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Modeling and Analysis of the Population Density Dependent Industrial Emissions of Toxic Air Pollutants and Their Control by External Species Sprayed in the Atmosphere

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Abstract It is well known that the human health is adversely affected by toxic air pollutants such as sulfur dioxide, nitrous oxide etc. present in the atmosphere. The removal of such pollutants from the atmosphere is, therefore, very much desirable. In this paper, a nonlinear mathematical model is proposed to study the population density dependent industrial emission of toxic air pollutants in the atmosphere and their removal by spraying liquid (water droplets) and particulate matter. In the modeling process, five variables are considered, namely; the cumulative concentration of toxic air pollutants, the density of human population affected by the toxic pollutants, the density of industrialization which is population density dependent, the number density of liquid droplets sprayed in the environment and the density of particulate matter sprayed in the environment. It is assumed that the emissions of toxic air pollutants are linearly related to the density of industrialization, the growth rate of which is directly proportional to the density of human population. It is also assumed that the growth rate of externally sprayed species in the environment is directly proportional to the concentration of toxic air pollutants in the environment. The model is analyzed using stability theory of nonlinear differential equations and numerical simulations. The model analysis shows that as the rate of spray of external species in the environment increases, the cumulative concentration of toxic air pollutants decreases. It is also found that as the rate of removal of toxic pollutants increases, the cumulative concentration of toxic air pollutants in the environment decreases. The effect of toxic air pollutants is observed to decrease the density of human population. The numerical simulation confirms analytical results.

Keywords: *non linear mathematical model, pollutant, particulate matter, industrialization, stability*

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1. Introduction

Various types of toxicants such as SO_2, NO_2, emitted from human population density dependent sources like manmade thermal power plants etc., affect human health and therefore it is very desirable to remove these toxic pollutants from the atmosphere. It is noted that nitric oxide (NO) converts to nitrogen oxide (NO_2) which in turn reacts with available moisture in the atmosphere to form nitric acid. Similarly sulfur dioxide converts into sulfuric acid. It has been shown that resource and hence population dependent on them may lead to extinction as a result of increased industrialization [1]. The most important techniques by which toxic air pollutants can be removed from the atmosphere are; the precipitation scavenging in which toxic air pollutants are precipitated by the use of liquid droplets or by using particulate matter such as calcium oxide (CaO). In precipitation scavenging, pollutants are absorbed / trapped in liquid droplets and as such these pollutants are precipitated on earth surface whereas using calcium oxide (CaO) as particulate matter with CO_2 results in forming calcium carbonate, thus removing pollutants from the atmosphere as per reaction below,

$$CO_2 + CaO \rightarrow CaCO_3$$

Since visibility is increased after rain, the same phenomenon is used artificially to remove pollutants from the atmosphere. Several experimental investigations have been made to study the removal of pollutants by the process of precipitation [2-5,16]. It can be seen that after rain the visibility always increases and the pollutants are removed from the atmosphere resulting in the enhanced visibility. In some studies around the cities of Kanpur, Varanasi, Pune in India [3,4,5] and Sheffield in United Kingdom [2] appreciable decline in the concentration of pollutants after rain is observed.

Many researchers have developed mathematical models and analyzed them to understand the scavenging of pollutants by precipitation [6-16]. A theoretical framework for scavenging of gases in the atmosphere using rain was developed [10]. A mathematical model to calculate the redistribution and washout of sulfur dioxide by raindrop spectra characteristic of drizzle and heavy rain was also presented [17]. A six dimensional mathematical model has been proposed to study the effect of the density of cloud droplets on the removal of pollutants, gaseous as well as particulate, from the atmosphere [18]. Some investigations have also been made to study the phenomenon of removal of gaseous pollutants and particulate matters by precipitation scavenging using nonlinear mathematical models [19-21].

Thus, in order to reduce the concentration of gaseous pollutants, particulate matters and dust particles which affect our environment considerably in various ways, using liquid droplets and particulate matters can be very significant removal mechanism to keep the environment clean.

From the above, it is observed that no study has been made to remove the pollutants from the atmosphere by using both the liquid droplets and particulate matter calcium oxide (CaO) associated with some human activity from the atmosphere. Therefore, in this paper, we propose and analyze a nonlinear mathematical model to study the removal of toxic air pollutants from the atmosphere using above concepts.

2. Mathematical Model

Consider that in a human habitat toxic air pollutants are emitted by human population dependent industrial sources which affect the human population. Let C be the cumulative concentration of toxic air pollutants, N be the density of human population governed by a logistic model, the growth rate of which decreases due to toxic air pollutants. Let I be the density of industrialization, the growth rate of which is directly proportional to the density of human population. It is further assumed that the growth rate of number densities of liquid droplets C_d and particulate matter C_p are proportional to the concentration of toxic air pollutants present in the environment. The effect of these externally sprayed species is to reduce the concentration of toxic air pollutants in the atmosphere.

Keeping these considerations in view, the model is proposed as follows.

