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Fixed Point Theorem for Non-self Mapping in Cone Metric Space

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Abstract In this paper, we prove a common fixed point theorem for coincidentally commuting non-self mappings for a generalized contraction condition in cone b-metric space.

Keywords: *cone b-metric space, common fixed point, non-self mapping, coincidentally commuting, coincidentally point*

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1. Introduction

Fixed point theory has various equal on fixed point theorems for self-mappings in metric and Banach spaces. Huang and Zhang [1] originated the conception of cone metric space by reconstituting the collection of real numbers by an ordered Banach space and attained few fixed point theorems for mappings gratifying disparate contractive conditions. Various originators like Abbas and Jungck [2], Rhoades [3], Raja and Vaezpour [4] have generalized the result of Huang and Zhang [1] and analyzed the origination of common fixed point in cone metric spaces. In [5], Bhakhtin acquainted b-metric spaces as a generalization of metric spaces and verified the contraction mappings equal in b-metric space that generalizes the familiar Banach contraction results in metric spaces. The analysis of fixed point results for non-self mappings in metrically convex metric space was initiated by Assad and Kirk [6]. B.E. Rhoades and S. Radenovic [7] have manifested fixed point theorems for non-self mappings satisfying generalized contraction condition in cone metric spaces. In this paper, we prove a common fixed point theorem for non-self mappings convincing contraction condition in cone b-metric space.

2. Preliminaries

Definition 2.1

Let B be a real Banach space. A subset C of B is called a cone if and only if

a) C is closed, nonempty and C$\neq \{0\}$,

b) p, q∈R, p, q≥ 0, a, b ∈C shows that $ap + bq$ ∈C,

c) x∩(−x)={0}.

In a cone C⊂B, we imply a partial ordering $\leq$ with respect to C by $a \leq b$which implies $b - a$ ∈C. A cone C is called normal if there is a number n> 0 such that for all a, b ∈B, 0$\leq a \leq$b shows $\|a\| \leq n\|b\|$.

The smallest positive number convincing the above inequality is called normal constant of C, while $a \ll b$ stands for $b - a \in$ int C (interior of C).

Definition 2.2

If Ӿis a non-void set then the mapping d:Ӿ × Ӿ → Econvincing these conditions

a) 0≤d(a, b) for all a, b ∈Ӿ and d$(a, b) = 0$ if and only if $a = b$

b) d$(a, b) =$ d(b, a) for all a, b ∈Ӿ

c) d$(a, b) \leq$d$(a, c) +$ d$(c, b)$for all a, b, c ∈Ӿ.

Called a cone metric in Ӿ and (Ӿ,d) is called a cone metric in Ӿ and (Ӿ,d) is called a cone metric space. The idea of a cone metric space is more familiar than that of a metric space.

Definition 2.3 [8]

If Ӿis a non-void set and $k\geq 1$ be a given real number then the mapping d:Ӿ × Ӿ →E is said to be cone b-metric if and only if for all a, b, c ∈Ӿ, the following conditions are satisfied:

a) 0≤d(a, b) for all a, b ∈Ӿ and d$(a, b) = 0$ if and only if $a = b$

b) d(a, b) =d(b, a) for all a, b ∈Ӿ

c) d$(a, b) \leq k$[d$(a, c) +$ d(c, b)] for alla, b, c ∈Ӿ.

The pair (Ӿ,d) is said to be a cone b-metric space.

Definition 2.4

If (Ӿ,d) is a cone b-metric space then we say that $\{a_n\}$ is

a) A cauchy sequence, if for every k in E with k≫ 0, there is N such that for all n, m> ₦, d(a_n, a_m)≪c

b) A convergent sequence, if for every k in E with k≫ 0, there is N such that for all n> ₦, d(a_n, a) ≪c for some fixed point a in Ӿ.

We say that a cone b-metric space Ẍ is said to be complete only if every Cauchy sequence in Ẍ is convergent in Ẍ. Also $\{a_n\}$ is convergent to a in Ẍ if and only if d$(a_n, a) \to 0$ as $n \to \infty$.

Remarks 2.5 [4]:

1. If a$\leq bandb \ll cthena \ll c.$
2. If a$\ll bandb \ll cthena \ll c.$
3. If 0$\leq a \ll pforeachp \in intCthena = 0.$

Remarks 2.6 [4]

If $p \in intC$, 0$\leq x_n$ and $x_n \to 0$, then there *existan* n_0 such that for all n> $n_0\, wehavex_n \ll p.$

