

Tests for Exponentiality: A Comparative Study

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Abstract The exponential distribution is considered in situations where intervals between events are considered as well as where a skewed distribution is appropriate. The exponential distribution also plays key role in survival analysis. Goodness-of-fit for exponentiality is crucial as, in the natural sciences, some of the commonly used distributions such as gamma and Weibull distributions are just translated versions of the exponential distributions. Several well known exponentiality tests are reviewed. A power comparison is performed using simulation.

Keywords: Ahsanullah characterization, Gini statistic, Pietra statistic, Rossberg characterization

Cite This Article: Mezbahur Rahman, and Han Wu, "Tests for Exponentiality: A Comparative Study." *American Journal of Applied Mathematics and Statistics*, vol. 5, no. 4 (2017): 125-135. doi: 10.12691/ajams-5-4-3.

1. Introduction

Assessing whether a data set is in compliance with a postulated distribution is termed as goodness-of-fit of a distribution. Tests for exponentiality is the general term for goodness-of-fit tests for exponential distributions. An exhaustive list of references for goodness-of-fit tests for exponential distributions are given in N. Balakrishnan and Asit P. Basu [1], K. Yu. Volkova [2], A. P. Rogozhnikov and B. Yu. Lemeshko [3], and the references there in.

In this paper, we implement several exponentiality tests, such as, Test for exponentiality based on Ahsanullah characterization [4,5], Atkinson test for exponentiality [6], Cox and Oakes test for exponentiality [7], Cramervon Mises test for exponentiality [7], Deshpande test for exponentiality [8], Test for exponentiality of Epps and Pulley ([7] section 2.8.1), Epstein test for exponentiality [9], Frozini test for exponentiality [10], test for exponentiality based on the Gini statistic [11], Gnedenko F-test of exponentiality [9], Harris modification of Gnedenko F-test [9], Hegazy-Green test for exponentiality [12], Hollander-Proshan test for exponentiality [13], Kimber-Michael test [14] and [15], Kochar test for exponentiality [16], Kolmogorov-Smirnov test ([7] section 2.1) forexponentiality, Lorenz test for exponentiality [17], Moran test for exponentiality [18,19], Pietra statistic [9], exponentiality based on Rossberg characterization [2], Shapiro-Wilk test for exponentiality [20], WE test for exponentiality [9], Wong and Wong test for exponentiality [9], and Anderson-Darling test for exponentiality [21].

2. Tests Descriptions

2.1. Test for Exponentiality Based on Ahsanullah Characterization

The test is based on the following statistic by Alexey Novikov and Ruslan Pusev,

$$I_n = \int_0^{\infty} (H_n(t) - G_n(t)) dF_n(t),$$

where F_n is the empirical distribution function,

$$H_n(t) = \frac{1}{n^2} \sum_{i,j=1}^n 1\{|X_i - X_j| < t\}, t \geq 0,$$

$$G_n(t) = \frac{1}{n^2} \sum_{i,j=1}^n 1\{2 \min(X_i - X_j) < t\}, t \geq 0.$$

Under exponentiality, one has

$$\sqrt{n} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{647}{4725}\right), \text{ as } n \rightarrow \infty$$

[4,5]. This test can be implemented either using the asymptotic normal distribution or using the simulated distribution under the exponentiality assumption. Here we represent them as *AHTT* and *AHTS*, respectively.

2.2. Atkinson Test for Exponentiality

The Atkinson test for exponentiality is based on the following statistic [6],

$$T_n = \sqrt{n} \left| \frac{\left(n^{-1} \sum_{i=1}^n X_i^p\right)^{1/p}}{\bar{X}} - (\Gamma(1+p))^{1/p} \right|.$$

The statistic is asymptotically (absolute) normal,

$$T_n(P) \xrightarrow{\mathcal{D}} \left| \mathcal{N}\left(0, \sigma^2(p)\right) \right|, \text{ as } n \rightarrow \infty,$$

where,

$$\sigma^2(p) = (\Gamma(1+p))^{2/p} \left(-1 - \frac{1}{p^2} + \frac{\Gamma(1+2p)}{p^2 \Gamma^2(1+p)} \right).$$

This test can be implemented either using the asymptotic normal distribution or using the simulated

distribution under the exponentiality assumption. Here we represent them as *AKTT* and *AKTS*, respectively.

2.3. Test for Exponentiality of Cox and Oakes

The Cox and Oakes test is a test for the composite hypothesis of exponentiality [7]. The test statistic is,

$$CO_n = n + \sum_{j=1}^n (1 - y_j) \log(Y_j),$$

where $Y_j = X_j / \bar{X}$. $(6/n)^{1/2} (CO_n / \pi)$ is asymptotically standard normal [7]. This test can be implemented either using the asymptotic normal distribution or using the simulated distribution under the exponentiality assumption. Here we represent them as *COTT* and *COTS*, respectively.

