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Some Constructions of Nearly μ-Resolvable Designs

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Abstract This paper deals with general construction methods of nearly μ - resolvable balanced incomplete block designs with illustration using nearly one resolvable balanced incomplete block (BIB) designs and some known group divisible (GD) designs.

Keywords: *balanced incomplete block design, symmetric balanced incomplete block design, resolvable design, μ – resolvable balanced block designs, nearly μ – resolvable balanced block designs, GD design*

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1. Introduction

An incomplete block design (IBD) is a pair (V, D) where V is a *v*-set of symbols and D is a collection of k – subsets of V called blocks where k < *v*. A balanced incomplete block design (BIBD) is an IBD (V, D) such that each pair of V is contained in exactly λ blocks. We denote such design as a (*v*, k, λ) – BIBD.

A near parallel class of an IBD (V, D), with respect to a symbol s, is a set of blocks that partitions the set V–{S} into (v – 1) / k blocks of that design. We call s the missing symbol / hole of this parallel class.

A design is (near) μ-resolvable if it has a (near) resolution such that any (v – 1) treatments occur μ-times in each of the resolution sets. When μ = 1, it leads to a near resolvable design in the sense of Abel and Furino [2]. The constructions and existence results of nearly resolvable designs can be seen in literatures as Morales et al. [13], Haanpaa and Kaski [8] and Abel and Funiro [2].

The existence of group divisible (GD) designs has been of interest over the years, going back to at least the work of Bose and Shimamoto [5], who began classifying such designs. A group divisible design is a 2-associates partially balanced incomplete block design based on v = mn treatments (being m groups of n treatments each), consisting of b blocks of size k (< v), such that each treatment appears in r blocks, and any two treatments (called the first associates) belonging to the same group occur together in λ_1 blocks whereas any two treatments (called the second associates) belonging to different groups occur together in λ_2 blocks. Furthermore, group divisible (GD) designs are classified into three types: (i) Singular (S) design for which r - λ_1=0; (ii) Semi regular (SR) design for which r - λ_1 > 0 and rk - $v\lambda_2$ = 0; (iii) Regular (R) design for which r - λ_1 > 0 and rk - $v\lambda_2$ > 0. For notations of parameters in a GD association scheme, we refer the reader to Raghavarao [17].

Bose and Nair [4] first introduced the concept of "dualisation" in the field of design of experiments. They derived a new class of block designs by interchanging the role of treatments and blocks in a block design. Shrikhande [18] applied the concept of dualisation to an asymmetric BIB design with λ = 1 or 2 to produce 2-associate PBIB designs. Dualisation of an incomplete block design with respect to unordered pairs of treatments could be found in Vanstone [19], who constructed a BIB design through symmetric BIB designs with λ ≥ 2. Using similar technique Mohan and Kageyama [14] constructed 2-associate group divisible designs. The generalization of the concept due to Vanstone [19] i.e. dualisation with respect to s-tuples could be seen in Kageyama and Mohan [10] in constructing PBIB designs for any s ≥ 1. Kageyama et al. [12] and Philip et al. [16] used the concept of "restricted dualisation" to construct some nested BIB designs and PBIB designs.

In the present paper, we have proposed some construction methods of nearly μ – resolvable BIB designs. In one section, we have proposed construction methods of getting nearly μ – resolvable balanced block designs having same parameters using two different techniques, firstly by rearrangements of blocks and secondly by using some restrictions in dualisation with respect to treatments of the blocks. In the next section we have discussed two elegant methods of getting nearly μ – resolvable balanced block designs using some known group divisible designs.

2. Method of Construction I

Consider a BIB design *D* with parameters (*v*, b, r, k, λ). Considering the r blocks containing any treatment θ_i, $i = 1,2,...,v$. Rearranging the r–blocks corresponding to any i-th treatment $(i = 1, 2, ..., v)$ as

$$
\begin{array}{c|c|c|c}
\overbrace{\theta_1,\ldots\ldots\ldots,\theta_1}^{r} & \overbrace{\theta_2,\ldots\ldots\ldots,\theta_2}^{r} & \ldots\ldots\ldots\ldots\ldots & \overbrace{\theta_v,\ldots\ldots\ldots,\theta_v}^{r} \\
\left\{\begin{array}{c} \vdots \\ remaining \\ (k-1)\ treatments \\ \vdots \end{array}\right. & \left\{\begin{array}{c} \vdots \\ remaining \\ (k-1)\ treatments \\ \vdots \end{array}\right. & \begin{array}{c} \\ \ldots\ldots\ldots\ldots\ldots \\ \\ \ldots\ldots\ldots\ldots\ldots \end{array} & \left\{\begin{array}{c} \vdots \\ remaining \\ (k-1)\ treatments \\ \vdots \end{array}\right.
\end{array}
$$

Now eliminate the treatment θ_i, $(i=1,2,...,v)$ and consider the remaining structure of the r - blocks as D_i, where $i=1,2,...,v$ as

$$D_1^* = [\ D_1 \ | \ D_2 \ | \ .... \ | \ D_v\ ] \quad (2.1.1)$$

This D_1^* denotes the nearly λ–resolvable block design with parameters

$$v_1^* = v,\ \ b_1^* = vr,\ \ r_1^* = r(k-1) \text{ or } \lambda\,(v-1), \quad (2.1.2)$$
$$k_1^* = k-1 \ \text{ and } \ \lambda_1^* = \lambda(k-2)$$

