative method.

Lemma 3.1. Let K be a non-empty, closed and convex subset of a complete metric space E and let $T_1,T_2,...T_N : K \to \Pr ox(E)$ be a finite family of multi valued mappings, for $u_k \in T_k x$, define $h_{u_k} : K \to \Re$ by

$$h_{u_k}(x) = \inf\left\{\lambda \in [0,1] : W(x,u_k,1-\lambda) \in K\right\}.$$

Then for any $x \in K$, $h_{u_k}(x) \in [0,1]$ and $h_{u_k}(x) = 0$ if and only if $u_k \in K$, whereas if $\beta \in [h_{u_k}(x),1]$, then $W(x,u_k,1-\beta) \in K$. Moreover, if T_k is inward mapping, then $h_{u_k}(x) < 1$, in addition, if $u_k \notin K$, then

$$W(x,u_k,1-h_{u_k}(x)) \in \partial K,$$

where ∂K is the boundary of K.

The proof of this lemma follows from, lemma 2.1 and 3.1 of Colao and Mariao and Tuffa and Zegeye in [8,23] respectively.

Theorem 3.2. Let $T_1,T_2,...T_N : K \to E$ be a family of nonself, nonexpansive and inward mappings on a non-empty, closed and convex subset K of a complete uniformly convex Hyperbolic space E with monotone modulus of uniformly convexity, $F = \bigcap_{k=1}^{N} F(T_k)$ non-empty and for each k, $T_k = T_{k(ModN)+1}$.

Then the sequence $\{x_n\}$ which is defined by Mann type of iterative method

$$\begin{cases} x_1 \in K, \alpha_1 = \left[\max\left\{\alpha, h_1(x_1)\right\}, c\right], \\ 0 < \alpha < c < 1, \\ x_{n+1} = W(x_n, T_n x_n, 1-\alpha_n), \\ \alpha_{n+1} = \left[\max\left\{\alpha_n, h_n(x_n)\right\}, c\right], \\ h_n(x_n) = \inf\left\{\lambda \ge 0 : W(x_n, T_n x_n, 1-\lambda) \in K\right\}. \end{cases} \tag{3.1}$$

is well-defined and

$$\lim_{n\to\infty} d(x_n, T_l x_n) = 0. \forall l \in \{1,2,3,...N\}.$$

Proof. By lemma 3.1, the sequence $\{x_n\}$ is well-defined and in K, thus, to prove the theorem first we prove $\{x_n\}$ is fejer monotone with respect to F, to do so, let $p \in F$. Then since each T_n is nonexpansive we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(x_n, T_n x_n, 1-\alpha_n), p) \\ &\le \alpha_n d(x_n, p) + (1-\alpha_n) d(x_n, p) \\ &\le d(x_n, p). \end{aligned} \tag{3.2}$$

Thus, the sequence $\{x_n\}$ is fejer monotone with respect to F, hence, the sequence $\{x_n\}$ is bounded.

Also, the sequence $\{d(x_n, p)\}$ is decreasing, hence it converges for all $p \in F$, thus, there exists $r \ge 0$ such that $\lim_{n\to\infty} d(x_n, p) = r$, hence, $\limsup_{n\to\infty} d(x_n, p) \le r$. Moreover, since $d(T_n x_n, p) \le d(x_n, p)$, taking $\limsup_{n\to\infty}$ both sides we have $\limsup_{n\to\infty} d(T_n x_n, p) \le r$. We also see that $\lim_{n\to\infty} d(x_{n+1}, p) = d(W(x_n, T_n x_n, 1-\alpha_n), p) = r$. Thus, by lemma 2.3 we have $\lim_{n\to\infty} d(x_n, T_n x_n) = 0$.

$$\begin{aligned} d(x_{n+1}, x_n) &= d(W(x_n, T_n x_n, 1-\alpha_n), x_n) \\ &\le \alpha_n d(x_n, x_n) + (1-\alpha_n) d(T_n x_n, x_n) \\ &= (1-\alpha_n) d(T_n x_n, x_n) \end{aligned} \tag{3.3}$$