3. Main Result

Theorem 3.1

If (Ẍ,d) is a complete cone b-metric space and Ṁ a non-empty closed subset of X such that for each $a \in$ Ṁ and $b \notin$ Ṁ there exist a point c$\in \partial$Ṁ such that

$$d(a,c)+d(c,b)=d(a,b). \quad (1)$$

Suppose that f, T: Ṁ → Ẍ are two non-self mappings satisfying for all a, b ∈ Ṁ$witha$ ≠b,

$$d(Ta,Tb) \leq \left\{d(fa,fb), \frac{d(fa,Ta)}{2}, \frac{d(fb,Tb)}{2}\right\} + \mu\{d(Ta,fb)+d(Tb,fa)\}. \quad (2)$$

For every a, b in Ṁ and λ, μ are positive real numbers such that (λ+2μ) < 1 and

$$h = max\left\{\frac{\lambda+\mu s}{1-\mu s}, \frac{\lambda+2\mu s}{2(1-\mu s)}, \frac{\mu s}{1-\frac{\lambda}{2}-\mu s}\right\}$$

$$and\ h = max\left\{\begin{array}{l}\frac{(\lambda+\mu s)(1+\mu)}{(1-\mu s)(1-\mu)}, \\ \frac{(\lambda+\mu s)\left(1+\frac{\lambda}{2}+\mu\right)}{(1-\mu s)(1-\mu)}, \\ \frac{(\lambda+\mu s)(1+\mu)}{(1-\mu s)\left(1-\frac{\lambda}{2}-\mu\right)}\end{array}\right\}$$

where s$\geq$ 1 and h" = max$\{h, h'\}$. Also assume that

(1) ∂Ṁ ⊆ fṀ, TṀ ∩ Ṁ ⊆ fṀ

(2) $fa \in \partial$Ṁ implies $Ta \in$ Ṁ

(3) fṀ is closed in Ẍ.

Then there exist a coincidence point of f *and* T *in* Ṁ. Moreover if f *and* T are weakly compatible, then f *and* T have a unique common fixed point in Ṁ.

Proof:

We construct the sequence $\{a_n\}$ and $\{b_n\}$ in M and a sequence $\{b_n\}$ in TṀ. Let $a \in \partial$Ṁ. Set up a point c_0= a. Also $c_0 \in \partial$Ṁ and from condition (1) ∂Ṁ ⊆ fṀ, we have c_0 = fa_0 for some $a_0 \in$ Ṁ. Now, since $fa_0 \in \partial$Ṁ from condition (2) we conclude that $Ta_0 \in$ Ṁ. Also clearly $Ta_0 \in T$Ṁ. Therefore $Ta_0 \in$ Ṁ ∩ TṀ and from (1) $Ta_0 \in f$Ṁ. Therefore for some $a_1 \in$ Ṁ, we have $fa_1 = Ta_0 \in$ Ṁ.

Set up c_1= $b_1 = Ta_0$= a_1 , $b_2 = Ta_1$. If $b_2 = T$Ṁ ∩ Ṁ, then condition (1) implies $b_2 \in f$Ṁ. Therefore for some point $a_2 \in$ Ṁ, we have $fa_2 = b_2 = c_2 = Ta_1$. Suppose if $b_2 = Ta_1 \notin$ Ṁ, then we denote a point c_2 in ∂Ṁ, ($c_2 \neq b_2$) such that

$$d(b_1,c_2)+d(c_2,b_2)=d(b_1,b_2)=d(Ta_0,Ta_1).$$

Next we set $b_3 = Ta_2$=c_3. Therefore for some point $c_3 \in$ Ṁ, we have $fa_3 = b_3 = c_3 = Ta_2$. Therefore, if $c_2 \neq b_2 = fa_1$, then we have $c_1 = b_1 = Ta_0$ and $c_3 = b_3 = Ta_2$. If we continue the process, we obtain three sequences$\{a_n\}$⊆Ṁ, $\{c_n\}$⊆Ṁ, $\{b_n\}$⊆TṀ which is in X, such a way that

a) $b_n = Ta_{n-1}$

b) c_n= fa_n

c) c_n= b_n if and only if $b_n \in$ Ṁ

d) If $c_n \neq b_n$, whenever $b_n \notin$ Ṁand then from equation (1), $c_n \in \partial$Ṁ and d(b_{n-1} ,c_n) + d(c_n, b_n) = $d(b_{n-1}, b_n)$.

If $c_n \neq b_n$, then $b_{n+1} = b_{n+1}$ and $c_{n-1} = b_{n-1}$. We discuss the case about d(c_n, c_{n+1}). If d(c_n, c_{n+1})=0, then it is clear that d(c_n, c_{n+k})=0 for all k$\geq$ 1. Now, if d(c_n, c_{n+1}) >0, for all n, then three cases are distinguished.