Theorem 2.1: The existence of BIB design with parameters v, b, r, k and λ implies the existence of nearly λ – resolvable block designs with parameters:

$$v_1^* = v,\ \ b_1^* = vr,\ \ r_1^* = r(k-1) \text{ or } \lambda\,(v-1), k_1^* = k-1$$
$$\text{and } \ \lambda_1^* = \lambda(k-2)$$

Proof: Let $\theta_1, \theta_2, \theta_3, \ldots, \theta_v$ denotes the v treatments of the chosen design D. Now rearrange the r blocks of D corresponding to any i-th treatment θ_i, (i = 1, 2,3, ..., v) and then eliminate θ_i, $(i=1,2,........,v)$. Doing the same for all the v treatments, we will get the required nearly λ–resolvable design D_1^*. The parameters v_1^*, b_1^*, r_1^* and k_1^* are obvious by construction. As in the original BIB design any pair of treatments say (θ, ϕ) occurs in λ blocks, the method produces this pair (θ, ϕ) will occur in (k–2) sets of D_i ; $i=1,2,...,(k-2)$ and hence the λ blocks, where (θ, ϕ) occurs in the original BIB design, produce $\lambda_1^* = \lambda(k-2)$. It is now obvious to note that in D_1^*, there is a natural partition of v resolution sets of r blocks each and every (v – 1) treatments occur λ times in i-th resolution set with a hole i.e the i-th resolution set do not contain $\theta_i, i=1,2,...,v$. This completes the proof.

Corollary 2.2: The existence of SBIB design with parameters (v, k, λ) implies the existence of nearly λ – resolvable block designs with parameters:

$$v_1^* = v,\ \ b_1^* = br,\ \ r_1^* = k(k-1) \text{ or } \lambda\,(b-1),\ \ k_1^* = k-1$$
$$\text{and } \ \lambda_1^* = \lambda(k-2)$$

Corollary 2.3: If λ = 1, then the structure define in (2.1.2) leads to a near resolvable design.

Example 2.4: Consider a SBIB design with parameters $v=b=16$, $r=k=6$ and $\lambda=2$. Then the theorem 2.1 yields a nearly 2 – resolvable block design with parameters $v_1^*=16$, $b_1^*=96$, $r_1^*=30$, $k_1^*=5$ and $\lambda_1^*=8$; whose blocks are given as

Table 2.1.

2	2	3	4	5	6
3	7	7	8	9	10
4	8	11	11	12	13
5	9	12	14	14	15
6	10	13	15	16	16

1	1	6	5	3	4
3	7	10	9	7	8
4	8	11	11	14	12
5	9	12	13	15	13
6	10	14	15	16	16

1	1	4	5	2	6
2	7	9	8	7	8
4	11	10	10	14	9
5	12	11	12	15	13
6	13	16	15	16	14

1	1	3	6	5	2
2	8	9	7	7	8
3	11	10	9	10	12
5	14	11	12	13	13
6	15	16	15	14	16

1	1	3	2	4	6
2	9	8	9	7	7
3	12	10	11	10	8
4	14	12	13	13	11
6	16	15	15	14	16

1	1	2	4	5	3
2	10	10	7	7	8
3	13	11	9	8	9
4	15	12	12	11	13
5	16	14	15	16	14

1	1	4	2	4	5
2	3	6	3	5	6
8	11	9	14	10	8
9	12	12	15	13	11
10	13	15	16	14	16

1	1	3	5	3	2
2	4	5	6	6	4
7	11	10	7	9	12
9	14	12	11	13	13
10	15	15	16	14	16

1	1	3	4	2	3
2	5	4	6	5	6
7	12	10	7	11	8
8	14	11	12	13	13
10	16	16	15	15	14

1	1	3	2	3	4
2	6	4	6	5	5
7	13	9	11	8	7
8	15	11	12	12	13
9	16	16	14	15	14

1	1	3	2	2	5
3	4	4	6	5	6
7	8	9	10	9	7
12	14	10	12	13	8
13	15	16	14	15	16

1	1	2	4	3	2
3	5	6	6	5	4
7	9	10	7	8	8
11	14	11	9	10	13
13	16	14	15	15	16

1	1	2	4	3	2
3	6	5	5	6	4
7	10	9	7	8	8
11	15	11	10	9	12
12	16	15	14	14	16

1	1	2	2	4	3
4	5	6	3	5	6
8	9	10	7	7	8
11	12	11	15	10	9
15	16	12	16	13	13

1	1	4	3	2	2
4	6	6	5	5	3
8	10	7	8	9	7
11	13	9	10	11	13
14	16	12	12	13	16

1	1	3	2	5	2
5	6	4	3	6	4
9	10	9	7	7	8
12	13	10	14	8	12
14	15	11	15	11	13

Example 2.5: Consider a SBIB design with parameters $v = b = 7$, $r = k = 4$ and $\lambda = 2$. Then the theorem 2.1 yields a nearly 2 – resolvable block designs with parameters $v_1^* = 7$, $b_1^* = 28$, $r_1^* = 12, k_1^* = 3$ and $\lambda_1^* = 4$, whose blocks are given as

Table 2.2

4	2	2	3
6	5	3	4
7	7	6	5

1	1	3	4
5	3	4	5
7	6	7	6

5	1	2	1
6	2	4	4
7	6	7	5

1	2	1	2
6	3	3	5
7	7	5	6

3	1	1	2
6	2	3	4
7	7	4	6

3	1	1	2
5	4	2	4
7	7	3	5

3	1	1	2
5	4	2	3
6	6	5	4

We can also obtained the nearly λ – resolvable block design defined in (2.1.1) with same parameters given in (2.1.2) using another technique of restricted dualisation with respect to the treatments of the blocks.

