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A Unique Fixed Point Theorem on a Generalized d – Cyclic Contraction Mapping in d-Metric Spaces

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Abstract In this paper, we prove a unique fixed point theorem for generalized d-cyclic contraction in dislocated metric spaces (d-metric spaces). Our result generalizes, extends and improves some known results existing in the references.

Keywords: *dislocated metric space, fixed point, cyclic mapping, d-cyclic contraction*

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1. Introduction

In 2000, Hitzler and Seda [2] have introduced the notion of dislocated metric space(also called d-metric space) and established some fixed point theorems in complete dislocated metric spaces, dis-located metric space plays an important role in Topology, Logical programming and in electronics engineering. In 2003, Kirk et al. [5] have introduced the notion of cyclic contraction and they obtained some fixed point theorems for cyclic contractions in dislocated metric spaces. In 2013, George et al. [1] have obtained some fixed point results on d-cyclic contractions in dislocated metric spaces. In this paper, we obtain a unique fixed point theorem for a generalized d-cyclic contraction in dislocated metric spaces.

Definition 1.1 [2]. Let X be a non-empty set and let $d: X \times X \to [0,\infty)$ be a function satisfying the following conditions

(d1) $d(x, y) = d(y, x)$.

(d2) $d(x, y) = d(y, x) = 0 \Rightarrow x = y$.

(d3) $d(x, y) \le d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called dislocated metric or d-metric on X.

Definition 1.2 [2]. A sequence $\{x_n\}$ in a d-metric space (X, d) is called a Cauchy sequence if for given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all m, n ≥ 0, we have $d(x_m, x_n) < \epsilon$.

Definition 1.3 [2]. A sequence $\{x_n\}$ in a d-metric space (X, d) d-converges with respect to d if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case x is called limit of $\{x_n\}$(in d) and we write $x_n \to x$.

Definition 1.4 [2]. A d-metric space (X, d) is called d-complete if every Cauchy sequence in it is d-convergent.

Definition 1.5 [5]. Let A and B be non-empty subsets of a metric space (X, d). A cyclic map

$T: A \cup B \to A \cup B$ is said to be cyclic map if $T(A) \subset B$ and $T(B) \subset A$.

Definition 1.6 [5]. Let A and B be non-empty subsets of a metric space (X, d). A cyclic map $T: A \cup B \to A \cup B$ is said to be a cyclic contraction if there exists $k \in (0,1)$ such that $d(Tx, Ty) \le kd(x, y)$ for all $x \in A$ and $y \in B$.

We define a generalized d-cyclic contraction mapping in the following way.

Definition 1.7. Let A and B be non-empty subsets of a d-metric space (X, d). A cyclic map $T: A \cup B \to A \cup B$ is said to be a generalized d-cyclic contraction if there exists $\alpha, \beta, \gamma > 0$ satisfying $\alpha + 2\beta + 4\gamma < 1$ such that

$$d(Tx,Ty) \le \alpha d(x,y) + \beta\left[d(Tx,x) + d(Ty,y)\right] + \gamma[d(Tx,y) + d(x,Ty)],$$

for all $x \in A$ and $y \in B$.

2. Fixed Point Theorem

Theorem 2.1. Let (X, d) be a complete d-metric space, A and B be non-empty subsets of X and $T: A \cup B \to A \cup B$ be a generalized d- cyclic contraction in X. Then T has a unique fixed point in $A \cap B$.

Proof. Fix $x \in A$. By the definition 1.7 there exists α, β, $\gamma > 0$, $\alpha + 2\beta + 4\gamma < 1$ such that

$$\begin{aligned} d(T^2x, Tx) &= d(T(Tx), Tx), \\ &\le \alpha d(Tx, x) + \beta\left[d(T^2x, Tx) + d(Tx, x)\right] \\ &+ \gamma\left[d(T^2x, x) + d(Tx, Tx)\right] \\ &\le \alpha d(Tx, x) + \beta\left[d(T^2x, Tx) + d(Tx, x)\right] \\ &+ \gamma\left[d(T^2x, Tx) + d(Tx, x) + d(Tx, x) + d(Tx, x)\right] \\ &\le (\alpha + \beta + 3\gamma) d(Tx, x) + (\beta + \gamma) d(T^2x, Tx) \\ &\Rightarrow \mathbf{1} - (\beta + \gamma) d(T^2x, Tx) \le (\alpha + \beta + 3\gamma) d(Tx, x) \\ &\le (\alpha + \beta + 3\gamma) / \mathbf{1} - (\beta + \gamma) d(Tx, x) \le bd(Tx, x), \end{aligned}$$

Where, $b = (\alpha + \beta + 3\gamma)$ /**1-** $(\beta+\gamma) < 1$.