Thus,

$$\begin{aligned} &\lim_{n\to\infty} d(x_{n+1}, x_n) \\ &= \lim_{n\to\infty} d(W(x_n, T_n x_n, 1-\alpha_n), x_n) \\ &\le \lim_{n\to\infty} \alpha_n d(x_n, x_n) + (1-\alpha_n) d(T_n x_n, x_n) \\ &= \lim_{n\to\infty} (1-\alpha_n) d(x_n, T_n x_n) = 0. \end{aligned} \tag{3.4}$$

By induction we have

$$\lim_{n\to\infty} d(x_{n+i}, x_n) = 0, \forall i = 1,2,..N.$$

$$d(x_n, T_{n+i} x_{n+i}) \le d(x_n, x_{n+i}) + d(x_{n+i}, T_{n+i} x_{n+i});$$

$$\begin{aligned} &d(x_n, T_{n+i} x_n) \\ &\le d(x_n, T_{n+i} x_{n+i}) + d(T_{n+i} x_{n+i}, T_{n+i} x_n) \\ &\le 2d(x_n, x_{n+1}) + d(x_n, T_{n+i} x_{n+i}) \\ &\le d(x_n, T_{n+i} x_{n+i}) + 2d(x_{n+i}, x_n). \end{aligned} \tag{3.5}$$

Thus, from equations (3.3) to (3.5) we have

$$\lim_{n\to\infty} d(x_n, T_{n+i} x_n) = 0, \forall i \in \{1,2,..N\},$$
$$\exists l \in \{1,2,..N\} \ni T_{n+i} = T_l$$

and vice versa, thus, we have

$$\lim_{n\to\infty} d(x_n, T_l x_n) = 0, \forall l = 1,2,...N.$$

Corollary 3.3. If $T_1 = T_2 =,.. = T_N = T : K \to E$, then the iterative method in theorem 3.2 is reduced to the following

$$\begin{cases} x_1 \in K, \alpha_1 = [\max\{\alpha, h_1(x_1)\}, c], \\ 0 < \alpha < c < 1, x_{n+1} = W(x_n, Tx_n, 1-\alpha_n), \\ \alpha_{n+1} = [\max\{\alpha_n, h_n(x_n)\}, c], \\ h_n(x_n) = \inf\{\lambda \geq 0 : W(x_n, Tx_n, 1-\lambda) \in K\}. \end{cases} \quad (3.6)$$

In this case, the sequence $\{x_n\}$ is well-defined and satisfies

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Theorem 3.4. In theorem 3.2, if $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$ and (F,K) satisfies condition(S), then the sequence $\{x_n\}$ converges strongly to some element p of F.

Proof. The sequence $\{x_n\}$ is bounded, that is, there exists $x \in E, m \geq 0$ such that $d(x_n, x) \leq m, \forall n$,

$$\begin{aligned} d(x_n, T_n x_n) &\leq d(x_n, p) + d(p, T_n x_n) \\ &\leq 2d(x_n, p) \\ &\leq M \end{aligned}$$

for some $M \geq 0$.

$$d(x_{n+1}, x_n) \leq (1-\alpha_n) d(x_n, T_n x_n),$$

thus, we have

$$\sum_{n=1}^{\infty} d(x_{n+1}, x_n) \leq \sum_{n=1}^{\infty}(1-\alpha_n) d(x_n, T_n x_n) < \infty. \quad (3.7)$$

Hence, the sequence $\{x_n\}$ is strongly Cauchy, hence Cauchy, in a complete metric space it converges. Thus, the sequence $\{x_n\}$ converges to some element $q \in K$. We need to show $p \in F$.