3. Method of Construction II

Consider a symmetric balanced incomplete block design D (v, k, λ) having v treatments $\theta_1, \theta_2, \theta_3, \ldots, \theta_v$ (with replication number r) and b number of blocks $B_1, B_2, B_3, \ldots, B_b$ (of size k). Choose any one block from it and call this chosen block as a fixed block (say B_j). Now ignore this fixed block and dualise D with respect to the treatments of this fixed block B_j. Then write the block numbers corresponding to any treatment of this fixed block. A new design so formed is called D_1. Doing this dualisation for all the v treatments, we get a required structure as follows

$$D_2^* = \left[D_1 \quad | \quad D_2 \quad | \quad \ldots \quad | \quad D_v \right] \quad (3.1.1)$$

This designs D_2^* denotes the nearly λ – resolvable designs with parameters same as defined in (2.1.2).

Theorem 3.1: The existence of SBIB design with parameters v = b, r = k and λ implies the existence of nearly λ – resolvable block designs with parameters:

$$v_2^* = v,\ b_2^* = vr,\ r_2^* = r(k-1) \text{ or } \lambda(v-1),\ k_2^* = k-1$$
$$\text{and } \lambda_2^* = \lambda(k-2)$$

Which are same as parameters given in (2.1.2).

Remark: The restricted dualisation used in Method II when made for Non-Symmetrical BIB designs; it produces 2-association PBIB designs.

Example 3.2: Consider a SBIB design with parameters $v = b = 11$, $r = k = 6$ and $\lambda = 3$. Then the theorem 3.1 yields a nearly 3 – resolvable block designs with parameters $v_2^* = 11$, $b_2^* = 66$, $r_2^* = 30$, $k_2^* = 5$ and $\lambda_2^* = 12$. This gives six multiple solution of (11, 5, 2) design with no repeated blocks. The structure is as follows

Table 3.1.

3	5	2	2	3	2
4	7	6	3	4	4
6	8	8	7	5	5
7	10	9	9	9	6
8	11	11	10	11	10

4	1	1	3	1	3
5	6	3	4	4	5
7	8	7	8	5	6
8	9	9	10	6	7
9	11	10	11	10	11

5	1	2	1	2	1
6	2	4	4	5	4
8	7	8	5	6	6
9	9	10	9	7	7
10	10	11	11	11	8

6	2	1	1	1	2
7	3	3	2	3	5
9	8	5	5	6	7
10	10	9	6	7	8
11	11	11	10	8	9

1	1	1	2	2	3
7	3	2	3	4	6
8	4	4	6	7	8
10	9	6	7	8	9
11	11	10	11	9	10

1	1	2	1	3	4
2	2	3	3	5	7
8	4	5	4	8	9
9	5	7	7	9	10
11	10	11	8	10	11

1	2	1	2	4	1
2	3	3	4	6	5
3	5	4	5	9	8
9	6	6	8	10	10
10	11	8	9	11	11

2	1	2	3	1	1
3	3	4	5	5	2
4	4	5	6	7	6
10	6	7	9	10	9
11	7	9	10	11	11

1	2	3	4	1	1
3	4	5	6	2	2
4	5	6	7	6	3
5	7	8	10	8	7
11	8	10	11	11	10

1	3	4	1	1	2
2	5	6	5	2	3
4	6	7	7	3	4
5	8	9	8	7	8
6	9	11	11	9	11

2	4	1	1	2	1
3	6	5	2	3	3
5	7	7	6	4	4
6	9	8	8	8	5
7	10	10	9	10	9

In the next sections, we have discussed two elegant methods of construction of nearly μ – resolvable balanced incomplete block designs.

Preposition 1: Malcolm [14] proved that the two multiple of a SBIB design v = b = 4t+3, r = k = 2t+1 and λ = t; where 4t+3 is a prime or a prime power is always near one resolvable balanced incomplete block design with parameters *v*' = 4t+3, b' = 2(4t+3), r' = 2(2t+1), k' = 2t+1 and λ' = 2t respectively.

4. Method of Construction III

Consider a two multiple of a SBIB (4t+3, 2t+1, t) design. This design *D*' is always near one resolvable balanced incomplete block design since one treatment is missing from each resolution set. Also *D*' gives column-wise BIB design D_c with parameters V_c = 4t+3, B_c = 2(4t+3), R_c = 2(2t+1), K_c = 2t+1, λ_c = 2t and row-wise BIB design D_r with parameters V_r = 4t+3, B_r = (2t+1) (4t+3), R_r = 2(2t+1), K_r = 2, λ_r = 1 respectively.