2.4. Cramer-von Mises Test for Exponentiality

The Cramer-von Mises test for exponentiality (Henze and Meintanis [7]) is based on the following statistic,

$$\omega_n^2 = \int_0^\infty (F_n(x) - (1 - \exp(-x)))^2 \exp(-x) dx,$$

where F_n is the empirical distribution function of the scaled data $Y_j = X_j / \bar{X}$. The p-value is computed by Monte Carlo simulation. Here we represent the test as *CMTS*.

2.5. Deshpande Test for Exponentiality

Deshpande test for the composite hypothesis of exponentiality [8], is based on the following statistic,

$$J = \frac{1}{n(n-1)} \sum_{i \neq j} 1\{x_i > bx_j\}.$$

Under exponentiality, one has

$$\sqrt{n} \left(J - \frac{1}{b+1} \right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 4\zeta_1), \text{ as } n \rightarrow \infty,$$

where

$$\zeta_1 = \frac{1}{4} \left(\frac{1 + \frac{b}{b+2} + \frac{1}{2b+1} + \frac{2(1-b)}{b+1}}{-\frac{2b}{b^2+b+1} - \frac{4}{(b+1)^2}} \right).$$

This test can be implemented either using the asymptotic normal distribution or using the simulated distribution under the exponentiality assumption. Here we represent them as *DPTT* and *DPTS*, respectively.

2.6. Test for Exponentiality of Epps and Pulley

The test statistic is

$$EP_n = (48n)^{1/2} \left(\frac{1}{n} \sum_{j=1}^n \exp(-Y_j) - \frac{1}{2} \right),$$

where $Y_j = X_j / \bar{X}$. EP_n is asymptotically standard normal ([7] section 2.8.1). This test can be implemented either using the asymptotic normal distribution or using the simulated distribution under the exponentiality assumption. Here we represent them as *EPTT* and *EPTS*, respectively.

2.7. Epstein Test for Exponentiality

The test [9] is based on the following statistic:

$$EPS_n = \frac{2n \left(\log \left(n^{-1} \sum_{i=1}^n D_i \right) - n^{-1} \sum_{i=1}^n \log(D_i) \right)}{1 + (n+1)/(6n)},$$

where

$$D_i = (n-i+1) (X_{(i)} - X_{(i-1)}),$$

$X_0 = 0, X_{(1)} \leq \dots \leq X_{(n)}$ are order statistics. Under exponentiality, EPS_n is approximately distributed as a chi-square with $n-1$ degrees of freedom. This test can be implemented either using the asymptotic normal distribution or using the simulated distribution under the exponentiality assumption. Here we represent them as *ESTT* and *ESTS*, respectively.

2.8. Frozini Test for Exponentiality

The Frozini test for exponentiality is based on the following statistic [10],

$$B_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left| 1 - \exp \left(-X_{(i)} / \bar{X} \right) - \frac{i-0.5}{n} \right|.$$

The p-value is computed by Monte Carlo simulation. Here we represent the test as *FRTS*.

2.9. Test for Exponentiality Based on the Gini Statistic

The test is based on the Gini statistic,

$$G_n = \frac{\sum_{i,j=1}^n |X_i - X_j|}{2n(n-1)\bar{X}}.$$

Under exponentiality, the normalized statistic $(12(n-1))^{1/2} (G_n - 0.5)$ is asymptotically standard normal [11]. This test can be implemented either using the asymptotic normal distribution or using the simulated distribution under the exponentiality assumption. Here we represent them as *GSTT* and *GSTS*, respectively.

2.10. Gnedenko F-test of Exponentiality

The test [9] is based on the following statistic,

$$Q_n(R) = \frac{\sum_{i=1}^R D_i / R}{\sum_{i=R+1}^n D_i / (n-R)},$$

where

$$D_i = (n - i + 1)(X_{(i)} - X_{(i-1)}),$$

$X_{(0)} = 0, X_{(1)} \leq \dots \leq X_{(n)}$ are order statistics. Under exponentiality, $Q_n(R)$ follows an F distribution with $2R$ and $2(n - R)$ degrees of freedom. Here we represent the test as *GNTT*.

2.11. Harris modification of Gnedenko F-test

The test [9] is based on the following statistic,

$$Q_n(R) = \frac{\left(\sum_{i=1}^R D_i + \sum_{i=n-R+1}^n D_i\right) / 2R}{\sum_{i=R+1}^{n-R} D_i / (n - 2R)},$$

where

$$D_i = (n - i + 1)(X_{(i)} - X_{(i-1)}),$$

$X_{(0)} = 0, X_{(1)} \leq \dots \leq X_{(n)}$ are order statistics. Under exponentiality, $Q_n(R)$ follows an F distribution with $4R$ and $2(n - 2R)$ degrees of freedom. Here we represent the test as *HGTT*.