Case (i)

If $c_n = b_n \in$ Ṁ and $c_{n+1} = b_{n+1} \in$ Ṁ then $c_n = b_n$ = Ta_{n-1} and $c_{n+1} = b_{n+1}$= Ta_n from (a)

And $c_{n-1} = fa_{n-1}$ from (b). Using contraction (2), we have,

$$\begin{aligned} d(c_n,c_{n+1}) &= d(b_n,b_{n+1}) \\ &= d(Ta_{n-1},Ta_n) \\ &\leq \lambda\left\{\begin{array}{l} d(fa_{n-1},fa_n), \\ \frac{d(fa_{n-1},Ta_{n-1})}{2}, \\ \frac{d(fa_n,Ta_n)}{2} \end{array}\right\} \\ &+\mu\{d(Ta_{n-1},fa_n)+d(Ta_n,fa_{n-1})\} \\ &= \lambda\left\{d(c_{n-1},c_n), \frac{d(c_{n-1},c_n)}{2}, \frac{d(c_n,c_{n+1})}{2}\right\} \\ &+\mu\{d(c_n,c_n)+d(c_{n+1},c_{n-1})\} \\ &= \lambda\left\{d(c_{n-1},c_n), \frac{d(c_{n-1},c_n)}{2}, \frac{d(c_n,c_{n+1})}{2}\right\} \\ &+\mu\{d(c_{n+1},c_{n-1})\}. \end{aligned}$$

Now, three subcases arises,

i) d$(c_n, c_{n+1}) \leq \lambda\{d(c_{n-1}, c_n)\} + \mu\{d(c_{n+1}, c_{n-1})\}$
$\leq \lambda\{d(c_{n-1}, c_n)\} + \mu\{s[d(c_{n+1}, c_n) + d(c_n, c_{n-1})]\}$
d$(c_n, c_{n+1})(1 - \mu s) \leq (\lambda + \mu s)\{d(c_{n-1}, c_n)\}$

$$d(c_n, c_{n+1}) \leq \frac{(\lambda+\mu s)}{(1-\mu s)} d(c_{n-1}, c_n)$$

(ii) d$(c_n, c_{n+1}) \leq \lambda \frac{d(c_{n-1},c_n)}{2} + \mu\{d(c_{n+1}, c_{n-1})\}$

$$\leq \lambda \frac{d(c_{n-1},c_n)}{2} + \mu\{s[d(c_{n+1},c_n) + d(c_n,c_{n-1})]\}$$

$$d(c_n,c_{n+1})(1-\mu s) \leq (\frac{\lambda}{2} + \mu s)\{d(c_{n-1},c_n)\}$$

$$d(c_n,c_{n+1}) \leq (\frac{\lambda+2\mu s}{2(1-\mu s)})\{d(c_{n-1},c_n)\}$$

(iii) $d(c_n,c_{n+1}) \leq \lambda\left\{\frac{d(c_n,c_{n+1})}{2}\right\} + \mu\{d(c_{n+1},c_{n-1})\}$

$$\leq \lambda\left\{\frac{d(c_n,c_{n+1})}{2}\right\} + \mu\left\{s\left[d(c_{n+1},c_n) + d(c_n,c_{n-1})\right]\right\}$$

$$d(c_n,c_{n+1})(1-\frac{\lambda}{2}-\mu s) \leq \mu s\{d(c_n,c_{n-1})\}$$

$$d(c_n,c_{n+1}) \leq \frac{\mu s}{\left(1-\frac{\lambda}{2}-\mu s\right)} d(c_n,c_{n-1}).$$

from subcases (i), (ii), (iii) we get

$$d(c_n,c_{n+1}) \leq hd(c_n,c_{n-1})$$

where $h=\max\left\{\frac{\lambda+\mu s}{1-\mu s}, \frac{\lambda+2\mu s}{2(1-\mu s)}, \frac{\mu s}{1-\frac{\lambda}{2}-\mu s}\right\}$.

Case ii

Let $c_n = b_n \in \dot{M}$ and $c_{n+1} \neq b_{n+1}$. Then $b_{n+1} \in \partial\dot{M}$ and

$$d(b_n,c_{n+1}) + d(c_{n+1},b_{n+1}) = d(b_n,b_{n+1}).$$

Therefore, $d(c_n,c_{n+1}) = d(b_n,c_{n+1}) = d(b_n,b_{n+1}) - d(c_{n+1},b_{n+1}) < d(b_n,b_{n+1})$.