If we consider rows of D_r as an association scheme, then there are (4t+3) such association schemes with (2t+1) groups of size two which meets with the association scheme of chosen GD design with parameters $v^*=2(2t+1)$, b^*, r^*, k^*, λ_1^*, λ_2^*, $m^*=(2t+1)$ and $n^*=2$. The association scheme for each resolution set is given as

$$\begin{array}{cccc} G_1 & G_2 & \ldots\ldots\ldots\ldots & G_{2t+1} \\ \hline \theta_1 & \theta_2 & \ldots\ldots\ldots\ldots & \theta_{2t+1} \\ \theta_{2t+2} & \theta_{2t+3} & \ldots\ldots\ldots\ldots & \theta_{2(2t+1)} \end{array} \qquad (4.1.1)$$

Considering all the (4t+3) resolution sets, we obtained the resultant design D_3^* as given below

$$D_3^* = [\ D_1 \ |\ D_2 \ |\ \ldots\ldots\ldots\ldots\ldots \ |\ D_v\ ] \quad (4.1.2)$$

This D_3^* gives nearly r^* – resolvable balanced incomplete block design with parameters

$$v_3^* = 4t+3, b_3^* = (4t+3)\, b^*, r_3^* = 2(2t+1)\, r^*,$$
$$k_3^* = k^* \text{ and } \lambda_3^* = \lambda_1^* + 4t\, \lambda_2^*.$$

Theorem 4.1: The existence of a row-wise BIB design and group divisible (GD) design with parameters V_r = 4t+3, B_r = (2t+1) (4t+3), R_r = 2 (2t+1), K_r = 2, λ_r = 1 and $v^*=2(2t+1)$, b^*, r^*, k^*, λ_1^*, λ_2^*, $m^*=(2t+1)$, $n^*=2$ respectively; implies the existence of nearly r^* – resolvable balanced incomplete block design with parameters

$$v_3^* = 4t+3, b_3^* = (4t+3)\, b^*, r_3^* = 2(2t+1)\, r^*,$$
$$k_3^* = k^* \text{ and } \lambda_3^* = \lambda_1^* + 4t\, \lambda_2^*$$

Proof: Let us consider a row-wise BIB design D_r having parameters V_r = 4t+3, B_r = (2t+1) (4t+3), R_r = 2(2t+1), K_r = 2, λ_r = 1 and considering rows of D_r as an association scheme which meets with the association scheme of chosen GD design having parameters $v^* = 2(2t+1)$ (since one treatment is missing from each resolution set in *D*'), b^*, r^*, k^*, λ_1^*, λ_2^*, m^* = $(2t+1)$, $n^* = 2$. In each resolution set, there are $2(2t+1)$ treatments arranged in m^* = $(2t+1)$ groups of size n^* = 2.

There are (4t+3) resolution sets in all. For each resolution set, using the association scheme defined in (4.1.1), we construct group divisible designs D_i (i = 1,2,...,*v*). Then their juxtaposition gives the resultant design D_3^* as defined in (4.1.2).

In D_3^*; $v_3^* = 4t+3$ and $k_3^* = k^*$ are obvious from the construction. Under the present method of construction, since each D_i (i = 1,2,...,*v*) contains b^* blocks and there are (4t+3) resolution sets in D_3^*. Thus in resultant design D_3^* there are $b_3^* = (4t+3)\, b^*$ blocks in all.

Furthermore any treatment, say θ_i, appears in r^* blocks of D_i (i = 1,2,...,*v*). These blocks contribute 2(2t+1) times in D_3^*, since one treatment is missing in each D_i (i = 1,2,...,*v*). Thus $r_3^* = 2(2t+1)\, r^*$.

Also any pair of treatment, say, (θ, ϕ), which is first associate in D_i, will contributes λ_1^* times in one resolution set and in the remaining 4t resolution sets, it will contributes λ_2^* times. Hence $\lambda_3^* = \lambda_1^* + 4t\ \lambda_2^*$.

It can be noted that the resultant design is nearly r^* – resolvable balanced incomplete block design as there is a natural partition of *v* resolution sets of r blocks each and every (*v* – 1) treatments occur λ times in i-th resolution set with a hole i.e the i-th resolution set do not contain θ_i, $i = 1,2,\ldots,v$. This completes the proof.

Example 4.2: Consider a SBIB design with parameters $v = b = 7$, $r = k = 3$ and $\lambda = 1$. A two multiple of this design yields a nearly one resolvable balanced incomplete block design *D*' as given below

$$D' = \left[\begin{array}{cc|cc|cc|cc|cc|cc|cc} 1 & 6 & 2 & 7 & 3 & 1 & 4 & 2 & 5 & 3 & 6 & 4 & 7 & 5 \\ 2 & 5 & 3 & 6 & 4 & 7 & 5 & 1 & 6 & 2 & 7 & 3 & 1 & 4 \\ 4 & 3 & 5 & 4 & 6 & 5 & 7 & 6 & 1 & 7 & 2 & 1 & 3 & 2 \end{array}\right].$$