Moreover, $d(x_n, x) \to 0$ as $n \to \infty$ and since for every $\beta \in [h_{u_n}(x), 1)$ we have $W(x, T_n x, 1-\beta) \in K$. Since $\lim_{n\to\infty} \alpha_n = 1$ there exists a sub sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} h_{u_{n_j}}(x_{n_j}) = 1$. Suppose $\{\beta_{n_j}\}$ is a sequence of real numbers such that $\beta_{n_j} < 1$ and the limit $\lim_{j\to\infty} \beta_{n_j} = 1$, then $\beta_{n_j} h_{u_{n_j}}(x_{n_j}) < h_{u_{n_j}}(x_{n_j})$, hence, we must have $W(x_{n_j}, T_{n_j} x_{n_j}, 1-\beta_{n_j} h_{u_{n_j}}(x_{n_j})) \notin K$ and its limit is x which is in K, thus, $x \in \partial K$, and assuming that (F, K) satisfies condition(S) we have $x \in F$.

Thus, the sequence $\{x_n\}$ converges strongly to some element $p \in \bigcap_{k=1}^{N} F(T_k)$.

Theorem 3.5. Let $T_1, T_2, ...T_N : K \to \Pr ox(E)$ be the family of nonself, multi valued, nonexpansive and inward mappings on a non-empty, closed and convex subset K of a complete uniformly convex hyperbolic space E with monotone modulus of uniformly convexity, $F = \bigcap_{k=1}^{N} F(T_k)$ non-empty, $T_k = T_{k(ModN)+1}$ and for every point $p \in F = \bigcap_{k=1}^{N} F(T_k)$, $T_K(p) = \{p\}$. Let $\{x_n\}$ be a sequence of Mann type defined by the iterative method

$$\begin{cases} u_1 \in T_1 x_1, \alpha_1 = \max\{\alpha, h_{u_1}(x_1)\}, \alpha > 0, \\ x_{n+1} = W(x_n, u_n, 1-\alpha_n), u_n \in T_n x_n \ni \\ \alpha_{n+1} = \max\left\{\alpha_n, h_{u_{n+1}}(x_{n+1}) - \dfrac{1}{(n+1)^2}\right\}, \\ h_{u_n}(x_n) = \inf\{\lambda \geq 0 : W(x_n, u_n, 1-\lambda) \in K\}. \end{cases} \quad (3.8)$$

Then the sequence $\{x_n\}$ is well-defined, furthermore, if (F,K) satisfies condition (S), then the sequence $\{x_n\}$ converges strongly to some p of $F = \bigcap_{k=1}^{N} F(T_k)$.

Proof. By lemma 3.1 the sequence $\{x_n\}$ is well-defined and is in K, thus, to prove the theorem first we prove that $\{x_n\}$ is fejer monotone with respect to F, to do so, let $p \in F$. Then since each T_n is nonexpansive we have $d(u_n, p) \leq D(T_n x_n, p)$ and $T_n p = \{p\}$ by lemma 2.1 and lemma 2.2, there exists a sequence satisfying equality;

$$\begin{aligned} d(x_{n+1}, p) &= d(W(x_n, u_n, 1-\alpha_n), p) \\ &\leq (1-\alpha_n) d(x_n, p) + \alpha_n d(u_n, p) \\ &\leq (1-\alpha_n) d(x_n, p) + \alpha_n D(T_n x_n, T_n p) \quad (3.9) \\ &\leq (1-\alpha_n) d(x_n, p) + \alpha_n d(x_n, p) \\ &= d(x_n, p). \end{aligned}$$

Thus, the sequence $\{(x_n\}$ is fejer monotone with respect to F.

Since $d(x_n, p)$ is decreasing and bounded below it converges, and hence $\{x_n\}$ and $\{u_n\}$ are bounded, thus, $\{d(x_n, u_n)\}$ is bounded.