Using contraction (2) we have,

$$d(b_n,b_{n+1}) = d(Ta_{n-1},Ta_n)$$

$$\leq \left\{d(fa_{n-1},fa_n), \frac{d(fa_{n-1},Ta_{n-1})}{2}, \frac{d(fa_n,Ta_n)}{2}\right\}$$

$$+\mu\{d(Ta_{n-1},fa_n) + d(Ta_n,fa_{n-1})\}$$

$$\leq \lambda\left\{d(c_{n-1},c_n), \frac{d(c_{n-1},c_n)}{2}, \frac{d(b_n,b_{n+1})}{2}\right\}$$

$$+\mu\{d(c_n,c_n) + d(b_{n+1},c_{n-1})\}$$

$$\leq \lambda\left\{d(c_{n-1},c_n), \frac{d(c_{n-1},c_n)}{2}, \frac{d(b_n,b_{n+1})}{2}\right\}$$

$$+\mu\{d(b_{n+1},c_{n-1})\}.$$

Now, three subcases arises,

i) $d(b_n,b_{n+1}) \leq \lambda\{d(c_{n-1},c_n)\} + \mu\{d(b_{n+1},c_{n-1})\}$
$\leq \lambda\{d(c_{n-1},c_n)\} + \mu\{s[d(b_{n+1},b_n) + d(b_n,c_{n-1})]\}$
$\leq \lambda\{d(c_{n-1},c_n)\} + \mu\{s[d(b_{n+1},b_n) + d(c_n,c_{n-1})]\}$
$d(b_n,b_{n+1})(1-\mu s) \leq (\lambda+\mu s)\{d(c_{n-1},c_n)\}$

$$d(b_n,b_{n+1}) \leq \frac{(\lambda+\mu s)}{(1-\mu s)} d(c_{n-1},c_n).$$

(ii) $d(b_n,b_{n+1}) \leq \lambda \frac{d(c_{n-1},c_n)}{2} + \mu\{d(b_{n+1},c_{n-1})\}$

$$\leq \lambda \frac{d(c_{n-1},c_n)}{2} + \mu\{s[d(b_{n+1},b_n) + d(b_n,c_{n-1})]\}$$

$$d(b_n,b_{n+1})(1-\mu s) \leq \left(\frac{\lambda}{2} + \mu s\right)\{d(c_{n-1},c_n)\}$$

$$d(b_n,b_{n+1}) \leq \left(\frac{\lambda+2\mu s}{2(1-\mu s)}\right)\{d(c_{n-1},c_n)\}$$

(iii) $d(b_n,b_{n+1}) \leq \lambda\left\{\frac{d(b_n,b_{n+1})}{2}\right\} + \mu\{d(b_{n+1},c_{n-1})\}$

$$\leq \lambda\left\{\frac{d(b_n,b_{n+1})}{2}\right\} + \mu\{s[d(b_{n+1},b_n) + d(b_n,c_{n-1})]\}$$

$$d(b_n,b_{n+1})(1-\frac{\lambda}{2}-\mu s) \leq \mu s\{d(b_n,c_{n-1})\}$$

$$d(b_n,b_{n+1}) \leq \frac{\mu s}{\left(1-\frac{\lambda}{2}-\mu s\right)} d(c_n,c_{n-1}).$$

from subcases (i), (ii), (iii) we get

$$d(c_n,c_{n+1}) \leq hd(c_n,c_{n-1})$$

where $h=\max\left\{\frac{\lambda+\mu s}{1-\mu s}, \frac{\lambda+2\mu s}{2(1-\mu s)}, \frac{\mu s}{1-\frac{\lambda}{2}-\mu s}\right\}$.

Case (iii)

Let $c_n \neq b_n \in \dot{M}$. Then $c_n \in \partial\dot{M}$ and

$$d(b_{n-1},c_n) + d(b_n,c_n) = d(b_{n-1},b_n) \quad (3)$$

and we have $c_{n+1} = b$ and $c_{n-1} = b_{n-1}$.

Now using triangle inequality, we get

$$d(c_n,c_{n+1}) = d(c_n,b_{n+1}) \leq d(c_n,b_n) + d(b_n,b_{n+1}),$$

$$d(c_n,c_{n+1}) = d(b_{n-1},b_n) \leq d(b_{n-1},b_n) + d(b_n,b_{n+1}). \quad (4)$$

We need to find $d(b_{n-1},b_n)$ and $d(b_n,b_{n+1})$. Now using contraction (2) we find $d(b_{n-1},b_n)$.

$$d(b_{n-1},b_n) = d(Ta_{n-2},Ta_{n-1})$$

$$\leq \left\{d(fa_{n-2},fa_{n-1}), \frac{d(fa_{n-2},Ta_{n-2})}{2}, \frac{d(fa_{n-1},Ta_{n-1})}{2}\right\}$$

$$+\mu\{d(Ta_{n-2},fa_{n-1}) + d(Ta_{n-1},fa_{n-2})\}$$

$$\leq \left\{d(c_{n-2},c_{n-1}), \frac{d(c_{n-2},b_{n-1})}{2}, \frac{d(c_{n-1},b_n)}{2}\right\}$$