Here the rows of *D*' forms a BIB design D_r with parameters V_r = 7, B_r = 21, R_r = 6, K_r = 2, λ_r = 1 and considering rows of this design D_r as an association scheme as follows

G_1	G_2	G_3	G_1	G_2	G_3	G_1	G_2	G_3	G_1	G_2	G_3	G_1	G_2	G_3	G_1	G_2	G_3	G_1	G_2	G_3
1	2	4	2	3	5	3	4	6	4	5	7	5	6	1	6	7	2	7	1	3
6	5	3	7	6	4	1	7	5	2	1	6	3	2	7	4	3	1	5	4	2

Each resolution set meets with the GD design having parameters $v^* = b^* = 6$, $r^* = k^* = 3$, $\lambda_1^* = 2$, $\lambda_2^* = 1$, $m^* = 3$ and $n^* = 2$.

Then the theorem 4.1 yields a nearly 3 – resolvable balanced incomplete block design with parameters $v_3^* = 7$, $b_3^* = 42$, $r_3^* = 18$, $k_3^* = 3$ and $\lambda_3^* = 6$; whose blocks are given as

Table 4.1.

1	2	4	6	5	3
2	4	6	5	3	1
6	5	3	1	2	4

2	3	5	7	6	4
3	5	7	6	4	2
7	6	4	2	3	5

3	4	6	1	7	5
4	6	1	7	5	3
1	7	5	3	4	6

4	5	7	2	1	6
5	7	2	1	6	4
2	1	6	4	5	7

5	6	1	3	2	7
6	1	3	2	7	5
3	2	7	5	6	1

6	7	2	4	3	1
7	2	4	3	1	6
4	3	1	6	7	2

7	1	3	5	4	2
1	3	5	4	2	7
5	4	2	7	1	3

5. Method of Construction IV

Consider a two multiple of a SBIB (4t+3, 2t+1, t) design. This design *D*' is always near one resolvable balanced incomplete block design since one treatment is missing from each resolution set. Also *D*' gives column-wise BIB design D_c with parameters V_c = 4t+3, B_c = 2(4t+3), R_c = 2(2t+1), K_c = 2t+1, λ_c = 2t and row-wise BIB design D_r with parameters V_r = 4t+3, B_r = (2t+1) (4t+3), R_r = 2(2t+1), K_r = 2, λ_r = 1 respectively.

If we consider columns of D_c as an association scheme, then there are (4t+3) such association schemes with two groups of size (2t+1) which meets with the association scheme of chosen GD design with parameters $v^* = 2(2t+1)$, b^*, r^*, k^*, λ_1^*, λ_2^*, $m^* = 2$ and $n^* = (2t+1)$. The association scheme for each resolution set is given as

$$\begin{array}{cc} G_1 & G_2 \\ \hline \theta_1 & \theta_{2(t+1)} \\ \theta_2 & \theta_{2t+3} \\ \vdots & \vdots \\ \vdots & \vdots \\ \theta_{2t+1} & \theta_{2(2t+1)} \end{array} \qquad (5.1.1)$$

Considering all the (4t+3) resolution sets, we obtained the resultant design D_4^* is given as

$$D_4^* = \left[\ D_1 \ \middle|\ D_2 \ \middle|\ .... \ \middle|\ D_v \ \right] \quad (5.1.2)$$

This D_4^* gives nearly r^* – resolvable balanced incomplete block design with parameters

$$v_4^* = 4t+3, b_4^* = (4t+3)b^*, r_4^* = 2(2t+1)r^*,$$
$$k_4^* = k^* \text{ and } \lambda_4^* = 2t\lambda_1^* + (2t+1)\lambda_2^*.$$

Theorem 5.1: The existence of a column-wise BIB design and group divisible (GD) design with parameters V_c = 4t+3, B_c = 2(4t+3), R_c = 2(2t+1), K_c = 2t+1, λ_c = 2t and $v^* = 2(2t+1)$, b^*, r^*, k^*, λ_1^*, λ_2^*, m^* = 2, $n^* = (2t+1)$ respectively, implies the existence of nearly r^*– resolvable balanced incomplete block design with parameters

$$v_4^* = 4t+3, b_4^* = (4t+3)b^*, r_4^* = 2(2t+1)r^*,$$
$$k_4^* = k^* \text{ and } \lambda_4^* = 2t\lambda_1^* + (2t+1)\lambda_2^*.$$

Proof: Let us consider a column-wise BIB design D_c having parameters V_c = 4t+3, B_c = 2(4t+3), R_c = 2(2t+1), K_c = 2t+1, λ_c = 2t and considering columns of D_c as an association scheme which meets with the association scheme of chosen GD design having parameters $v^* = 2(2t+1)$ [since one treatment is missing from each resolution set in *D*'], b^*, r^*, k^*, λ_1^*, λ_2^*, m^* = 2, $n^* = (2t+1)$. In each resolution set, there are $2(2t+1)$ treatments arranged in $m^* = 2$ groups of size $n^* = (2t+1)$.

There are (4t+3) resolution sets in all. For each resolution set, using the association scheme defined in (5.1.1), we construct group divisible designs D_i (i = 1,2,...,*v*). Then their juxtaposition gives the resultant design D_4^* as defined in (5.1.2).