2.12. Hegazy-Green Test for Exponentiality

The Hegazy-Green test for exponentiality [12] is based on the following statistic,

$$T_1 = n^{-1} \sum_{i=1}^n \left| X_{(i)} + \ln \left(1 - \frac{i}{n+1} \right) \right|.$$

The p-value is computed by Monte Carlo simulation. Here we represent the test as *HFTS*.

2.13. Hegazy-Green Test for Exponentiality

The Hegazy-Green test for exponentiality [12] is based on the following statistic,

$$T_2 = n^{-1} \sum_{i=1}^n \left(X_{(i)} + \ln \left(1 - \frac{i}{n+1} \right) \right)^2.$$

The p-value is computed by Monte Carlo simulation. Here we represent the test as *HRTS*.

2.14. Hollander-Proshan Test for Exponentiality

The test [13] is based on the following statistic,

$$J_n = \frac{1}{n(n-1)(n-2)} \sum_{i \neq j, k; j < k} 1\{x_i > x_j + x_k\}.$$

Under exponentiality, one has

$$\sqrt{n} \left(J_n - \frac{1}{4} \right) \xrightarrow{\mathcal{D}} \mathcal{N} \left(0, \frac{5}{432} \right), \text{ as } n \rightarrow \infty.$$

This test can be implemented either using the asymptotic normal distribution or using the simulated distribution under the exponentiality assumption. Here we represent them as *HPPT* and *HPTS*, respectively.

2.15. Kimber-Michael Test for Exponentiality

The Kimber-Michael test [14,15] for exponentiality is based on the following statistic,

$$D = \max_i |r_i - s_i|,$$

where

$$s_i = \frac{2}{\pi} \arcsin \sqrt{1 - \exp(-X_{(i)} / \bar{X})},$$

$$r_i = \frac{2}{\pi} \arcsin \sqrt{(i - 0.5) / n}.$$

The p-value is computed by Monte Carlo simulation. Here we represent the test as *KMTS*.

2.16. Kochar Test for Exponentiality

The Kochar test for exponentiality [16] is based on the following statistic,

$$T = \sqrt{\frac{108n}{17}} \frac{\sum_{i=1}^n J(i / (n+1)) X_{(i)}}{\sum_{i=1}^n X_i},$$

where

$$J(u) = 2(1-u)(1 - \log(1-u)) - 1.$$

The statistic T is asymptotically standard normal. This test can be implemented either using the asymptotic normal distribution or using the simulated distribution under the exponentiality assumption. Here we represent them as *KCTT* and *KCTS*, respectively.

2.17. Kolmogorov-Smirnov Test for Exponentiality

The Kolmogorov-Smirnov test ([7] section 2.1) for exponentiality is based on the following statistic,

$$KS_n = \sup_{x \geq 0} |F_n(x) - (1 - \exp(-x))|,$$

where, F_n is the empirical distribution function of the scaled data $Y_j = X_j / \bar{X}$. The p-value is computed by Monte Carlo simulation. Here we represent the test as *KSTS*.

2.18. Lorenz Test for Exponentiality

The Lorenz test for exponentiality [17] for exponentiality is based on the following statistic,

$$L = \sum_{i=1}^{np} X_{(i)} / \sum_{i=1}^n X_{(i)}.$$

The statistic $\sqrt{n}(L-p-(1-p)\log(1-p))$ is asymptotically standard normal. This test can be implemented either using the asymptotic normal distribution or using the simulated distribution under the exponentiality assumption. Here we represent them as *LZTT* and *LZTS*, respectively.

2.19. Moran Test for Exponentiality

The Moran test for exponentiality [18,19] is based on the following statistic,

$$T_n^+ = \gamma + \frac{1}{n} \sum_{i=1}^n \log \frac{X_i}{\bar{X}},$$

where, where γ is Euler-Mascheroni constant. The statistic is asymptotically normal,

$$\sqrt{n}T_n^+ \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{\pi^2}{6} - 1\right), \text{ as } n \rightarrow \infty.$$

This test can be implemented either using the asymptotic normal distribution or using the simulated distribution under the exponentiality assumption. Here we represent them as *MRTT* and *MRTS*, respectively.

2.20. Test for Exponentiality Based on the Pietra statistic

The test is based on the following Pietra statistic [9],

$$P_n = \sum_{i=1}^n \frac{|X_i - \bar{X}|}{2n\bar{X}}.$$

The p-value is computed by Monte Carlo simulation. Here we represent the test as *PSTS*.