$$+\mu\{d(b_{n-1},c_{n-1}) + d(b_n,c_{n-2})\}.$$

As $c_{n-1} = b_{n-1}$ and $c_{n-2} = b_{n-2}$ we have,

$$d(b_{n-1},b_n) \leq \{d(c_{n-2},c_{n-1})\} + \mu\{d(b_n,c_{n-2})\}$$

$$\leq \{d(c_{n-2},c_{n-1})\} + \mu\left\{s\begin{bmatrix} d(b_n,c_{n-1}) \\ +d(c_{n-1},c_{n-2}) \end{bmatrix}\right\} d(b_{n-1},b_n) \quad (5)$$

$$\leq \frac{(\lambda+\mu s)}{(1-\mu s)} d(c_{n-2},c_{n-1}).$$

Next we have to find $d(b_n, b_{n+1})$.

$$d(b_n, b_{n+1}) = d(Ta_{n-1}, Ta_n)$$

$$\leq \lambda\left\{d(fa_{n-1}, fa_n), \frac{d(fa_{n-1}, Ta_{n-1})}{2}, \frac{d(fa_n, Ta_n)}{2}\right\}$$

$$+\mu\{d(Ta_{n-1}, fa_n) + d(Ta_n, fa_{n-1})\}$$

$$\leq \lambda\left\{d(c_{n-1}, c_n), \frac{d(c_{n-1}, b_n)}{2}, \frac{d(c_{n,}, b_{n+1})}{2}\right\}$$

$$+\mu\{d(b_n, c_n) + d(b_{n+1}, c_{n-1})\}.$$

Now, we find separately what is $d(b_n, c_n) + d(b_{n+1}, c_{n-1})$ from above equation

$$d(b_n, c_n) + d(b_{n+1}, c_{n-1})$$

$$= d(b_n, c_n) + d(c_{n+1}, c_{n-1})$$

$$= d(b_n, b_{n-1}) - d(c_{n-1}, c_n) + d(c_{n+1}, c_{n-1})$$

$$\leq d(b_n, b_{n-1}) - d(c_{n-1}, c_n) + d(c_{n-1}, c_n) + d(b_n, b_{n-1})$$

$$= d(b_n, b_{n-1}) + d(b_n, b_{n-1})$$

As $b_{n-1} = c_{n-1}, b_{n+1} = c_{n+1}$ and

$$d(b_n, b_{n-1}) \leq \frac{(\lambda + \mu s)}{(1-\mu s)} d(c_{n-2}, b_{n-1}).$$

We get,

$$d(b_n, b_{n+1}) \leq \lambda\left\{\begin{array}{l} d(c_{n-1}, c_n), \\ \frac{\lambda + \mu s}{2(1-\mu s)} d(c_{n-2}, c_{n-1}), \frac{d(c_n, c_{n+1})}{2} \end{array}\right\} \tag{6}$$

$$+\mu\left\{\frac{\lambda + \mu s}{(1-\mu s)} d(c_{n-2}, c_{n-1}) + d(c_{n,}, c_{n+1})\right\}.$$

Substituting (5) and (6) in (4), we get

$$d(c_n, c_{n+1})$$

$$\leq \frac{(\lambda + \mu s)}{(1-\mu s)} d(c_{n-2}, c_{n-1}) - d(c_{n-1}, c_n)$$

$$+\lambda\left\{d(c_{n-1}, c_n), \frac{\lambda + \mu s}{2(1-\mu s)} d(c_{n-2}, c_{n-1}), \frac{d(c_n, c_{n+1})}{2}\right\}$$

$$+\mu\left\{\frac{\lambda + \mu s}{(1-\mu s)} d(c_{n-2}, c_{n-1}) + d(c_{n,}, c_{n+1})\right\}.$$

Again three subcases follows,

(i)

$$d(c_n, c_{n+1}) \leq \frac{(\lambda + \mu s)}{(1-\mu s)} d(c_{n-2}, c_{n-1})$$

$$-d(c_{n-1}, c_n) + \lambda(d(c_{n-1}, c_n))$$

$$+\mu\left\{\frac{(\lambda + \mu s)}{(1-\mu s)} d(c_{n-2}, c_{n-1}) + d(c_n, c_{n+1})\right\}.$$

Therefore,

$$d(c_n, c_{n+1})(1-\mu) \leq \frac{(\lambda + \mu s)}{(1-\mu s)} d(c_{n-2}, c_{n-1})$$

$$+\mu\left\{\frac{(\lambda + \mu s)}{(1-\mu s)} d(c_{n-2}, c_{n-1})\right\} d(c_n, c_{n+1})$$

$$\leq d(c_{n-2}, c_{n-1})\left\{\frac{(\lambda + \mu s)}{(1-\mu s)(1-\mu)} + \mu\frac{(\lambda + \mu s)}{(1-\mu s)(1-\mu)}\right\}.$$