In D_4^* ; $v_4^* = 4t+3$ and $k_4^* = k^*$ are obvious from the construction. Under the present method of construction, there are (4t+3) resolution sets in the resultant design and each resolution set contains b^* blocks. Thus in resultant design D_4^* there are $b_4^* = (4t+3)b^*$ blocks in all.

Furthermore any treatment, say θ_i, appears in r^* blocks of D_i (i = 1,2,...,*v*). These blocks contribute 2(2t+1) times in D_4^*, since one treatment is missing in each D_i (i= 1,2,...,*v*). Thus $r_4^* = 2(2t+1)r^*$.

Also any pair of treatment, say (θ, ϕ), which is first associate in D_i, will contributes λ_1^* times in 2t resolution set and in the remaining (2t+1) resolution sets, it will contributes λ_2^* times. Hence $\lambda_4^* = 2t\lambda_1^* + (2t+1)\lambda_2^* = 2t(\lambda_1^* + \lambda_2^*) + \lambda_2^*$.

It can be noted that the resultant design is nearly r^*– resolvable balanced incomplete block design as there is a natural partition of *v* resolution sets of r blocks each and every (*v* – 1) treatments occur λ times in i-th resolution set with a hole i.e the i-th resolution set do not contain $\theta_i, i = 1,2,...,v$. This completes the proof.

Example 5.2: Consider a SBIB design with parameters $v = b = 11$, $r = k = 5$ and $\lambda = 2$. A two multiple of this design yields a nearly one resolvable balanced incomplete block design *D*' as given below

$$D' = \left[\begin{array}{cc|cc|cc|cc|cc|cc|cc|cc|cc|cc|cc} 1 & 10 & 2 & 11 & 3 & 1 & 4 & 2 & 5 & 3 & 6 & 4 & 7 & 5 & 8 & 6 & 9 & 7 & 10 & 8 & 11 & 9 \\ 3 & 8 & 4 & 9 & 5 & 10 & 6 & 11 & 7 & 1 & 8 & 2 & 9 & 3 & 10 & 4 & 11 & 5 & 1 & 6 & 2 & 7 \\ 4 & 7 & 5 & 8 & 6 & 9 & 7 & 10 & 8 & 11 & 9 & 1 & 10 & 2 & 11 & 3 & 1 & 4 & 2 & 5 & 3 & 6 \\ 5 & 6 & 6 & 7 & 7 & 8 & 8 & 9 & 9 & 10 & 10 & 11 & 11 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 \\ 9 & 2 & 10 & 3 & 11 & 4 & 1 & 5 & 2 & 6 & 3 & 7 & 4 & 8 & 5 & 9 & 6 & 10 & 7 & 11 & 8 & 1 \end{array}\right]$$

Here the columns of *D*' forms a BIB design D_c with parameters $V_c = 11$, $B_c = 22$, $R_c = 10$, $K_c = 5$, $\lambda_c = 4$ and considering columns of this design D_c as an association scheme as follows

G_1	G_2	G_1	G_2	G_1	G_2	G_1	G_2	G_1	G_2	G_1	G_2	G_1	G_2	G_1	G_2	G_1	G_2	G_1	G_2	G_1	G_2
1	10	2	11	3	1	4	2	5	3	6	4	7	5	8	6	9	7	10	8	11	9
3	8	4	9	5	10	6	11	7	1	8	2	9	3	10	4	11	5	1	6	2	7
4	7	5	8	6	9	7	10	8	11	9	1	10	2	11	3	1	4	2	5	3	6
5	6	6	7	7	8	8	9	9	10	10	11	11	1	1	2	2	3	3	4	4	5
9	2	10	3	11	4	1	5	2	6	3	7	4	8	5	9	6	10	7	11	8	1

Each resolution set meets with the GD design having parameters $v^* = 11$, $b^* = 25$, $r^* = 10$, $k^* = 4$, $\lambda_1^* = 5$, $\lambda_2^* = 2$, $m^* = 2$ and $n^* = 5$.

Then the theorem 5.1 yields a nearly 10 – resolvable balanced incomplete block design with parameters $v_4^* = 11$, $b_4^* = 275$, $r_4^* = 100$, $k_4^* = 4$ and $\lambda_4^* = 30$; whose blocks are given as

Table 5.1.

1	1	1	1	10	10	3	1	1	10	10	3	3	8	8	1	1	1	1	3	10	10	10	10	8
10	10	3	8	3	8	8	7	5	4	7	4	7	4	4	3	3	3	4	4	8	8	8	7	7
3	4	4	6	6	5	7	6	9	6	5	9	5	7	5	4	4	5	5	5	7	7	6	6	6
8	7	5	9	9	2	2	2	2	2	9	2	6	9	6	5	9	9	9	9	6	2	2	2	2

2	2	2	2	11	11	4	2	2	11	11	4	4	9	9	2	2	2	2	4	11	11	11	11	9
11	11	4	9	4	9	9	8	6	5	8	5	8	5	5	4	4	4	5	5	9	9	9	8	8
4	5	5	7	7	6	8	7	10	7	6	10	6	8	6	5	5	6	6	6	8	8	7	7	7
9	8	6	10	10	3	3	3	3	3	10	3	7	10	7	6	10	10	10	10	7	3	3	3	3