Also, from the method of proof of theorem 3.2 we have

$$d(x_{n+1}, x_n) \leq (1-\alpha_n) d(x_n, u_n).$$

Again, since $\alpha_n \geq 1 - \dfrac{1}{(n+1)^2}$ we get

$$\sum_{n=1}^{\infty}(1-\alpha_n) \leq \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} < \infty.$$

Since $\alpha_n > \alpha$ and α is positive we have

$$\sum_{n=1}^{\infty}(1-\alpha_n) d(x_n, u_n) \leq \sum_{n=1}^{\infty}(1-\alpha_n) K < \infty.$$

From the proof of theorem 3.2 we have

$$\begin{aligned} d(x_{n+1},x_n) &= d(W(x_n,u_n,1-\alpha_n),x_n) \\ &\le (1-\alpha_n)d(x_n,u_n). \end{aligned} \tag{3.10}$$

Thus, the sequence $\{x_n\}$ is strongly Cauchy, hence, it is Cauchy and Cauchy sequence converges in the complete space E, thus, there exists $x \in K$ such that $d(x_n,x) \to 0$ as $n \to \infty$ and for each x_n there corresponds $\{u_n\}$. Since the coefficient $\beta \in [h_u(x),1)$ and $u \in Tx$ we have $W(x,u,1-\beta) \in K$. Also since $\lim_{n\to\infty} \alpha_n = 1$ there exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $\lim_{j\to\infty} h_{u_{n_j}}(x_{n_j}) = 1$, suppose the sequence $\{\beta_{n_j}\}$ is sequence of real numbers $\beta_{n_j} < 1$ and $\lim_{j\to\infty} \beta_{n_j} = 1$, in particular, $\beta_{n_j} h_{u_{n_j}}(x_{n_j}) < h_{u_{n_j}}(x_{n_j})$, hence, the sequence $W(x_{n_j},u_{n_j},1-\beta_{n_j} h_{u_{n_j}}(x_{n_j})) \notin K$. Thus, we have

$$\begin{aligned} &\lim_{j\to\infty} d(W(x_{n_j},u_{n_j},1-\beta_{n_j} h_{u_{n_j}}(x_{n_j})),x) \\ &\le \lim_{j\to\infty} \beta_{n_j} h_{u_{n_j}}(x_{n_j}) d(x_{n_j},x) + \\ &\lim_{j\to\infty} (1-\beta_{n_j} h_{u_{n_j}}(x_{n_j})) d(u_{n_j},x) = 0, \end{aligned}$$

thus, the limit is x which is in K, thus, $x \in \partial K$, since the pair (F, K) satisfies condition(S) we have $x \in F$.

Thus, the sequence $\{x_n\}$ converges strongly to some element $p \in \bigcap_{k=1}^{N} F(T_k)$.

Corollary 3.6. Let $T_1,T_2,...T_N : K \to E$ be a finite family of nonself, single valued, nonexpansive and inward mappings on a non-empty, closed and convex subset K of a complete uniformly convex hyperbolic space E with monotone modulus of uniformly convexity, such that $F = \bigcap_{k=1}^{N} F(T_k)$ non-empty, $T_k = T_{k(ModN)+1}$ and for all $p \in F = \bigcap_{k=1}^{N} F(T_k)$. Let $\{x_n\}$ be a sequence of Mann type defined by the iterative method

$$\begin{cases} x_1 \in K, \alpha_1 = \max\{\alpha, h_1(x_1)\}, \alpha > 0, \\ x_{n+1} = W(x_n, T_n x_n, 1-\alpha_n), \\ \alpha_{n+1} = \max\left\{\alpha_n, h_{n+1}(x_{n+1}) - \dfrac{1}{(n+1)^2}\right\}, \\ h_n(x_n) = \inf\{\lambda \ge 0 : W(x_n, T_n x_n, 1-\lambda) \in K\}. \end{cases} \tag{3.11}$$

Then the sequence $\{x_n\}$ is well-defined, furthermore, if (F,K) satisfies condition(S), then the sequence $\{x_n\}$ converges strongly to some p of $F = \bigcap_{k=1}^{N} F(T_k)$.