We get,

$$d(c_n, c_{n+1}) \leq d(c_{n-2}, c_{n-1})\left\{\frac{(\lambda + \mu s)(1+\mu)}{(1-\mu s)(1-\mu)}\right\}$$

(ii)

$$d(c_n, c_{n+1})$$

$$\leq \frac{(\lambda + \mu s)}{(1-\mu s)} d(c_{n-2}, c_{n-1}) - d(c_{n-1}, c_n)$$

$$+\lambda\left\{\frac{(\lambda + \mu s)}{2(1-\mu s)} d(c_{n-2}, c_{n-1})\right\}$$

$$+\mu\left\{\frac{(\lambda + \mu s)}{(1-\mu s)} d(c_{n-2}, c_{n-1}) + d(c_n, c_{n+1})\right\}$$

Therefore,

$$d(c_n, c_{n+1})(1-\mu)$$

$$\leq d(c_{n-2}, c_{n-1})\left\{\frac{(\lambda + \mu s)}{(1-\mu s)} + \frac{\lambda}{2}\frac{(\lambda + \mu s)}{(1-\mu s)}\right\}$$

$$+\mu\left\{\frac{(\lambda + \mu s)}{(1-\mu s)} d(c_{n-2}, c_{n-1})\right\}.$$

(iii)

$$d(c_n, c_{n+1})$$

$$\leq \frac{(\lambda + \mu s)}{(1-\mu s)} d(c_{n-2}, c_{n-1})$$

$$-d(c_{n-1}, c_n) + \lambda\left\{\frac{d(c_n, c_{n+1})}{2}\right\}$$

$$+\mu\left\{\frac{(\lambda + \mu s)}{(1-\mu s)} d(c_{n-2}, c_{n-1}) + d(c_n, c_{n+1})\right\}$$

$$\therefore d(c_n, c_{n+1})\left(1 - \frac{\lambda}{2} - \mu\right)$$

$$\leq d(c_{n-2}, c_{n-1})\left\{\frac{\mu(\lambda + \mu s)}{(1-\mu s)} + \frac{(\lambda + \mu s)}{(1-\mu s)}\right\}$$

Therefore, $d(c_n, c_{n+1}) \leq \frac{(\lambda + \mu s)(1+\mu)}{(1-\mu s)\left(1 - \frac{\lambda}{2} - \mu\right)} d(c_{n-2}, c_{n-1}).$

From all the above three subcases we have,

$$d(c_n, c_{n+1}) \leq h' d(c_{n-2}, c_{n-1})$$

Where

$$h' = \max\left\{\begin{array}{l}\frac{(\lambda+\mu s)(1+\mu)}{(1-\mu s)(1-\mu)},\\ \frac{(\lambda+\mu s)\left(1+\frac{\lambda}{2}+\mu\right)}{(1-\mu s)(1-\mu)},\\ \frac{(\lambda+\mu s)(1+\mu)}{(1-\mu s)\left(1-\frac{\lambda}{2}-\mu\right)}\end{array}\right\}$$

In all cases (i), (ii), (iii) we get

$$dc_n, c_{n+1} \le h'' w_n$$

Where

$$h'' = \max\{h, h'\}$$

and

$$w_n \in \{d(c_{n-2}, c_{n-1}), d(c_{n-1}, c_n)\}.$$

Now, following the induction procedure of Assad & Kirk [3] it can be showed that by induction for $n > 1$,

$$d(z_n, z_{n+1}) \le h''^{\frac{n-1}{2}} w_2 \qquad (7)$$

Where, $w_2 \in \{d(c_0, c_1), d(c_1, c_2)\}$ from (7) and by triangle inequality for $n > m$, we have

$$\begin{aligned} d(c_n, c_m) &\le d(c_n, c_{n-1}) + d(c_{n-1}, c_{n-2}) \\ &\quad + \ldots + d(c_{m+1}, c_m) \\ &\le (h''^{\frac{n-1}{2}} + h''^{\frac{n-2}{2}} + \ldots h''^{\frac{m-1}{2}}) w_2 \\ &\le \frac{\sqrt{(h'')}^{m-1}}{1-\sqrt{(h'')}} w_2 \to 0 \ as \ m \to \infty. \end{aligned}$$

From remark 2.5 and 2.6, $d(c_n, c_m) \ll c_t$, where c_t is constant, (i.e), $\{c_n\}$ is a Cauchy sequence. Since $c_n = fa_n \in \dot{M} \cap \partial\dot{M}$ and $\dot{M} \cap \partial\dot{M}$ is complete, there is some point $c \in \dot{M} \cap \partial\dot{M}$ such that $c_n \to c$. Let $w \in \dot{M}$ be such that $fw = c$. By construction of $\{c_n\}$ there is subsequence $\{c_{n(k)}\}$ such that $c_{n(k)} = b_{n(k)} = T_{n(k)-1}$ and so $T_{n(k)-1} \to c$, we have to prove $Tw = c$.