The emission of toxic air pollutants C is governed by the equation (1), wherein Q is the constant emission rate of pollutants in the atmosphere. The growth rate of toxic air pollutants is enhanced by population density dependent industrialization and therefore it is assumed that $\delta > 0$ is the growth rate coefficient of toxic air pollutants in the atmosphere due to increase in industrialization. The constant $\delta_0 > 0$ is natural depletion rate coefficient of toxic air pollutants in the atmosphere. Some of the pollutants are removed from the atmosphere by the use of liquid droplets C_d in atmosphere, $\delta_1 > 0$, being the depletion rate coefficient of pollutants due to externally introduced liquid species. Particulate matters are also used to remove the toxic air pollutants from the atmosphere and therefore the removal of toxic air pollutants is taken in the direct proportion of number density of external species as well as the concentration of these pollutants as in equation (1), $\delta_2 > 0$ being the depletion rate coefficients of toxic air pollutants due to particulate matters. The constant $\delta_3 > 0$ is the depletion rate coefficient of toxic air pollutants due to self awareness of human beings about the adverse effects of these pollutants. In equation (2), N is the population density, the growth rate of which is assumed to follow logistic equation. Let r be the intrinsic growth rate of N with carrying capacity K . Since toxic air pollutants emitted from industries have adverse effect on human population and therefore it is reasonable to assume $r_1 > 0$ as depletion rate coefficient of population due to toxic air pollutants. As population increases, demand and supply equations get changed. Thus, in order to fulfill demand and supply equations, more industries are to be established and therefore the growth of industrialization is assumed to be proportional to the density of human population as shown in equation (3). Therefore, in this equation, the constant $\lambda > 0$ is assumed to be growth rate coefficient of industrialization and $\lambda_0 > 0$ is its natural depletion rate coefficient. In equation (4), $\mu > 0$ is the growth rate coefficient of number density of liquid droplets used to reduce the concentration of toxic air pollutants in the atmosphere. Since some of the liquid droplets decays themselves and hence $\mu_0 > 0$ is taken as natural depletion rate coefficient of liquid droplets. The constant $\mu_1 > 0$ is the depletion coefficient of liquid droplets due to toxic air pollutants. In equation (5), $\nu > 0$ is the growth rate coefficient of particulate matters. Since some of the particulate matters are depleted itself and hence it is assumed that $\nu_0 > 0$ is the natural depletion rate coefficient of particulate mattes. The constant $\nu_1 > 0$ is the depletion coefficient of particulate matters due to toxic air pollutants.

Thus, in view of the above, the system is assumed to be governed by the following nonlinear ordinary differential equations,

$$\frac{dC}{dt} = Q + \delta I - \delta_0 C - \delta_1 C C_d - \delta_2 C_p C - \delta_3 NC \quad (1)$$

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - r_1 NC \quad (2)$$

$$\frac{dI}{dt} = \lambda N - \lambda_0 I \quad (3)$$

$$\frac{dC_d}{dt} = \mu C - \mu_0 C_d - \mu_1 C_d C \quad (4)$$

$$\frac{dC_p}{dt} = \nu C - \nu_0 C_p - \nu_1 C_p C \qquad (5)$$

$C(0) \geq 0$, $I(0) \geq 0$, $N(0) \geq 0$, $C_d(0) \geq 0$, $C_p(0) \geq 0$.

Remark 1:

It is noted from equations (2), (4) and (5) that $r - r_1 C$, $\mu - \mu_1 C_d$ and $\nu - \nu_1 C_p$ are growth rates of population, liquid droplets and particulate matters respectively and hence must be positive for all time $t > 0$.

2.1. Lemma

The region of attraction of the model system (1) – (5) is given as follows,

$$\Omega = \left\{ (C^*, N^*, I^*, C_d^*, C_p^*) \in R^4 : 0 \leq C \leq C_{\max}, \; 0 \leq N \leq K, 0 \leq I \leq \frac{\lambda K}{\lambda_0}, \; 0 \leq C_d \leq \frac{\mu}{\mu_0} C_{\max}, 0 \leq C_p \leq \frac{\nu}{\nu_0} C_{\max} \right\},$$

where $C_{\max} = \dfrac{Q\lambda_0 + \lambda K \delta}{\lambda_0 \delta_0}$

Proof:

From equation (2), we have

$$\frac{dN}{dt} \leq rN\left(1 - \frac{N}{K}\right)$$

implying that $0 \leq N \leq K$.

From equation (3), we note that $0 \leq I \leq \dfrac{\lambda}{\lambda_0} K$.

From equation (1) we have,

$$\frac{dC}{dt} \leq Q + \delta I - \delta_0 C \leq Q + \frac{\lambda \delta K}{\lambda_0} - \delta_0 C$$

which gives, $0 \leq C \leq C_{\max}$ (say)

where $C_{\max} = \dfrac{Q\lambda_0 + \lambda K \delta}{\lambda_0 \delta_0}$.

From equation (4), we have

$$\frac{dC_d}{dt} \leq \mu C - \mu_0 C_d \leq \mu C_{\max} - \mu_0 C_d$$

implying $0 \leq C_d \leq \dfrac{\mu}{\mu_0} C_{\max}$.

In a similar manner, we can show from equation (5) that $0 \leq C_p \leq \dfrac{\nu}{\nu_0} C_{\max}$.

2.2. Equilibrium Analysis

The model system has following non-negative equilibria,

1. $E_0\left(\dfrac{Q}{\delta_0}, 0, 0, 0, 0\right)$

Existence of E_0 is obvious.

2. $E_1(\bar{C}, 0, 0, \bar{C}_d, \bar{C}_P)$.

To show the existence of E_1,

Let

$$f(C) = Q - \delta_0 C - \frac{\delta_1 \mu C^2}{\mu_0 + \mu_1 C} - \frac{\delta_2 \nu C^2}{\nu_0 + \nu_1 C} = 0 \qquad (6)$$

From equation (6), we note that,

(i) $f(0) > 0$

(ii) $f\left(\dfrac{Q}{\delta_0}\right) < 0$

(iii) $f'(C) < 0$.