3	3	3	3	1	1	5	3	3	1	1	5	5	10	10	3	3	3	3	5	1	1	1	1	10
1	1	5	10	5	10	10	9	7	6	9	6	9	6	6	5	5	5	6	6	10	10	10	9	9
5	6	6	8	8	7	9	8	11	8	7	11	7	9	7	6	6	7	7	7	9	9	8	8	8
10	9	7	11	11	4	4	4	4	4	11	4	8	11	8	7	11	11	11	11	8	4	4	4	4

4	4	4	4	2	2	6	4	4	2	2	6	6	11	11	4	4	4	4	6	2	2	2	2	11
2	2	6	11	6	11	11	10	8	7	10	7	10	7	7	6	6	6	7	7	11	11	11	10	10
6	7	7	9	9	8	10	9	1	9	8	1	8	10	8	7	7	8	8	8	10	10	9	9	9
11	10	8	1	1	5	5	5	5	5	1	5	9	1	9	8	1	1	1	1	9	5	5	5	5

5	5	5	5	3	3	7	5	5	3	3	7	7	1	1	5	5	5	5	7	3	3	3	3	1
3	3	7	1	7	1	1	11	9	8	11	8	11	8	8	7	7	7	8	8	1	1	1	11	11
7	8	8	10	10	9	11	10	2	10	9	2	9	11	9	8	8	9	9	9	11	11	10	10	10
1	11	9	2	2	6	6	6	6	6	2	6	10	2	10	9	2	2	2	2	10	6	6	6	6

6	6	6	6	4	4	8	6	6	4	4	8	8	2	2	6	6	6	6	8	4	4	4	4	2
4	4	8	2	8	2	2	1	10	9	1	9	1	9	9	8	8	8	9	9	2	2	2	1	1
8	9	9	11	11	10	1	11	3	11	10	3	10	1	10	9	9	10	10	10	1	1	11	11	11
2	1	10	3	3	7	7	7	7	7	3	7	11	3	11	10	3	3	3	3	11	7	7	7	7

7	7	7	7	5	5	9	7	7	5	5	9	9	3	3	7	7	7	7	9	5	5	5	5	3
5	5	9	3	9	3	3	2	11	10	2	10	2	10	10	9	9	9	10	10	3	3	3	2	2
9	10	10	1	1	11	2	1	4	1	11	4	11	2	11	10	10	11	11	11	2	2	1	1	1
3	2	11	4	4	8	8	8	8	8	4	8	1	4	1	11	4	4	4	4	1	8	8	8	8

8	8	8	8	6	6	10	8	8	6	6	10	10	4	4	8	8	8	8	10	6	6	6	6	4
6	6	10	4	10	4	4	3	1	11	3	11	3	11	11	10	10	10	11	11	4	4	4	3	3
10	11	11	2	2	1	3	2	5	2	1	5	1	3	1	11	11	1	1	1	3	3	2	2	2
4	3	12	5	5	9	9	9	9	9	5	9	2	5	2	12	5	5	5	5	2	9	9	9	9

9	9	9	9	7	7	11	9	9	7	7	11	11	5	5	9	9	9	9	11	7	7	7	7	5
7	7	11	5	11	5	5	4	2	1	4	1	4	1	1	11	11	11	1	1	5	5	5	4	4
11	1	1	3	3	2	4	3	6	3	2	6	2	4	2	1	1	2	2	2	4	4	3	3	3
5	4	2	6	6	10	10	10	10	10	6	10	3	6	3	2	6	6	6	6	3	10	10	10	10

10	10	10	10	8	8	1	10	10	8	8	1	1	6	6	10	10	10	10	1	8	8	8	8	6
8	8	1	6	1	6	6	5	3	2	5	2	5	2	2	1	1	1	2	2	6	6	6	5	5
1	2	2	4	4	3	5	4	7	4	3	7	3	5	3	2	2	3	3	3	5	5	4	4	4
6	5	3	7	7	11	11	11	11	11	7	11	4	7	4	3	7	7	7	7	4	11	11	11	11

11	11	11	11	9	9	2	11	11	9	9	2	2	7	7	11	11	11	11	2	9	9	9	9	7
9	9	2	7	2	7	7	6	4	3	6	3	6	3	3	2	2	2	3	3	7	7	7	6	6
2	3	3	5	5	4	6	5	8	5	4	8	4	6	4	3	3	4	4	4	6	6	5	5	5
7	6	4	8	8	1	1	1	1	1	8	1	5	8	5	4	8	8	8	8	5	1	1	1	1

6. Results and Discussion

The following Table 6.1 provides the list of nearly μ – resolvable incomplete block designs for Methods I, II which are obtained by using certain known SBIB designs with k ≤15.

Table 6.1.