Proof**.** From the method of proof of theorem 3.4, we put $u_n = T_n x_n$, hence, the proof can be made in similar fashion.

Furthermore, strong convergence result can be obtained with suitable conditions on the mappings such as condition (I).

Definition 3.1. The finite family of mappings $\{T_i\}_{i=1}^{N}$ where $T_i : K \to 2^E$ with the intersection of sets of fixed points $F = \bigcap_{k=1}^{N} F(T_k)$ is said to satisfy condition (I) if there exists a non decreasing non negative function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$, $g(r) > 0$, $\forall r \in (0,\infty)$ such that the following holds

$$d(x_n, T_i x_n) \ge g(d(x_n, F)), F \ne \emptyset. \tag{3.12}$$

Theorem 3.7. Let $T_1,T_2,...T_N : K \to prox(E)$ be a family of nonself, multi valued, nonexpansive and inward mappings satisfying condition (I) on a non-empty, closed and convex subset K of a complete uniformly convex Hyperbolic space E with monotone modulus of uniformly convexity, $F = \bigcap_{k=1}^{N} F(T_k)$ non-empty, $T_k = T_{k(ModN)+1}$ and for all $p \in F = \bigcap_{k=1}^{N} F(T_k)$, $T_K(p) = \{p\}$. Let $\{x_n\}$ be a sequence of Mann type defined by the iterative method

$$\begin{cases} u_1 \in T_1 x_1, \alpha_1 = \max\{\alpha, h_{u_1}(x_1)\}, \alpha > 0, \\ x_{n+1} = W(x_n, u_n, 1-\alpha_n), u_n \in T_n x_n, \\ \alpha_{n+1} = \max\{\alpha_n, h_{u_{n+1}}(x_{n+1})\}, \\ h_{u_n}(x_n) = \inf\{\lambda \ge 0 : W(x_n, u_n, 1-\lambda) \in K\}. \end{cases}$$

Then the sequence $\{x_n\}$ is well-defined and in K, and if $\{\alpha_n\} \subseteq [\varepsilon, 1-\varepsilon] \subset (0,1)$ for $\varepsilon > 0$, then the sequence $\{x_n\}$ converges strongly to some fixed point element p of

$$F = \bigcap_{k=1}^{N} F(T_k). \tag{3.13}$$

Proof. From the method proof of theorem 3.2 we have $\lim_{n\to\infty} d(x_n, T_n x_n) \le \lim_{n\to\infty} d(x_n, u_n) = 0$, hence, we have $\lim_{n\to\infty} d(x_n, T_l x_n) = 0, \forall l \in \{1,2,...N\}$. Furthermore, since the mappings satisfy condition (I), there exists a non decreasing function $g : [0,\infty) \to [0,\infty)$ satisfying the conditions $g(0) = 0, g(r) > 0$, $\forall r \in (0,\infty)$ such that $d(x_n, T_i x_n) \ge g(d(x_n, F)), F \ne \emptyset$, hence, we have

$$\liminf_{n\to\infty} d(x_n, F) = 0.$$

Thus, the monotonicity $d(x_{n+1},F)\le d(x_n,F)$ gives $\lim_{n\to\infty} d(x_n,F)=0,$ hence, for $n>m$ and for all $p\in F$ we have

$$d(x_n,x_m)\le d(x_n,p)+d(p,x_m)\le 2d(x_m,p).$$

Taking infimum over all $p\in F$ we get

$$d(x_n,x_m)\le 2d(x_m,F)\to 0, n,m\to\infty,$$

hence, the sequence $\{x_n\}$ is Cauchy sequence, thus, it converges to some $q\in K.$ Moreover, we have

$$\begin{aligned} d(q,T_lq) &\le d(x_n,q)+d(x_n,T_lx_n)+d(T_lx_n,T_lq) \\ &\le 2d(x_n,q)+d(x_n,T_lx_n)\to 0, n\to\infty. \end{aligned} \tag{3.14}$$

Since T_iq is closed we have $q\in F$ which completes the proof.