This implies that there exists a unique positive root (say $\bar{C}$) of $f(C) = 0$ in $0 \leq C \leq \dfrac{Q}{\delta_0} < C_{\max}$. Using this value we can find the value of other variables $\bar{C}_d = \dfrac{\mu \bar{C}}{\mu_0 + \mu_1 \bar{C}}$ and $\bar{C}_p = \dfrac{\nu \bar{C}}{\nu_0 + \nu_1 \bar{C}}$,

3. $E^*(C^*, N^*, I^*, C_d^*, C_p^*)$.

2.3. Existence of the Equilibrium E^*

Equilibrium values of different variables in E^* are given by the following algebraic equations

$$Q + \delta I - \delta_0 C - \delta_1 C C_d - \delta_2 C C_p - \delta_3 N C = 0 \qquad (7)$$

$$N = \frac{K}{r}(r - r_1 C) \qquad (8)$$

$$I = \frac{\lambda}{\lambda_0} N \qquad (9)$$

$$C_d = \frac{\mu C}{\mu_0 + \mu_1 C} \qquad (10)$$

$$C_p = \frac{\nu C}{\nu_0 + \nu_1 C}. \qquad (11)$$

Using equations (8) – (11) in equation (7), we get

$$F(C) = Q + \frac{\delta \lambda K}{\lambda_0 r}(r - r_1 C) - \delta_0 C - \frac{\delta_1 \mu C^2}{\mu_0 + \mu_1 C} - \frac{\delta_2 \nu C^2}{\nu_0 + \nu_1 C} - \delta_3 \frac{K}{r}(r - r_1 C) C = 0. \qquad (12)$$

From equation (12), we note that,

(i) $F(0) > 0$

(ii) $F(C_{\max}) < 0$ in view of remark 1.

Since we have $F(0) > 0$ and $F(C_{\max}) < 0$ this implies that equilibrium level can be attained and thus equilibrium E^* exists. We also note from equation (12) that

(iii) $F'(C) < 0$, which implies that E^* is unique.

Thus, $F(C)=0$ has a unique positive root (say C^*) in $0 \le C \le C_{\max}$. Using this value we can find the value of other variables from equations (8) – (11).

2.4. Variation of Different Variables with Relevant Parameters

2.4.1. Variation of C with δ

Differentiating equation (12) with respect to δ we note that

$$\frac{dC}{d\delta}=\frac{\frac{\lambda K}{\lambda_0 r}(r-r_1C)}{\left[\begin{array}{l}\frac{K\lambda\delta r_1}{\lambda_0 r}+\delta_0+\frac{\mu\delta_1 C(2\mu_0+\mu_1C)}{(\mu_0+\mu_1C)^2}\\+\frac{\nu\delta_2C(2\nu_0+\nu_1C)}{(\nu_0+\nu_1C)^2}+\delta_3\frac{K}{r}(r-2r_1C)\end{array}\right]}>0$$

provided

$$C \le \frac{r}{2r_1} < \frac{r}{r_1}. \tag{13}$$

This implies that concentration of toxic air pollutants in the atmosphere increases as the growth rate of toxic air pollutants due to population density dependent industrialization increases.

2.4.2. Variation of N with δ

From equation (8) we get

$$\frac{dN}{dC}=-\frac{Kr_1}{r}$$

Hence using the relation $\frac{dN}{d\delta}=\frac{dN}{dC}\frac{dC}{d\delta}$ and noting that $\frac{dC}{d\delta}>0$ we get $\frac{dN}{d\delta}<0$.

This implies that as the growth rate of toxic air pollutants increases, the growth of population density decreases.

2.4.3. Variation of C with μ

Differentiating equation (12) with respect to μ, then in view of (13), we get

$$\frac{dC}{d\mu}=-\frac{\frac{\delta_1C^2}{(\mu_0+\mu_1C)}}{\left[\begin{array}{l}\delta_0+\frac{K\lambda\delta r_1}{\lambda_0 r}+\frac{\delta_1\mu C(2\mu_0+\mu_1C)}{(\mu_0+\mu_1C)^2}\\+\frac{\delta_2\nu C(2\nu_0+\nu_1C)}{(\nu_0+\nu_1C)^2}+\delta_3\frac{K}{r}(r-2r_1C)\end{array}\right]}<0.$$

This implies that the concentration of toxic air pollutants decreases with increase in the rate of increase of externally introduced liquid droplets.

2.4.4. Variation of C with ν

Differentiating equation (12) with respect to ν, then in view of (13), we get,

$$\frac{dC}{d\nu}=-\frac{\frac{\delta_2C^2}{(\nu_0+\nu_1C)}}{\left[\begin{array}{l}\delta_0+\frac{K\lambda\delta r_1}{\lambda_0 r}+\frac{\delta_1\mu C(2\mu_0+\mu_1C)}{(\mu_0+\mu_1C)^2}\\+\frac{\delta_2\nu C(2\nu_0+\nu_1C)}{(\nu_0+\nu_1C)^2}+\delta_3\frac{K}{r}(r-2r_1C)\end{array}\right]}<0$$

which also implies that the concentration of toxic air pollutants decreases with increase in the rate of increase of particulate matters in the atmosphere.