2.21. Test for Exponentiality Based on Rossberg Characterization

The test is based on the following statistic [2],

$$S_n = \int_0^\infty (H_n(t) - G_n(t)) dF_n(t),$$

where F_n is the empirical distribution function,

$$H_n(t) = \left(C_n^3\right)^{-1} \sum_{1 \leq i < j < k \leq n} 1\{X_{2,\{i,j,k\}} - X_{1,\{i,j,k\}} < t\},$$

$$t \geq 0,$$

$$G_n(t) = \left(C_n^2\right)^{-1} \sum_{1 \leq i < j \leq n} 1\{\min(X_i - X_j) < t\}, t \geq 0.$$

Here $X_{2,\{i,j,k\}}, s=1,2$, denotes the s_{th} order statistic of X_i, X_j, X_k . The p-value is computed from the limit null distribution. Under exponentiality, one has

$$\sqrt{n}S_n \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{52}{1125}\right), \text{ as } n \rightarrow \infty.$$

This test can be implemented either using the asymptotic normal distribution or using the simulated distribution under the exponentiality assumption. Here we represent them as *RCTT* and *RCTS*, respectively.

2.22. Shapiro-Wilk Test for Exponentiality

The Shapiro-Wilk test for exponentiality [20] is based on the following statistic,

$$W = \frac{n(\bar{X} - X_{(1)})^2}{(n-1) \sum_{i=1}^n (X_i - \bar{X})^2}.$$

The p-value is computed by Monte Carlo simulation. Here we represent the test as *SWTS*.

2.24. Wong and Wong Test for Exponentiality

The Wong and Wong test for the composite hypothesis of exponentiality [9] is based on the following statistic,

$$Q = \frac{X_{(n)}}{X_{(1)}},$$

where $X_{(1)}$ and $X_{(n)}$ are the smallest and the largest order statistics respectively. The p-value is computed by Monte Carlo simulation. Here we represent the test as *WWTS*.

2.25. Anderson-Darling Upper Tail Test

Anderson-Darling upper tail test is based on the statistic [21]

$$LA = -n - 2 \sum_{i=1}^n \left\{ \frac{2i-1}{2n} \log Z_i + \left(1 + \frac{2i-1}{2n}\right) \log(1 - Z_i) \right\},$$

where $Z_i = 1 - e^{-X_i}$ and X_i is the i^{th} ordered measurement in the sample. The p-value is computed by Monte Carlo simulation. Here we represent the test as *ADTS*.

3. Power Comparison

Samples are generated from standard exponential distribution to check whether the proportions of rejections match with the respective levels of significances. And samples are generated from some selected alternative distributions, such as, standard Uniform (0, 1), standard half-normal, Weibull (1.0, 1.4), Gamma (2.0, 1.0), Beta (2.0, 1.0), Gamma (0.5, 1.0), and Log Normal (0.0, 0.8), distributions to investigate how powerful the tests are. We have considered sample sizes, 12, 20, and 28. Levels of significances considered are 1%, 5%, and 10%. In each cases, 10,000 samples are considered.

Table 1. Abbreviations of the Tests

<i>AHTT</i>	Ahsanullah characterization test using normal approximation
<i>AHTS</i>	Ahsanullah characterization test using simulation
<i>AKTT</i>	Atkinson test using normal approximation
<i>AKTS</i>	Atkinson test using simulation
<i>COTT</i>	Cox and Oakes test using normal approximation
<i>COTS</i>	Cox and Oakes test using simulation
<i>CMTS</i>	Cramer-von Mises test using simulation
<i>DPTT</i>	Deshpande test using normal approximation
<i>DPTS</i>	Deshpande test using simulation
<i>EPPT</i>	Epps and Pulley test using normal approximation
<i>EPTS</i>	Epps and Pulley test using simulation
<i>ESTT</i>	Epstein test using normal approximation
<i>ESTS</i>	Epstein test using simulation
<i>FRTS</i>	Frozini test using simulation
<i>GSTT</i>	Gini statistic test using normal approximation
<i>GSTS</i>	Gini statistic test using simulation
<i>GNTT</i>	Gnedenko F-test
<i>HGTT</i>	Harris-Gnedenko F-test
<i>HFTS</i>	Hegazy-Green test using simulation
<i>HRTS</i>	Hegazy-Green alternative test using simulation
<i>HPPT</i>	Hollander-Proshan test using normal approximation
<i>HPTS</i>	Hollander-Proshan test using simulation
<i>KMTS</i>	Kimber-michael test using simulation
<i>KCTT</i>	Kochar test using normal approximation
<i>KCTS</i>	Kochar test using simulation
<i>KSTS</i>	Kolmogorov-Smirnov test using simulation
<i>LZTT</i>	Lorenz test using normal approximation
<i>LZTS</i>	Lorenz test using simulation
<i>MRTT</i>	Moran test using normal approximation
<i>MRTS</i>	Moran test using simulation
<i>PSTS</i>	Pietra statistic test using simulation
<i>RCTT</i>	Rossberg characterization test using normal approximation
<i>RCTS</i>	Rossberg characterization test using simulation
<i>SWTS</i>	Shapiro-Wilk test using simulation
<i>WETS</i>	WE test using simulation
<i>WWTS</i>	Wong and Wong test using simulation
<i>ADTS</i>	Anderson-Darling test using simulation