S. No.	Used Design			Resultant Design						Reference No.**
	Symmetric balanced incomplete block design			Nearly μ – resolvable balanced incomplete block design						
	v	k	λ	v_1^*	b_1^*	r_1^*	k_1^*	λ_1^*	μ	
1	15	7	3	15	105	42	6	15	3	R(43), MH(16)
2	15	8	4	15	120	56	7	24	4	R(44)
3	16	10	6	16	160	90	9	48	6	R(49)
4*	19	9	4	19	171	72	8	28	4	R(55), MH(30)
5	25	9	3	25	225	72	8	21	3	R(67), MH(31)
6	27	13	6	27	351	156	12	66	6	R(71), MH(72)
7	31	10	3	31	310	90	9	24	3	R(76), MH(40)
8	37	9	2	37	333	72	8	14	2	R(81), MH(34)

*In the Table 2.1 of Kageyama et al. [11]; design no. 29 with v=19 was 8-resolvable BIB design but here in our construction it is near 4-resolvable BIB design.
**The symbols R(α) and MH(α) denote the reference number α in Raghavrao [17] and Marshal Hall's [9] list.

The following Table 6.2 provides the list of nearly μ – resolvable incomplete block designs for Method III which are obtained by using nearly one-resolvable balanced incomplete block design and certain known GD designs from Clatworthy table [6].

Table 6.2.

S. No.	Designs used		Resultant design					
			Nearly μ-Resolvable BIBD					
	Nearly one-resolvable BIBD (when rows are taken as an association scheme)	GD design	v_3^*	b_3^*	r_3^*	k_3^*	λ_3^*	μ
1	v’ = 11, b’ = 22, r’ = 10, k’ = 5 and λ’ = 4	SR52	11	88	80	5	16	8
2		SR53	11	132	60	5	24	6
3		SR54	11	110	80	5	32	8
4		SR55	11	220	100	5	40	10
5		R69	11	220	60	3	12	6
6		R106	11	220	80	4	24	8
7		R108	11	275	100	4	30	10
8	v’ = 19, b’ = 38, r’ = 18, k’ = 9 and λ’ = 8	S37	19	228	72	6	20	4
9		SR99	19	228	108	9	48	6

*Since the other resultant designs have high replication numbers so we have ignored those cases for method III and method IV as well.

References

[1] Abel R.J.R. (1994). Forty-Three Balanced Incomplete Block Designs. *Journal of Combinatorial Theory*, Series A 65, 252-267.

[2] Abel R.J.R. and Furino S.F. (2007). Resolvable and near resolvable designs, in: *The CRC Handbook of Combinatorial Designs*, (2nd edition), (C.J. Colbourn and J.H. Dinitz, eds.), CRC Press, Boca Raton FL, U.S.A., 87-94.

[3] Bose, R. C. (1942). A note on resolvability of balanced incomplete block designs. *Sankhya*, 6, 105-110.

[4] Bose, R. C. and Nair, K. R. (1939). Partially balanced block designs. *Sankhya*, 4, 337-372.

[5] Bose R.C., Shimamoto T. (1952). Classification and analysis of partially balanced incomplete block designs with two associate classes, *Journal of the American Statistical Association*, 47, 151-184.

[6] Clatworthy, W. H. (1973). Tables of two-associate class partially balanced designs. *NBS Applied Mathematics*, Series 63.

[7] Dey, A. (1986). *Theory of block designs*. Wiley Eastern Limited, New Delhi.

[8] Haanpaa H. and Kaski P. (2005). The near resolvable 2-(13,4,3) designs and thirteen-player whist tournaments. *Des. Codes Cryptog*r. 35, 271-285.

[9] Hall, M. Jr. (1986). *Combinatorial Theory*. John Wiley, New York.

[10] Kageyama, S. and Mohan, R. N. (1984). Dualizing with respect to s-tuples. *Proc. Japan Acad.* Ser. A 60, 266-268.

[11] Kageyama, S., Majumdar, A. And Pal, S. (2001). A new series of μ-resolvable BIB designs. *Bull. Grad. School Educ. Hiroshima Univ*., Part II, No. 50, 41-45.

[12] Kageyama, S., Philip, J. and Banerjee, S. (1995). Some constructions of nested BIB and 2-associate PBIB designs under restricted dualization. *Bull. Fac. Sch. Educ. Hiroshima Univ*., Part II, 17, 33-39.

[13] Luis B. Morales, Rodolfo San Agust´ın,1 Carlos Velarde (2007). Enumeration of all $(2k + 1, k, k - 1)$-NRBIBDs for $3 \leq k \leq 13$. *JCMCC*, 60, 81-95

[14] Malcolm Greig, Harri Haanpaa and Petteri Kaski (2005). On the coexistence of conference matrices and near resolvable 2-(2k+1, k, k-1) designs. *Journal of Combinatorial Theory*, Series A, 113, 703-711.

[15] Mohan, R. N. and Kageyama, S. (1983). A method of construction of group divisible designs. *Utilitas Math*., 24, 311-316.

[16] Philip, J., Banerjee, S. and Kageyama, S., (1997). Constructions of nested t-associate PBIB designs under restricted dualization. *Utilitas Mathematica*, 51, 27-32.

[17] Raghavarao, D. (1971). *Constructions and Combinatorial Problems in Designs of Experiments*. John Wiley, New York.

[18] Shrikhande, S. S. (1952). On the dual of some balanced incomplete block designs. *Biometrics*, 8, 66-72.

[19] Vanstone, A. (1975). A note on a construction of BIBD's. *Utilitas Math.*, 7, 321-322.