2.5. Stability Analysis

In order to establish the local stability behavior of equilibrium, we compute the Jacobian matrix M for the model system (1) – (5)

$$M=\begin{bmatrix}\begin{pmatrix}-\delta_0-\delta_1C_d\\-\delta_2C_p-\delta_3N\end{pmatrix} & -\delta_3C & \delta & -\delta_1C & -\delta_2C\\ -r_1N & r-\frac{2rN}{K}-r_1C & 0 & 0 & 0\\ 0 & \lambda & -\lambda_0 & 0 & 0\\ \mu-\mu_1C_d & 0 & 0 & -(\mu_0+\mu_1C) & 0\\ \nu-\nu_1C_p & 0 & 0 & 0 & -(\nu_0+\nu_1C)\end{bmatrix}.$$

From the above matrix we note that,

(i) Equilibrium $E_0\left(\frac{Q}{\delta_0},0,0,0,0\right)$ is unstable as one eigenvalue $r-\frac{r_1Q}{\delta_0}$ of the Jacobian matrix M corresponding to E_0 is positive.

(ii) Equilibrium $E_1(\bar{C},0,0,\bar{C}_d,\bar{C}_p)$ is unstable as one eigenvalue $r-r_1\bar{C}$ of the Jacobian matrix M corresponding to E_1 is positive.

In the following, we state the local and nonlinear stability theorems for the equilibrium E^*.

2.5.1. Theorem 1

The equilibrium E^* is locally asymptotically stable provided the following conditions are satisfied,

$$r_1\delta_3C^{*2}<\frac{1}{8}\frac{r}{K}(Q+\delta I^*) \tag{14}$$

$$r_1\lambda^2\delta^2<\frac{1}{2}\frac{\lambda_0^2 r\delta_3}{K}(Q+\delta I^*) \tag{15}$$

(See Appendix A for proof).

2.5.2. Theorem 2

The equilibrium E^* is nonlinearly stable inside the region of attraction Ω provided the following conditions are satisfied,

$$S_1=\frac{\delta_0 r}{8K}-r_1\delta_3C^*>0 \tag{16}$$

$$S_2 = \frac{1}{2}\frac{\lambda_0^2 r \delta_3}{K}(Q + \delta I^*) - r_1 \lambda^2 \delta^2 > 0 \qquad (17)$$

(See Appendix B for proof).

It is noted from the above theorems that if population density dependent growth rate coefficient of industrialization (δ), growth rate coefficient of industrialization (λ) and depletion rate coefficient (r_1) of human population growth tend to zero, the local and nonlinear stability conditions, stated in Theorems 1 and 2, will be satisfied automatically. This implies that δ, λ and r_1 have destabilizing effect on the model system.

2.6. Numerical Simulation

In this section, we perform some numerical simulations to study the local and nonlinear stability behavior of equilibria and feasibility of the model system (1)-(5) numerically using MAPLE 18 by choosing the following set of parameter values, $\delta_1 = 0.0003$, $\delta_2 = 0.0004$, $\delta_3 = 0.0001$, $\delta_0 = 0.2$, $\delta = 0.3$, $Q = 40$, $r = 1.6$, $K = 25000$, $r_1 = 0.0001$, $\mu = 0.3$, $\mu_0 = 0.11$, $\mu_1 = 0.01$, $\nu = 0.28$, $\nu_0 = 0.12$, $\nu_1 = 0.01$, $\lambda = 0.0001$, $\lambda_0 = 0.1$.

The equilibrium values of different variables in E^* corresponding to above data are given as $C^* = 17.528269$, $N^* = 24972.61208$, $I^* = 24.972612$, $C_d^* = 18.432526$, $C_p^* = 16.621073$.

The eigenvalues of the Jacobean matrix corresponding to $E^*(C^*, N^*, I^*, C_d^*, C_p^*)$ for the model system $(1)-(5)$ are, -1.598676, -0.459821, $-0.291041 + 0.001306i$, $-0.291041 - 0.001306i$, -0.100137. Since all eigenvalues are negative or having negative real part and hence the interior equilibrium $E^*(C^*, N^*, I^*, C_d^*, C_p^*)$ is locally asymptotically stable. The nonlinear stability behavior of E^* is shown in the Figure 1. This figure depicts that the solution trajectories that start at any point within the region of attraction approach to equilibrium E^*. In Figure 2, the variation of concentration of toxic air pollutants (C) with time t for different values of δ, the growth rate coefficient of toxic air pollutants due to population density dependent industrialization, is plotted. It is observed from the figure that as δ increases, the concentration of toxic air pollutants increases in the atmosphere. In Figure 3, the variation of concentration of toxic air pollutants (C) with time t for different values of δ_1, the depletion rate coefficient of pollutants due to liquid droplets is plotted. It is seen that as δ_1 increases, the concentration of toxic air pollutants C decreases. In Figure 4, the variation of toxic pollutants C with time t for different values of δ_2, the depletion rate coefficient of pollutants due to particulate matters is plotted. From this figure, it is noted that as δ_2 increases, the concentration of toxic pollutants decreases in the atmosphere. Thus, the level of toxic air pollutants increases with increase in the industrialization level but it decreases when liquid droplets or particulate matters are introduced in the atmosphere. As the depletion rate coefficient δ_3 of toxic pollutants due to human activity increases, the concentration of toxic pollutants C decreases in the atmosphere, (Figure 5). In Figure 6, the variation of human population N with time t for different values of r_1, the depletion rate coefficient of population due to toxic air pollutants is shown and it is observed that as r_1 increases, the growth of population N decreases. This implies that the abundance of toxic air pollutants in the atmosphere adversely affects the human population. Figure 7 shows the variation of toxic pollutants concentration C with time t for different values of μ, the growth rate coefficient of liquid droplets in the atmosphere. It is observed that as the rate of introduction of liquid droplets increases, the concentration of toxic air pollutants decreases in the atmosphere. Similar phenomenon of decrease of toxic air pollutants is observed when particulate matters are introduced in the atmosphere with different rates (Figure 8). In Figure 9, the variation of toxic pollutants C with time t for different values of λ is plotted. It is found that the rate of industrialization λ increases due to human activities, the concentration of toxic air pollutants increases in the atmosphere.