Table 2. Test Performance Summary

	Worst	Good	Better	Best
Table 3	AHTS HPPT LZTT RCTT	AHTT AKTT DPTT KCTT MRTT	PSTS RCTS	AKTS COTS CMTS DPTS EPPT EPTS ESTT ESTS FRTS GSTT GSTS GNTT HGTT HFTS HRTS HPTS KMTS KCTS KSTS LZTS MRTS SWTS WETS WWTS ADTS
Table 4	AHTS HFTS HRTS HPPT, LZTT RCTT SWTS	AKTT AKTS COTT COTS DPTT DPTS EPPT EPTS FRTS GSTT GSTS GNTT HPTS KMTS PSTS WETS	AHTT KCTT ADTS	KCTT
Table 5	AHTS HFTS HRTS LZTT RCTT SWTS		AHTT KCTT	KCTT
Table 6	AHTS HFTS HRTS HPPT LZTT RCTT SWTS WWTS	AKTT COTT COTS DPTT DPTS EPPT EPTS FRTS GSTT GSTS GNTT HPTS KCTS LZTS MRTT MRTS PSTS WETS ADTS	AHTT KCTT	KCTT
Table 7	AHTS ESTT ESTS HPPT LZTT RCTT SWTS WWTS	AKTT AKTS COTT COTS DPTS EPPT EPTS FRTS GSTT GSTS GNTT HPTS KCTS LZTS MRTT MRTS RCTS WETS	DPTT KCTT PSTS ADTS	AHTT HFTS HRTS
Table 8	AHTS HFTS HRTS LZTT RCTT SWTS WWTS	HGTT HPPT AKTT AKTS COTT CMTS ESTT ESTS KMTS KSTS	AHTT COTS DPTT DPTS EPPT EPTS FRTS GSTT GSTS GNTT HPTS KCTT KCTS LZTS MRTS PSTS RCTS WETS ADTS	AHTT DPTT EPTS FRTS GSTT GSTS HPTS KCTT PSTS WETS and ADTS
Table 9	AHTS GNTT HGTT HFTS HRTS HPPT LZTT PSTS RCTT ADTS	AKTT AKTS DPTT DPTS EPPT EPTS FRTS GSTT GSTS HPTS KCTS KSTS RCTS SWTS WETS	CMTS KMTS LZTS WWTS COTT COTS MRTT MRTS	MRTT MRTS
Table 10	AHTS ESTT ESTS HPPT KMTS LZTT RCTT SWTS WWTS	DPTS FRTS HFTS MRTS RCTS	DPTT KCTT	AHTT

Table 3. Rejection Proportions for Standard Exponential Data

Tests	n = 12			n = 20			n = 28			
	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	
AHTT	0.0865	0.1754	0.2455	0.0568	0.1375	0.1895	0.0448	0.1107	0.1669	⊗
AHTS	0.0107	0.0499	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗ ⊗
AKTT	0.0058	0.0223	0.0605	0.0056	0.0313	0.0745	0.0073	0.0343	0.0833	⊗
AKTS	0.0094	0.0492	0.0972	0.0104	0.0457	0.0982	0.0098	0.0516	0.0992	
COTT	0.0078	0.0279	0.0753	0.0085	0.0387	0.0837	0.0077	0.0381	0.0894	
COTS	0.0109	0.0498	0.0986	0.0113	0.0510	0.0969	0.0089	0.0472	0.0982	
CMTS	0.0109	0.0476	0.1037	0.0101	0.0500	0.1004	0.0108	0.0509	0.1025	
DPTT	0.0381	0.0128	0.1419	0.0235	0.0744	0.1266	0.0197	0.0635	0.1156	⊗
DPTS	0.0116	0.0585	0.1104	0.0096	0.0513	0.1085	0.0115	0.0518	0.1112	
EPTT	0.0059	0.0338	0.0855	0.0057	0.0423	0.0922	0.0066	0.0423	0.0962	
EPTS	0.0108	0.0486	0.0973	0.0089	0.0502	0.0997	0.0087	0.0506	0.0984	
ESTT	0.0070	0.0442	0.0917	0.0078	0.0473	0.0920	0.0079	0.0419	0.0898	
ESTS	0.0108	0.0538	0.0972	0.0119	0.0546	0.1059	0.0091	0.0538	0.1003	
FRTS	0.0097	0.0477	0.1017	0.0090	0.0529	0.1003	0.0106	0.0499	0.1055	
GSTT	0.0088	0.0453	0.0979	0.0081	0.0490	0.0984	0.0080	0.0468	0.1010	
GSTS	0.0115	0.0460	0.0986	0.0083	0.0506	0.0988	0.0085	0.0501	0.1008	
GNTT	0.0092	0.0503	0.0991	0.0088	0.0496	0.0988	0.0091	0.0484	0.1044	
HGTT	0.0094	0.0505	0.1063	0.0095	0.0502	0.1045	0.0094	0.0536	0.1018	
HFTS	0.0104	0.0534	0.0967	0.0092	0.0502	0.0998	0.0106	0.0500	0.0965	
HRTS	0.0125	0.0527	0.0958	0.0069	0.0501	0.0945	0.0090	0.0491	0.0968	
HPTT	0.0000	0.0005	0.0107	0.0000	0.0000	0.0033	0.0000	0.0000	0.0003	⊗ ⊗
HPTS	0.0113	0.0414	0.1032	0.0097	0.0525	0.0984	0.0097	0.0497	0.1020	
KMTS	0.0102	0.0473	0.1080	0.0126	0.0537	0.1013	0.0113	0.0486	0.1005	
KCTT	0.0335	0.2043	0.3477	0.0251	0.1465	0.2657	0.0235	0.1179	0.2173	⊗
KCTS	0.0106	0.0443	0.0968	0.0073	0.0483	0.0988	0.0093	0.0513	0.0994	
KSTS	0.0108	0.0486	0.1061	0.0096	0.0521	0.0967	0.0101	0.0533	0.1055	
LZTT	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗ ⊗
LZTS	0.0106	0.0485	0.1035	0.0113	0.0500	0.1014	0.0093	0.0538	0.1026	
MRTT	0.0108	0.0309	0.0717	0.0119	0.0395	0.0835	0.0094	0.0376	0.0877	⊗
MRTS	0.0100	0.0507	0.0968	0.0125	0.0522	0.0963	0.0091	0.0476	0.0977	
PSTS	0.0001	0.0511	0.1018	0.0101	0.0480	0.1021	0.0100	0.0518	0.0984	
RCTT	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗ ⊗
RCTS	0.1574	0.1632	0.1031	0.0116	0.0500	0.1003	0.0105	0.0553	0.1014	
SWTS	0.0109	0.0446	0.0995	0.0079	0.0480	0.0946	0.0080	0.0465	0.1023	
WETS	0.0106	0.0466	0.0952	0.0074	0.0502	0.1033	0.0090	0.0537	0.0994	
WWTS	0.0090	0.0486	0.0955	0.0111	0.0478	0.0922	0.0073	0.0497	0.1016	
ADTS	0.0104	0.0516	0.0985	0.0092	0.0493	0.1080	0.0087	0.0513	0.0952	