Theorem 3.8. Let $T_1,T_2,...T_N : K\to CB(E)$ be a family of nonself, multi valued, nonexpansive and inward mappings satisfying condition (I) on a non-empty, closed and convex subset K of a complete uniformly convex hyperbolic space E with monotone modulus of uniformly convexity, $F=\bigcap_{k=1}^{N} F(T_k)$ non-empty, $T_k=T_{k(ModN)+1}$ and for all $p\in F=\bigcap_{k=1}^{N} F(T_k)$, $T_K(p)=\{p\}$. Let $\{x_n\}$ be a sequence of Mann type defined by the iterative method

$$\begin{cases} u_1\in T_1x_1, \alpha_1=\max\{\alpha, h_{u_1}(x_1)\}, \alpha>0, \\ x_{n+1}=W(x_n,u_n,1-\alpha_n), u_n\in T_nx_n, \\ \alpha_{n+1}=\max\{\alpha_n, h_{u_{n+1}}(x_{n+1})\}, \\ h_{u_n}(x_n)=\inf\{\lambda\ge 0: W(x_n,u_n,1-\lambda)\in K\}. \end{cases}$$

Then the sequence $\{x_n\}$ is well-defined and in $K,$ and if $\{\alpha_n\}\subseteq[\varepsilon,1-\varepsilon]\subset(0,1)$ holds for some $\varepsilon>0$, then the sequence $\{x_n\}$ converges strongly to some element p of

$$F=\bigcap_{k=1}^{N} F(T_k). \tag{3.15}$$

Proof. Since lemma 3.1 is applicable if $Prox(E)$ is replaced by $CB(E).$ Thus, the proof can be made in similar way.

Theorem 3.9. Let $T:K\to E$ be a nonself, single valued and inward mapping satisfying both condition (C) and condition (I) on a non-empty, closed and convex subset K of a complete uniformly convex hyperbolic space E with monotone modulus of uniformly convexity and $F=F(T)$ is non-empty. Let $\{x_n\}$ be a sequence of Mann type defined by the iterative method

$$\begin{cases} x_1\in K, \alpha_1=\max\{\alpha, h_1(x_1)\}, \alpha>0, \\ x_{n+1}=W(x_n,T_nx_n,1-\alpha_n), \\ \alpha_{n+1}=\max\{\alpha_n, h_{n+1}(x_{n+1})\}, \\ h_n(x_n)=\inf\{\lambda\ge 0: W(x_n,T_nx_n,1-\lambda)\in K\}. \end{cases}$$

Then the sequence $\{x_n\}$ is well-defined and in $K,$ and if $\{\alpha_n\}\subseteq[\varepsilon,1-\varepsilon]\subset(0,1)$ holds for some $\varepsilon>0$, then the sequence $\{x_n\}$ converges strongly to some element p of

$$F=F(T). \tag{3.16}$$

Proof. Let $p\in F$. Then since

$$\frac{1}{2}d(p,Tp)=0\le d(p,x_n)\Rightarrow d(Tx,Tp)\le d(x_n,p).$$

We have,

$$\begin{aligned} d(x_{n+1},p) &= d(W(x_n,T_nx_n,1-\alpha_n),p) \\ &\le \alpha_n d(x_n,p)+(1-\alpha_n)d(T_nx_n,T_np) \\ &\le \alpha_n d(x_n,p)+\alpha_n d(x_n,p)=d(x_n,p). \end{aligned} \tag{3.17}$$

Hence, from method of proof of theorem 3.2 and 3.8 we have the sequence $\{x_n\}$ is Cauchy sequence, hence it converges to some point $q\in K.$ It suffices to show that $q\in F(T).$

But

$$\begin{aligned} d(q,Tq) &\le d(x_n,q)+d(x_n,Tx_n)+d(Tx_n,Tq) \\ &\le 2d(x_n,q)+d(x_n,Tx_n)\to 0, n\to\infty. \end{aligned}$$

Since $T\ q=q$ we have $q\in F$ which completes the proof.