We have also plotted stability condition with respect to crucial parameters to study the effect of these variables on stability condition. Figure 10 and Figure 11 show the variation of nonlinear stability condition S_1 and S_2 with respect to parameters r_1 and λ respectively. It is apparent from Figure 10 that S_1 remains positive for $r_1 < 0.00091281$ and negative for $r_1 > 0.00091281$. This implies that the stability condition is satisfied for $0 < r_1 < 0.00091281$ and for higher values of r_1 it will not be satisfied. Hence, r_1 has destabilizing effect on the model system. Likewise, from Figure 11 we infer that λ has destabilizing effect on the model system.

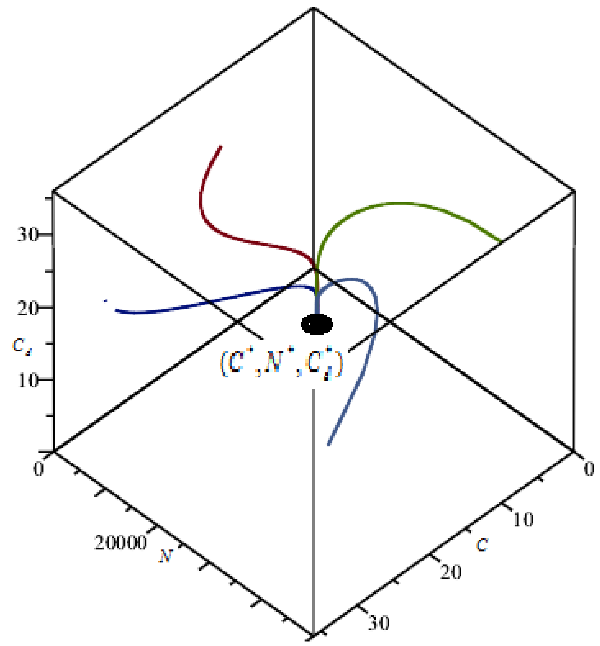

Figure 1. Nonlinear stability in $C - N - C_d$ Plane

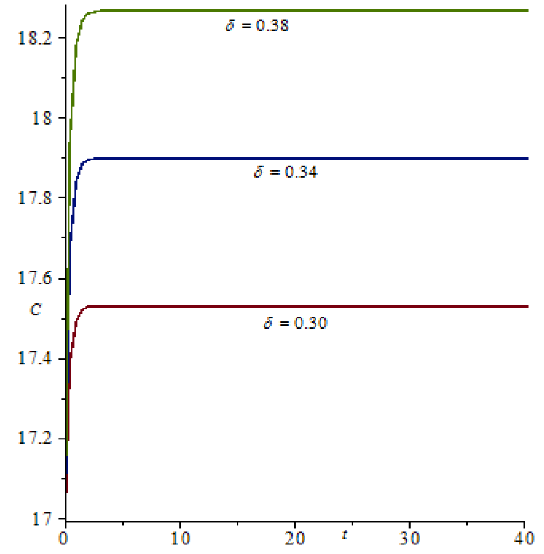

Figure 2. Variation of C with time t for different values of δ

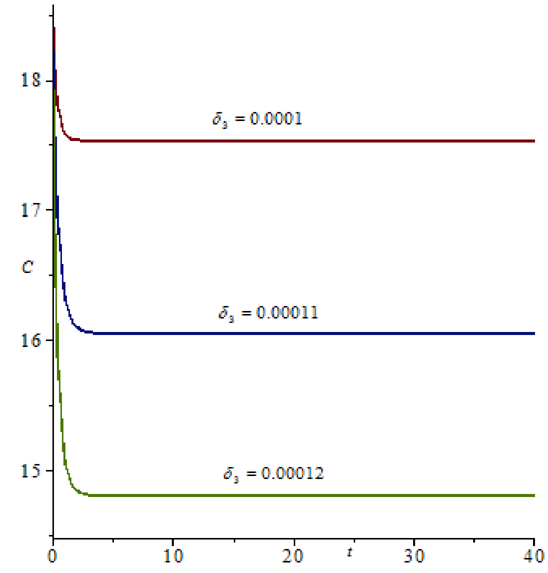

Figure 5. Variation of C with time t for different values of δ_3

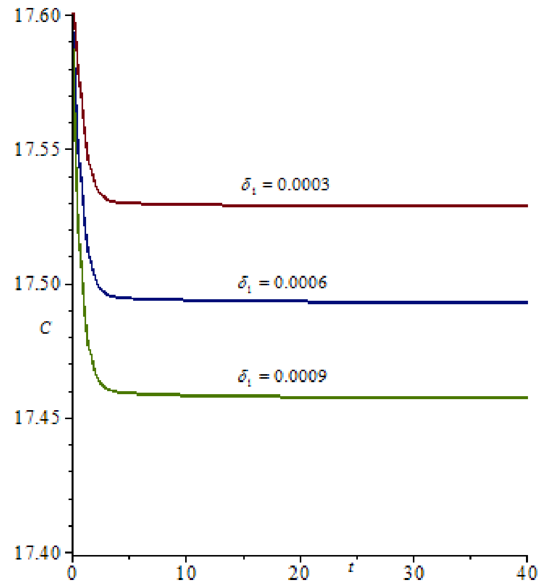

Figure 3. Variation of C with time t for different values of δ_1

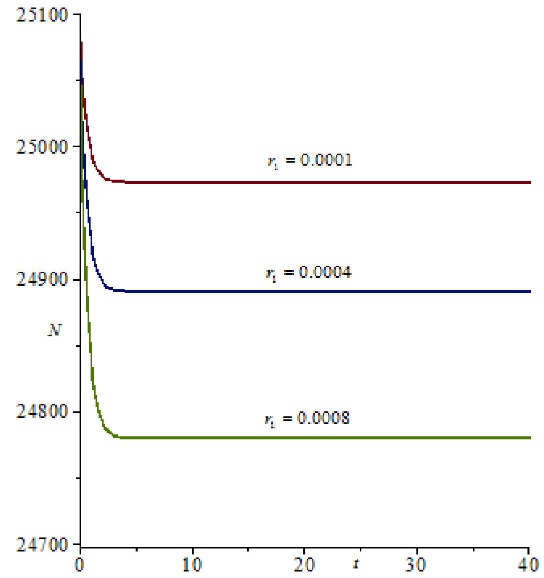

Figure 6. Variation of N with time t for different values of r_1

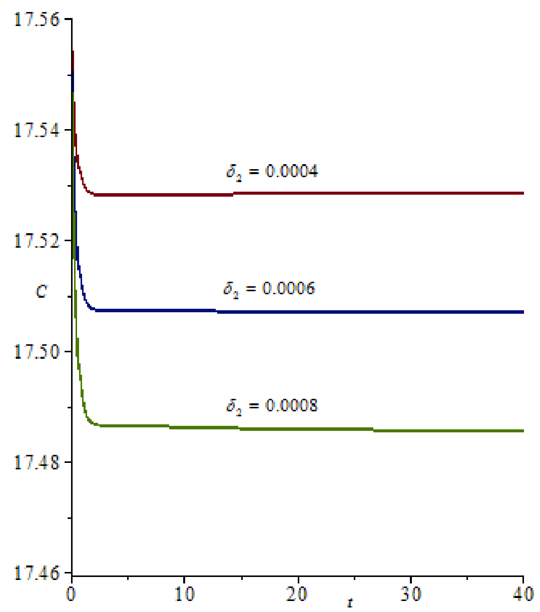

Figure 4. Variation of C with time t for different values of δ_2

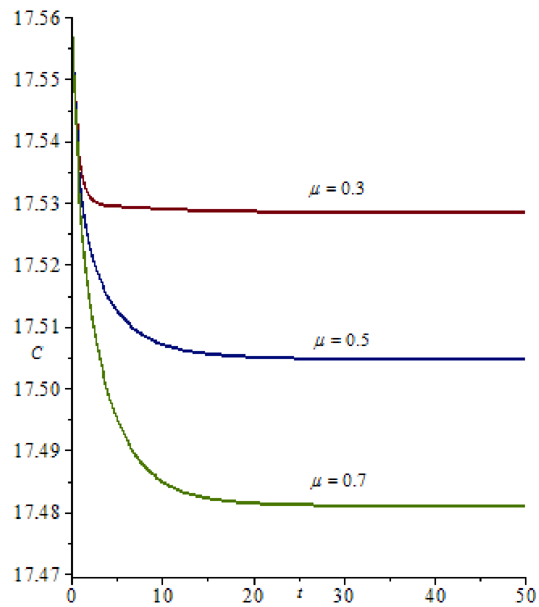

Figure 7. Variation of C with time t different values of μ

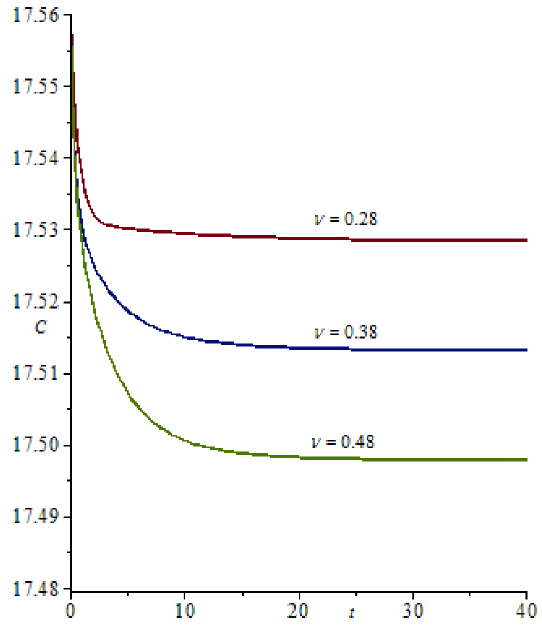

Figure 8. Variation of C with time t for different values of ν

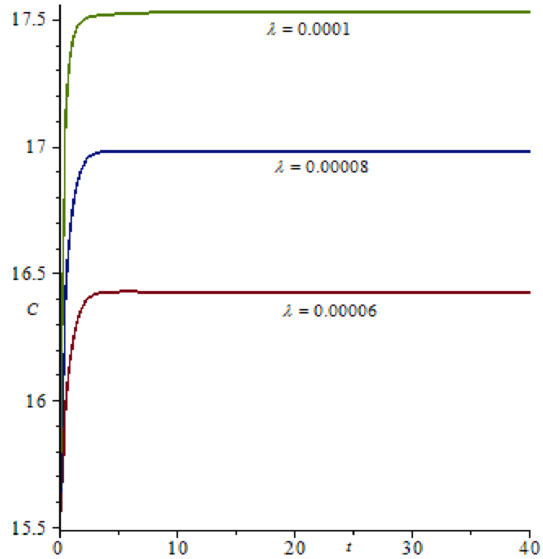

Figure 9. Variation of C with time t for different values of λ

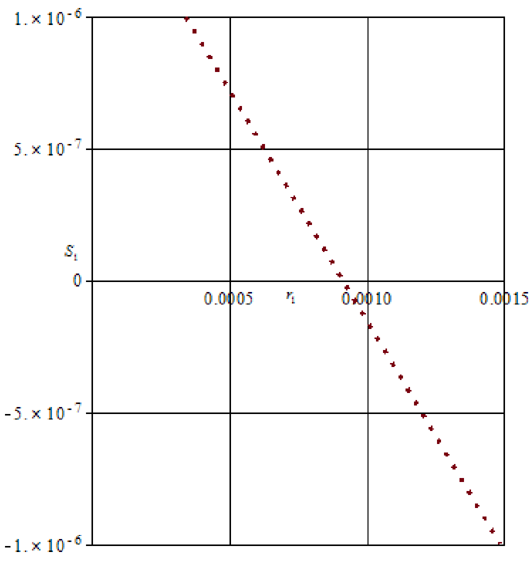