⊗ on the right panel indicates very low power, robust against the alternative considered. The more ⊗s the lower the power.

Table 4. Rejection Proportions for Standard Uniform Data

	Tests	n = 12			n = 20			n = 28			
		$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	
**	AHTT	0.5802	0.7404	0.8101	0.7164	0.8453	0.8972	0.8070	0.9093	0.9368	
	AHTS	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
*	AKTT	0.0016	0.1928	0.4305	0.0925	0.5195	0.7389	0.2872	0.7591	0.8807	
*	AKTS	0.0013	0.1907	0.3759	0.0733	0.4193	0.6368	0.1395	0.6477	0.8082	
*	COTT	0.0005	0.1935	0.3971	0.0892	0.4401	0.6272	0.2331	0.6234	0.7672	
*	COTS	0.1488	0.3321	0.4558	0.3050	0.5211	0.6515	0.3932	0.6888	0.7780	
*	CMTS	0.0002	0.0488	0.1501	0.0359	0.2264	0.4167	0.1246	0.4517	0.6438	
*	DPTT	0.3926	0.5221	0.5714	0.5298	0.6814	0.7596	0.6541	0.7998	0.8453	
*	DPTS	0.1542	0.3515	0.4739	0.3044	0.5248	0.6804	0.4444	0.7062	0.7987	
*	EPTT	0.0366	0.3417	0.5476	0.2408	0.6431	0.8016	0.4724	0.8264	0.9090	
*	EPTS	0.1764	0.4042	0.5451	0.4243	0.6575	0.7830	0.5660	0.8389	0.8999	
*	ESTT	0.0509	0.1777	0.2898	0.0992	0.2839	0.4337	0.1583	0.4067	0.5526	
*	ESTS	0.0308	0.1182	0.1954	0.0754	0.2096	0.3075	0.1146	0.3154	0.4311	
*	FRTS	0.2022	0.4499	0.6035	0.4547	0.6919	0.8208	0.6032	0.8652	0.9214	
*	GSTT	0.1878	0.4364	0.5736	0.4454	0.7183	0.8262	0.6513	0.8711	0.9229	
*	GSTS	0.1950	0.4327	0.5846	0.4702	0.6990	0.8166	0.6287	0.8731	0.9229	
*	GNTT	0.2597	0.5159	0.6519	0.4786	0.7240	0.8278	0.6442	0.8528	0.9115	
	HGTT	0.0149	0.0596	0.1121	0.0156	0.0672	0.1253	0.0182	0.0657	0.1193	
	HFTS	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
	HRTS	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
	HPTT	0.0000	0.0657	0.3323	0.0000	0.0232	0.3010	0.0000	0.0096	0.2694	⊗
*	HPTS	0.1979	0.4320	0.5723	0.4404	0.6493	0.7731	0.5736	0.8244	0.8842	
*	KMTS	0.0137	0.1836	0.3714	0.2850	0.6679	0.8314	0.7208	0.9290	0.9700	
***	KCTT	0.5699	0.9034	0.9614	0.8234	0.9706	0.9914	0.9322	0.9930	0.9982	
	KCTS	0.2451	0.5037	0.6527	0.5835	0.7911	0.8821	0.7621	0.9416	0.9654	
	KSTS	0.0237	0.1206	0.2344	0.1051	0.3206	0.4803	0.2425	0.5212	0.6878	
	LZTT	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
	LZTS	0.1697	0.3597	0.4909	0.3672	0.5593	0.6854	0.4857	0.7370	0.8092	
	MRTT	0.0003	0.1182	0.3165	0.0373	0.3174	0.5093	0.1265	0.4747	0.6403	
	MRTS	0.1327	0.2923	0.4036	0.2460	0.4345	0.5530	0.3016	0.5761	0.6713	
*	PSTS	0.2489	0.5117	0.6568	0.4926	0.7162	0.8284	0.6109	0.8606	0.9166	
	RCTT	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
	RCTS	0.0608	0.1505	0.2369	0.1106	0.2491	0.3778	0.1621	0.3643	0.4778	
	SWTS	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
*	WETS	0.2152	0.4681	0.6152	0.5188	0.7557	0.8636	0.6989	0.9272	0.9578	
	WWTS	0.0035	0.0162	0.0304	0.0025	0.0136	0.0288	0.0026	0.0157	0.0249	
**	ADTS	0.3265	0.5825	0.7416	0.6084	0.8155	0.8998	0.7446	0.9326	0.9640	