$$d\left(T^2x, Tx\right) \le bd(Tx, x).$$

By induction, we have $d(T^{n+1}x, T^nx) \le b^n d(Tx, x)$, more generally, for m>n, we have

$$\begin{aligned}
&d\left(T^m x, T^n x\right) \\
&\le d\left(T^m x, T^{m-1}x\right) + d\left(T^{m-1}x, T^{m-2}x\right) + \ldots \\
&+ d\left(T^{n+1}x, T^n x\right) \\
&\le \left(b^{m-1} + b^{m-2} + \ldots + b^n\right) d(Tx, x) \\
&= b^n\left(1 + b + b^2 + \ldots + b^{m-n-1}\right) d(Tx, x).
\end{aligned}$$

Since, $b < 1$, so as m, n $\to \infty$ we have

$$b^n\left(1 + b + b^2 + \ldots + b^{m-n-1}\right) \to 0.$$

Hence, $d(T^mx, T^nx) \to 0$,as m, n $\to \infty$.

Therefore, $\{T^nx\}$ is a Cauchy sequence. Since (X, d) is complete so $\{T^nx\}$ converge to some point $z \in X$. Since $\{T^{2n}x\} \subseteq A$ and $\{T^{2n-1}x\} \subseteq B$ and so $z \in A \cap B$.

We claim that $Tz = z$.

$$\begin{aligned}
&d\left(Tz, T^{2n}x\right) = d\left(Tz, T\left(T^{2n-1}x\right)\right), \\
&\le \alpha d\left(z, T^{2n-1}x\right) + \beta\left[d(Tz, z) + d\left(T^{2n}x, T^{2n-1}x\right)\right] \\
&+ \gamma\left[d\left(Tz, T^{2n-1}x\right) + d\left(T^{2n}x, T^{2n-1}x\right)\right].
\end{aligned}$$

Taking limit n→∞, we obtain

$$\begin{aligned}
&d(Tz, z) \\
&\le \alpha\, d(z, z) + \beta\left[d(Tz, z) + d(z, z)\right] \\
&+ \gamma\left[d(Tz, z) + d(z, z)\right] \\
&\le (\alpha + \beta + \gamma) d(z, z) + (\beta + \gamma) d(Tz, z) \\
&\le (\alpha + \beta + \gamma)\left[d(Tz, z) + d(Tz, z)\right] + (\beta + \gamma) d(Tz, z) \\
&\le (\alpha + \beta + \gamma) 2d(Tz, z) + (\beta + \gamma) d(Tz, z) \\
&\le (2\alpha + 2\beta + 2\gamma + \beta + \gamma) d(Tz, z) \\
&\le (2\alpha + 3\beta + 3\gamma) d(Tz, z)
\end{aligned}$$

$$\begin{aligned}
&1 - (2\alpha + 3\beta + 3\gamma) d(Tz, z) \le 0. \\
&\Rightarrow d(Tz, z) = 0. \\
&\Rightarrow Tz = z.
\end{aligned}$$

Thus, z is a fixed point.

To show the uniqueness, let us assume that there exists two fixed points say z_1 and z_2 such that $Tz_1 = z_1$ and $Tz_2 = z_2$.

Now,

$$\begin{aligned}
&d(Tz_1, Tz_2) \\
&\le \alpha\, d(z_1, z_2) + \beta[d(Tz_1, z_1) \\
&+ d(Tz_2, z_2)] + \gamma[d(Tz_1, z_2) + d(Tz_2, z_1)] \\
&\le \alpha d(z_1, z_2) + \beta\left[d(z_1, z_1) + d(z_2, z_2)\right] \\
&\quad + \gamma\left[d(z_1, z_2) + d(z_2, z_1)\right] \\
&\le \alpha d(z_1, z_2) + 2\beta d(z_1, z_1) + 2\gamma d(z_1, z_2) \\
&\le (\alpha + 2\beta + 2\gamma) d(z_1, z_2). \\
&\Rightarrow 1 - (\alpha + 2\beta + 2\gamma) d(z_1, z_2) \le 0.
\end{aligned}$$

Since, $\alpha + 2\beta + 2\gamma < 1$.

$$\begin{aligned}
&\Rightarrow d(z_1, z_2) \le 0. \\
&\Rightarrow z_1 = z_2.
\end{aligned}$$

Therefore, T has a unique fixed point in $A \cap B$.

This completes the proof of the theorem.

Remark 2.2. If we choose $\beta = \gamma = 0$ in the above theorem then we get the d-cyclic contraction theorem3.3 in [1].

Remark 2.3. If we choose $\alpha = \gamma = 0$ in the above theorem then we get the Kannan type d-cyclic contraction theorem3.6 in [1].

Remark 2.4. If we choose $\alpha = \beta = 0$ in the above theorem then we get the Chatterjee type d-cyclic contraction theorem3.8 in [1].

3. Conclusion

The above Theorem 2.1 is a generalization of Theorems 3.3., 3.6., and 3.8., in [1].

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