Figure 10. Variation of stability condition S_1 with r_1

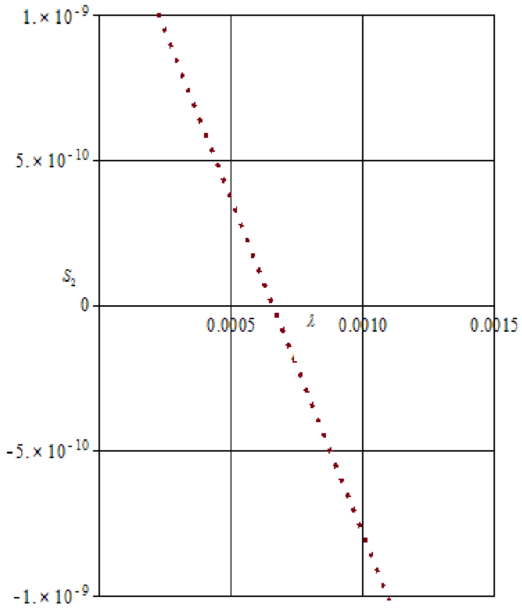

Figure 11. Variation of stability condition S_2 with λ

3. Conclusions

In this paper, a nonlinear mathematical model has been proposed to study the population density dependent industrial emissions of toxic air pollutants in the atmosphere and their removal by liquid droplets and particulate matters. In the modeling process, the following variables have been considered,

(1) The cumulative concentration of toxic air pollutants which is discharged by population density dependent industrialization in the atmosphere.

(2) The density of human population, the growth rate of which decreases due to cumulative density of toxic air pollutants.

(3) The density of industrialization, the growth rate of which is directly proportional to the density of human population.

(4) The density of liquid droplets sprayed in the atmosphere, the growth rate of which is assumed to be proportional to the cumulative concentration of toxic air pollutants.

(5) The concentration of particulate matter which is assumed to be proportional to the cumulative concentration of toxic air pollutants. It is assumed that the droplets and the particulate phases, formed in the atmosphere due to interaction of toxic air pollutants with spraying liquid droplets and particulate matter, remove the toxic air pollutants in the atmosphere in the same proportion by which their concentration/density get increased.