⊗ on the right panel indicates very low power, robust against the alternative considered.

★ on the left panel indicates large power and the more ★s the larger the power.

Table 7. Rejection Proportions for Gamma (2,1) Data

	Tests	n = 12			n = 20			n = 28			
		$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	
***	AHTT	0.5320	0.7158	0.8088	0.6622	0.8292	0.8965	0.7796	0.9047	0.9454	
	AHTS	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
*	AKTT	0.0012	0.1138	0.3423	0.0329	0.3573	0.5956	0.1382	0.5597	0.7531	
*	AKTS	0.0015	0.1276	0.3015	0.0066	0.3002	0.5015	0.0445	0.4532	0.6273	
*	COTT	0.0002	0.1483	0.4032	0.0507	0.4379	0.6800	0.2039	0.6619	0.8324	
*	COTS	0.1010	0.2972	0.4609	0.2892	0.5551	0.7105	0.4625	0.7229	0.8448	
	CMTS	0.0002	0.0276	0.1189	0.0167	0.1950	0.3743	0.0927	0.3951	0.5646	
**	DPTT	0.3243	0.4707	0.5309	0.4417	0.6257	0.7311	0.5946	0.7569	0.8329	
*	DPTS	0.1009	0.2783	0.4158	0.2096	0.4890	0.6243	0.3638	0.6502	0.7705	
*	EPTT	0.0131	0.2245	0.4358	0.1105	0.4557	0.6537	0.2558	0.6324	0.7870	
*	EPTS	0.0944	0.2610	0.4301	0.2370	0.5016	0.6374	0.3888	0.6669	0.7802	
	ESTT	0.0124	0.0647	0.1215	0.0197	0.0851	0.1667	0.0262	0.1070	0.1911	⊗
	ESTS	0.0114	0.0491	0.0976	0.0139	0.0628	0.1293	0.0248	0.0780	0.1287	⊗
*	FRTS	0.0937	0.2917	0.4471	0.2082	0.4972	0.6426	0.3567	0.6656	0.7674	⊗
*	GSTT	0.0830	0.2664	0.4101	0.2071	0.4732	0.6219	0.3590	0.6387	0.7626	
*	GSTS	0.0794	0.2547	0.4191	0.2102	0.4849	0.6225	0.3836	0.6558	0.7681	
*	GNTT	0.0696	0.2486	0.4043	0.1341	0.3988	0.5730	0.2225	0.5243	0.6929	
	HGTT	0.0274	0.1169	0.2142	0.0482	0.1643	0.2892	0.0616	0.2244	0.3559	
***	HFTS	0.6895	0.9005	0.9460	0.9150	0.9839	0.9938	0.9777	0.9977	0.9994	
***	HRTS	0.3849	0.7019	0.8321	0.6095	0.8732	0.9436	0.7763	0.9499	0.9854	
	HPTT	0.0000	0.0224	0.1842	0.0000	0.0030	0.1268	0.0000	0.0006	0.0836	⊗
*	HPTS	0.0844	0.2481	0.4040	0.2046	0.4829	0.6307	0.3626	0.6540	0.7705	
**	KMTS	0.0003	0.0246	0.0774	0.0159	0.0887	0.1629	0.0376	0.1586	0.2351	
**	KCTT	0.2843	0.7087	0.8534	0.4232	0.7814	0.9015	0.5516	0.8486	0.9289	
*	KCTS	0.0801	0.2234	0.3722	0.1636	0.4125	0.5466	0.2995	0.5682	0.6802	
	KSTS	0.0107	0.0871	0.1743	0.0629	0.2404	0.3680	0.1450	0.3957	0.5341	
	LZTT	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
*	LZTS	0.0807	0.2523	0.3874	0.1905	0.4593	0.6176	0.3797	0.6429	0.7685	
*	MRTT	0.0000	0.0869	0.3479	0.0209	0.3842	0.6562	0.1458	0.6325	0.8260	
*	MRTS	0.1052	0.3020	0.4679	0.2994	0.5613	0.7213	0.4618	0.7418	0.8514	
**	PSTS	0.1400	0.3755	0.5474	0.2643	0.5813	0.7478	0.4637	0.7382	0.8535	
	RCTT	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
*	RCTS	0.0724	0.2331	0.3370	0.1920	0.4084	0.5772	0.3438	0.6005	0.7154	⊗
*	SWTS	0.0005	0.0057	0.0182	0.0001	0.0021	0.0079	0.0003	0.0015	0.0034	⊗
*	WETS	0.0918	0.2433	0.3911	0.1915	0.4349	0.5690	0.3282	0.5686	0.6803	⊗
	WWTS	0.0000	0.0004	0.0011	0.0000	0.0001	0.0010	0.0000	0.0000	0.0006	⊗
**	ADTS	0.1524	0.3898	0.5678	0.2991	0.5917	0.7553	0.4827	0.7524	0.8636	