$$d(Tw, c) \le s\left[d\left(Tw, Ta_{n(k)-1}\right) + d\left(Ta_{n(k)-1}, c\right)\right]$$

$$d(Tw, c) \le s\left[\begin{array}{l}\lambda\left\{\begin{array}{l} d\left(fw, fa_{n(k)-1}\right), \frac{d(fw, Tw)}{2}, \\ \frac{d\left(fa_{n(k)-1}, Ta_{n(k)-1}\right)}{2}\end{array}\right\} \\ +\mu\left\{\begin{array}{l} d\left(Tw, fa_{n(k)-1}\right) \\ +d\left(Ta_{n(k)-1}, fw\right)\end{array}\right\} + d\left(Ta_{n(k)-1}, c\right)\end{array}\right].$$

And on using definition and the fact that

$$c_{n(k)} = b_{n(k)} = Ta_{n(k)-1} \to c \ as \ k \to \infty,$$

we obtain again three subcases.

Sub case (i)

$$\begin{aligned} &d(Tw, c) \\ &\le s\left[\begin{array}{l}\lambda\left\{d\left(fw, fa_{n(k)-1}\right)\right\} \\ +\mu\left\{\begin{array}{l} d\left(Tw, fa_{n(k)-1}\right) \\ +d\left(Ta_{n(k)-1}, c\right)\end{array}\right\} \\ +d\left(Ta_{n(k)-1}, c\right)\end{array}\right] \\ &\le s\left[\begin{array}{l}\lambda\left\{d\left(fa_{n(k)-1}, c\right)\right\} \\ +\mu\left\{\begin{array}{l} s\left[d(Tw, c) + d\left(c, fa_{n(k)-1}\right)\right] \\ +d\left(Ta_{n(k)-1}, c\right)\end{array}\right\} \\ +d\left(Ta_{n(k)-1}, c\right)\end{array}\right] \end{aligned}$$

$$\begin{aligned} &d(Tw, c)\left(1-\mu s^2\right) \\ &\le d\left(fx_{n(k)-1}, c\right)\left(\lambda s + \mu s^2\right) \\ &\quad + d\left(Ta_{n(k)-1}, c\right)\left(s + \mu s^2\right) \end{aligned}$$

$$\begin{aligned} &d(Tw, c) \\ &\le \frac{c_t}{2\frac{\left(\lambda s+\mu s^2\right)}{\left(1-\mu s^2\right)}}\frac{\left(\lambda s+\mu s^2\right)}{\left(1-\mu s^2\right)} + \frac{c_t}{2\frac{\left(\lambda s+\mu s^2\right)}{\left(1-\mu s^2\right)}}\frac{\left(s+\mu s^2\right)}{\left(1-\mu s^2\right)} \\ &\le \frac{c_t}{2} + \frac{c_t}{2}. \end{aligned}$$

$$d(Tw, c) \le c_t.$$

Sub case (ii)

$$d(Tw, c) \le s\left[\begin{array}{l}\lambda\left\{\frac{d(fw, Tw)}{2}\right\} \\ +\mu\left\{\begin{array}{l} d\left(Tw, fa_{n(k)-1}\right) \\ +d\left(Ta_{n(k)-1}, fw\right)\end{array}\right\} \\ +d\left(Ta_{n(k)-1}, c\right)\end{array}\right]$$

$$\begin{aligned} d(Tw, c) &\le \lambda s\left\{\frac{d(c, Tw)}{2}\right\} \\ &+\mu s\left\{s\left[d(Tw, c) + d\left(c, fa_{n(k)-1}\right)\right] + d\left(Ta_{n(k)-1}, c\right)\right\} \\ &+s\left[d\left(Ta_{n(k)-1}, c\right)\right]. \end{aligned}$$

$$\begin{aligned} &d(Tw, c)\left(1-\frac{\lambda s}{2}-\mu s^2\right) \\ &\le d\left(Tx_{n(k)-1}, c\right)(s+\mu s) + \mu s d(c, fa_{n(k)-1}) \end{aligned}$$

$$d(Tw,c)$$
$$\leq d\left(Ta_{n(k)-1},c\right)\frac{(s+\mu s)}{\left(1-\frac{\lambda s}{2}-\mu s^2\right)}+\mu s\frac{d\left(c,fx_{n(k)-1}\right)}{\left(1-\frac{\lambda s}{2}-\mu s^2\right)}$$
$$\leq \frac{c_t}{\frac{(s+\mu s)}{\left(1-\frac{\lambda s}{2}-\mu s^2\right)}}\frac{(s+\mu s)}{\left(1-\frac{\lambda s}{2}-\mu s^2\right)}$$
$$+\frac{c_t}{2\left(\frac{\mu s}{\left(1-\frac{\lambda s}{2}-\mu s^2\right)}\right)}\frac{\mu s}{\left(1-\frac{\lambda s}{2}-\mu s^2\right)}\leq \frac{c_t}{2}+\frac{c_t}{2}.$$
$$d(Tw,c)\leq c_t.$$