The model has been analyzed by using the stability theory of differential equations. The existence of interior equilibrium is established and its local as well as nonlinear stability has been studied. It has been shown further that due to population density depended emissions (mainly industries), the cumulative concentration of toxic air pollutants in the atmosphere increases. It has also been shown that cumulative concentration of toxic air pollutants decreases considerably by spraying liquid droplets and particulate matters. The model has also been analyzed using numerical simulation which confirms the above analytical results.

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Appendix A

To establish the local stability of E^* let us consider the following positive definite function

$$V = \frac{1}{2}(m_1 C_1^2 + m_2 N_1^2 + m_3 I_1^2 + m_4 C_{d1}^2 + m_5 C_{p1}^2) \tag{A1}$$

where C_1, N_1, I_1, C_{d1} and C_{p1} are the small perturbations about E^* described below,

$$C = C^* + C_1\ ,\ N = N^* + N_1\ , I = I^* + I_1\ C_d = C_d^* + C_{d1}\ ,\ C_p = C_p^* + C_{p1}$$

Differentiating equation (A1) with respect to $'t'$ we get

$$\frac{dV}{dt} = m_1 C_1 \frac{dC_1}{dt} + m_2 N_1 \frac{dN_1}{dt} + m_3 I_1 \frac{dI_1}{dt} + m_4 C_{d1} \frac{dC_{d1}}{dt} + m_5 C_{p1} \frac{dC_{p1}}{dt} \tag{A2}$$

The linearized system of the model system (1) – (5) corresponding to E^* is written as follows

$$\begin{bmatrix} \dot{C}_1 \\ \dot{N}_1 \\ \dot{I}_1 \\ \dot{C}_d \\ \dot{C}_p \end{bmatrix} = \begin{bmatrix} -\dfrac{Q+\delta I^*}{C^*} & -\delta_3 C^* & \delta & -\delta_1 C^* & -\delta_2 C^* \\ -r_1 N^* & \dfrac{-rN^*}{K} & 0 & 0 & 0 \\ 0 & \lambda & -\lambda_0 & 0 & 0 \\ \mu - \mu_1 C_d^* & 0 & 0 & -(\mu_0 + \mu_1 C^*) & 0 \\ \nu - \nu_1 C_p^* & 0 & 0 & 0 & -(\nu_0 + \nu_1 C^*) \end{bmatrix} \begin{bmatrix} C_1 \\ N_1 \\ I_1 \\ C_{d1} \\ C_{p1} \end{bmatrix}$$