The results can be extended to the class of quasi nonexpansive mappings too.

4. Conclusion

Authors constructed Mann type of iterative methods to approximate common fixed point for the finite family of nonself and nonexpansive mappings with inward condition by lowering the computation for metric projection, which doesn't exist in general Banach spaces and more general nonlinear spaces, even in Hilbert spaces, it requires additional computational techniques. Our theorems extended many results in the literature, in particular, we extended the result of [8,25-30] to a common fixed point for the family of nonexpansive and Suzi type of mappings to uniformly convex hyperbolic space which is more general than uniformly convex Banach spaces and CAT(0) spaces. We also extended many results to nonself single valued and multi valued mappings. Authors proved strong convergence result which is stronger than that of delta and weak convergence results.

Open questions. Finally we propose open questions for

a) the possibility to extend results of this work to more general classes of contractive mappings.

b) the possibility to lower condition (I) and condition(S) by imposing weaker conditions. If so, under what suitable conditions?

Authors' Contributions

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

Competing Interests

The authors declare that they have no competing interests.

References

[1] Dhompongsa. S, Kirk. W.A, Panyanak.B., Nonexpansive set-valued mappings in metric and Banach spaces, J.Nonlinear Convex Anal. 8 (2007) 35-45.

[2] Imdad.M, Dashputre.S., Fixed point approximation of Picard normal S-iteration process for generalized nonexpansive mappings in hyperbolic spaces, Math Sci. 10 (2016) 131-138.

[3] Khan. A.R, Fukhar-Ud-Din. H, Khan. M.A., An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces., Fixed Point Theory Appl. 2012 (2012) 54.

[4] Kirk.W.A., Geodesic geometry and fixed point theory I. In Seminar of Mathematical Analysis, Univ. Sevilla Secr. Publ., Sev. 64 (2003) 195-225.

[5] Kirk.W.A., Geodesic geometry and fixed point theory II. In International Conference on Fixed Point Theory and Applications, Yokohama Publ., Yokohama. (2004) 113-142.

[6] Leustean.L., Nonexpansive iterations in uniformly convex W-hyperbolic spaces. In: Leizarowitz A, Mordukhovich BS, Shafrir I, Zaslavski A(eds.) Contemp Math Am, Nonlinear Analysis and Optimization I: Nonlinear Analysis Math Soc AMS, Contemp Math Am Math Soc AMS. 513 (2010) 193-209 .

[7] Takahashi.W.A., A convexity in metric space and nonexpansive mappings, I. Kodai Math. Sem. Rep. 22 (1970) 142-149.

[8] Tufa. A.R, Zegeye. H, Thuto.M., Convergence Theorems for Non-self Mappings in CAT(0)Spaces, Numer. Funct. Anal. Optim. 38 (2017) 705-722.

[9] Ustean.LIE., A quadratic rate of asymptotic regularity for CAT(0)-spaces, Math. Anal. Appl. 325 (2007) 386-399.

[10] Wan.Li-Li., Demiclosed principle and convergence theorems for total asymptotically nonexpansive nonself mappings in hyperbolic spaces, Fixed Point Theory Appl. 2015 (2015).

[11] Abkar.A, Eslamian.M., Fixed point theorems for Suzuki generalized non-expansive multivalued mappings in Banach Space, Fixed Point Theory Appl. 2010 (2010) 10 pages.

[12] Kuratowski. K., Topology, Academic press, 1966.

[13] Kohlenbach. U., Some logical metathorems with applications in functional analysis, Trans. Am. Math. Soc. 357 (2005) 89-128.

[14] Bridson.M.R, Haefliger.A., Metric Spaces of Non-positive Curvature, Springer, Berlin, germany, 1999.