Sub case (iii)

$$d(Tw,c)$$
$$\leq s\left[\begin{array}{l}\lambda\left\{\frac{d\left(fa_{n(k)-1},Ta_{n(k)-1}\right)}{2}\right\}+\mu\left\{\begin{array}{l}d\left(Tw,fa_{n(k)-1}\right)\\+d\left(Ta_{n(k)-1},fw\right)\end{array}\right\}\\+d\left(Ta_{n(k)-1},c\right)\end{array}\right]$$
$$d(Tw,c)\left(1-\mu s^2\right)$$
$$\leq d\left(fa_{n(k)-1},c\right)\left(\frac{\lambda s^2}{2}+\mu s^2\right)+d\left(Ta_{n(k)-1},c\right)\left(s+\frac{\lambda s^2}{2}+\mu s\right)$$
$$d(Tw,c)\leq\left(\frac{\left(\frac{\lambda s^2}{2}+\mu s^2\right)}{\left(1-\mu s^2\right)}\right)d\left(fa_{n(k)-1},c\right)$$
$$+d\left(Ta_{n(k)-1},c\right)\left(\frac{\left(s+\frac{\lambda s^2}{2}+\mu s\right)}{\left(1-\mu s^2\right)}\right)$$
$$\leq\frac{c_t}{2\left(\frac{\left(\frac{\lambda s^2}{2}+\mu s^2\right)}{\left(1-\mu s^2\right)}\right)}\left(\frac{\left(\frac{\lambda s^2}{2}+\mu s^2\right)}{\left(1-\mu s^2\right)}\right)$$
$$+\frac{c_t}{2\left(\frac{\left(s+\frac{\lambda s^2}{2}+\mu s\right)}{\left(1-\mu s^2\right)}\right)}\left(\frac{\left(s+\frac{\lambda s^2}{2}+\mu s\right)}{\left(1-\mu s^2\right)}\right)\leq\frac{c_t}{2}+\frac{c_t}{2}.$$

∴ d(Tw,c) ≤ c_t

In all subcases (i), (ii), (iii), we obtain d(Tw,c) ≪ c_t for each c_t ∈int C and using result, it follows that d(Tw,c)=0 or Tw=c. If T and f are coincidentally commuting then c=Tw=fw which implies Tc=Tfw=fTw=fc. From contraction (2) we have,

$$d(Tc,c)=d(Tc,Tw)$$
$$\leq\lambda\left\{d(fc,fw),\frac{d(fc,Tc)}{2},\frac{d(fw,Tw)}{2}\right\}$$
$$+\mu\left\{d(Tc,fw)+d(Tw,fc)\right\}$$
$$\leq\lambda\left\{d(Tc,c),\frac{d(Tc,Tc)}{2},\frac{d(c,c)}{2}\right\}$$
$$+\mu\left\{d(Tc,c)+d(c,Tc)\right\}$$
$$\leq\lambda\left\{d(Tc,c),0,0\right\}+\mu\left\{2d(Tc,c)\right\}$$
$$=(\lambda+2\mu)d(Tc,c)$$

Since $(\lambda+2\mu)<1$ as mentioned in the conditions of theorem, it follows that Tc=c, that is c is a common fixed point of f and T. Hence, Uniqueness of common fixed point easily follows from contraction (2).

Corollary 3.2

If (Ẋ,d) is a complete cone b-metric space and Ṁ be a nonempty closed subset of Ẋ such that for each $a\in$ Ṁ and $b\notin$ Ṁ there exist a point c∈ ∂Ṁ such that

$$\mathbf{d}(\mathbf{a},\mathbf{c})+\mathbf{d}(\mathbf{c},\mathbf{b})=\mathbf{d}(\mathbf{a},\mathbf{b}).\qquad(1)$$

Suppose that T: Ṁ → Ẋ satisfying the condition

$$\mathbf{d}(\mathbf{Ta},\mathbf{Tb})\leq\lambda\left\{d(a,b),\frac{d(a,Ta)}{2},\frac{d(b,Tb)}{2}\right\}+\mu\left\{d(Ta,b)+d(Tb,a)\right\}.\qquad(2)$$

For all a, b in Ṁ and λ,μ are positive real numbers such that (λ+2 μ) < 1. If T has additional property that for each $a\in\partial$Ṁ, $Ta\in$ Ṁ, then T has a unique fixed point.

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