Using the above linearized system in equation (A2) and after simplification we have

$$\dot{V} = -m_1 \frac{Q+\delta I^*}{C*} C_1^2 - m_2 \frac{rN^*}{K} N_1^2 - m_3\lambda_0 I_1^2 - m_4(\mu_0 + \mu_1 C^*)C_{d1}^2 - m_5(\nu_0 + \nu_1 C^*)C_{p1}^2$$
$$+ m_1\delta C_1 I_1 - (m_1\delta_3 C^* + m_2 r_1 N^*)C_1 N_1 + [-m_1\delta_1 C^* + m_4(\mu - \mu_1 C_d^*)]C_1 C_{d1}$$
$$+[-m_1\delta_2 C^* + m_5(\nu - \nu_1 C_p^*)]C_1 C_{p1} + \lambda m_3 I_1 N_1. \tag{A3}$$

Now, $\frac{dV}{dt}$ will be negative definite under the following conditions

$$m_1 < \frac{1}{2}\lambda_0 \frac{(Q+\delta I^*)}{\delta^2 C^*} m_3 \tag{A4}$$

$$(m_1\delta_3 C^* + m_2 r_1 N^*)^2 < \frac{1}{2}\left(\frac{Q+\delta I^*}{C^*}\right)\frac{rN^*}{K} m_1 m_2 \tag{A5}$$

$$\lambda^2 m_3 < \lambda_0 \frac{rN^*}{K} m_2 \tag{A6}$$

$$[-m_1\delta_1 C^* + m_4(\mu - \mu_1 C_d^*)]^2 < \frac{4}{5} m_1 m_4 \left(\frac{Q+\delta I^*}{C*}\right)(\mu_0 + \mu_1 C^*) \tag{A7}$$

$$[-m_1\delta_2 C^* + m_5(\nu - \nu_1 C_p^*)]^2 < \frac{4}{5} m_1 m_5 \left(\frac{Q+\delta I^*}{C^*}\right)(\nu - \nu_1 C_p^*) \tag{A8}$$

After some algebraic manipulations and choosing $m_1 = 1$, $m_2 = \frac{\delta_3 C^*}{r_1 N^*}$, $m_4 = \frac{\delta_1 C^*}{\mu - \mu_1 C_d^*}$ and $m_5 = \frac{\delta_2 C^*}{\nu - \nu_1 C_p^*}$ $\frac{dV}{dt}$ will be negative definite provided the condition (14) – (15) are satisfied showing that V is Liaponuv function and hence the theorem.

Appendix B

Consider the following positive definite function about E^*

$$U = \frac{1}{2}k_1(C - C^*)^2 + k_2\left(N - N^* - N^*\log\frac{N}{N^*}\right) + \frac{1}{2}k_3(I - I^*)^2 + \frac{1}{2}k_4(C_d - C_d^*)^2 + \frac{1}{2}k_5(C_p - C_p^*)^2 \tag{B1}$$

Differentiating (B1) with respect to t we get

$$\frac{dU}{dt} = -k_1(\delta_0 + \delta_1 C_d + \delta_2 C_p + \delta_3 N)(C - C^*)^2 - k_2\frac{r}{K}(N - N^*)^2 - k_3\lambda_0(I - I^*)^2$$
$$-k_4(\mu_0 + \mu_1 C)(C_d - C_d^*)^2 - k_5(\nu_0 + \nu_1 C)(C_P - C_p^*)^2 + k_1\delta(C - C^*)(I - I^*)$$
$$-(k_2 r_1 + k_1\delta_3 C^*)(C - C^*)(N - N^*) + k_3\lambda(N - N^*)(I - I^*)$$
$$+[-k_1\delta_1 C^* + k_4(\mu - \mu_1 C_d^*)](C_d - C_d^*)(C - C^*) + [-k_1\delta_2 C^* + k_5(\nu - \nu_1 C_p^*)](C_p - C_p^*)(C - C^*)$$

Now $\frac{dU}{dt}$ will be negative definite under the following conditions

$$k_1\delta^2 < \frac{k_3\delta_0\lambda_0}{2} \tag{B2}$$

$$\left(k_1\delta_3 C^* + k_2 r_1\right)^2 < \frac{k_1 k_2 r\delta_0}{2K} \tag{B3}$$

$$k_3\lambda^2 < \frac{k_2 r\lambda_0}{K} \tag{B4}$$

$$[-k_1\delta_1 C^* + k_4(\mu - \mu_1 C_d^*)]^2 < \frac{4}{5} k_1 k_4 \delta_0 \mu_0 \quad \text{(B5)}$$

$$[-k_1\delta_2 C^* + k_5(\nu - \nu_1 C_p^*)]^2 < \frac{4}{5} k_1 k_5 \delta_0 \nu_0 \quad \text{(B6)}$$

Maximizing left hand side and minimizing right hand side and taking $k_1 = 1$, $k_2 = \frac{\delta_3 C^*}{r_1}$, $k_4 = \frac{\delta_1 C^*}{(\mu - \mu_1 C_d^*)}$, $k_5 = \frac{\delta_2 C^*}{\nu - \nu_1 C_p^*}$, $\frac{dU}{dt}$ will be negative definite provided the condition (16) – (17) are satisfied inside the region of attraction Ω showing that U is Liapunov function hence the theorem.