[15] Goebel.K, Kirk. W.A., Iteration processes for nonexpansive mappings. In: Singh, S.P., Thomeier, S., Watson, B. (eds) Topological Methods in Nonlinear Functional Analysis (Toronto, 1982), pp. 115-123. Contemporary Mathematics, vol 21., Amer- ican Mathematical Society, New York, 1983.

[16] Goebel.K, Rreich.S., Uniformly Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker Inc, New York, 1984.

[17] Reish. S, Shafirir. J., Nonexpansive iterations in hyperbolic spaces, Nonlinear Anal. 15 (1990) 537-558.

[18] Suzuk.T., Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, J. Math. Anal. Appl. 340 (2008) 1088-1095.

[19] Lim. T.C., Remarks on some fixed point theorems, Proc. Amer. Math. Soc. 60 (1976) 179-182.

[20] Chidume. C.E, Okpala. M.E., Fixed point iteration for acountable family of multivalued strictly pseudocontractive-type mappings, Springer Plus. 2015 (2015).

[21] Chidume.C.E, Zegeye.H and Shahzad.N., Convergence theorems for a Common fixed point of a finite family of nonself nonexpansive mappings, Fixed Point Theory Appl. 233-241 (2005) 233-241.

[22] Zegeye.H, Shahzad.N., An algorithm for a common fixed point of a family of pseudo contractive mappings, Fixed Point Theory Appl. 234 (2013) 234.

[23] Colao.V, Marino.G., Krasnoselskii-Mann method for nonself mappings, Fixed Point Theory Appl. 39 (2015).

[24] Guo, M; Li,X; Su.Y., on an open question of V. Colao and G. Marino presented in the paper “Krasnoselskii－Mann method for non-self mappings, Springer Plus. (2016).

[25] Takele.M.H, Reddy.B.K., Fixed point theorems for approximating a common fixed point for a family of nonself, strictly pseudocontractive and inward mappings in real Hilbert spaces, Glob. J. Pure Appl. Math. 13 (2017) 3657-367.

[26] Takele.M.H, Reddy.B.K., Iterative Method For Approximating a Common Fixed Point of Infinite Family of Strictly Pseudo Contractive Mappings in Real Hilbert Spaces, Int. J. Comput. Appl. Math. 12 (2017) 293-303.

[27] Takele. M.H, Reddy. B.K., Approximation of common fixed point of finite family of nonself and nonexpansive mappings in Hilbert space, IJMMS. 13 (2017) 177-201.

[28] Takele. M.H, Reddy. B.K., Approximation for a Common Fixed Point for Family of Multivalued Nonself Mappings, Am. J. Appl. Math. Stat. 5 (2017) 175-190.

[29] Takele. M.H, Reddy. B.K., Convergence theorems for common fixed point of the family of nonself and nonexpansive mappings in real Banach spaces AAM: Special isuee□No 4 (2019) 176-195.

[30] Tufa.A.R, Zegeye. H., Mann and Ishikawa-Type Iterative Schemes for Approximating Fixed Points of Multi-valued Non-Self Mappings, Mediterr.J.Math. 6 (2016).

[31] Bauschke. HH, Combettes. PL., A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces, Math. Oper. Res. 26 (2001) 248-26.

[32] Nadler. S.B. JR., Multi-valued contraction mappings, Pac. J. Math. 30 (1969) 475-488.

[33] Isiogugu.F.O, Osilike.M.O., Convergence theorems for new classes of multivalued hemi contractive-type mappings, Fixed Point Theory Appl. (2014).

[34] Khan. AR., Ishikawa Iteration Process for Non-self and non-expansive Maps in Banach Spaces, Int. J. Appl. Math. 38 (2008) 3.

[35] Chidume. C.E, Shahzad. N., Strong convergence of an implicit iteration process for a finite family of nonexpansive mappings, Nonlinear Anal. 65 (2005) 1149-1156.