⊗ on the right panel indicates very low power, robust against the alternative considered.
 ★ on the left panel indicates large power and the more ★s the larger the power.

Table 8. Rejection Proportions for Beta (2,1) Data

	Tests	n = 12			n = 20			n = 28			
		$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	
****	AHTT	0.9950	0.9993	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	
	AHTS	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
**	AKTT	0.0584	0.9041	0.9872	0.9269	0.9985	1.0000	0.9990	1.0000	1.0000	
**	AKTS	0.0035	0.8951	0.9804	0.7923	0.9974	0.9997	0.9890	1.0000	1.0000	
**	COTT	0.0514	0.8899	0.9797	0.9088	0.9963	0.9999	0.9957	0.9998	1.0000	
***	COTS	0.8487	0.9610	0.9861	0.9898	0.9980	0.9999	0.9994	1.0000	1.0000	
**	CMTS	0.0493	0.5963	0.8416	0.8063	0.9794	0.9970	0.9852	0.9997	1.0000	
****	DPTT	0.9586	0.9857	0.9906	0.9977	0.9994	0.9999	1.0000	1.0000	1.0000	
***	DPTS	0.8085	0.9504	0.9810	0.9794	0.9976	0.9993	0.9992	1.0000	1.0000	
***	EPTT	0.5733	0.9699	0.9950	0.9872	0.9995	1.0000	0.9999	1.0000	1.0000	
****	EPTS	0.9039	0.9827	0.9936	0.9963	0.9998	1.0000	1.0000	1.0000	1.0000	
**	ESTT	0.3294	0.6445	0.7902	0.6781	0.8979	0.9571	0.8836	0.9766	0.9929	
**	ESTS	0.2610	0.5269	0.6775	0.6119	0.8143	0.9101	0.8430	0.9585	0.9801	
****	FRTS	0.9194	0.9878	0.9970	0.9982	0.9998	1.0000	1.0000	1.0000	1.0000	
****	GSTT	0.9028	0.9872	0.9967	0.9983	0.9998	1.0000	1.0000	1.0000	1.0000	
****	GSTS	0.9246	0.9867	0.9965	0.9978	0.9998	1.0000	1.0000	1.0000	1.0000	
***	GNTT	0.8310	0.9633	0.9865	0.9851	0.9979	0.9998	0.9989	1.0000	1.0000	
*	HGTT	0.1638	0.4010	0.5574	0.3211	0.6044	0.7329	0.4759	0.7442	0.8502	
	HFTS	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
	HRTS	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
*	HPTT	0.0000	0.7220	0.9711	0.0000	0.7539	0.9932	0.0000	0.7954	0.9984	
****	HPTS	0.9192	0.9854	0.9969	0.9978	0.9996	0.9999	1.0000	1.0000	1.0000	
**	KMTS	0.3566	0.8741	0.9716	0.9892	0.9997	1.0000	1.0000	1.0000	1.0000	
****	KCTT	0.9959	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	
***	KCTS	0.9457	0.9913	0.9982	0.9992	1.0000	1.0000	1.0000	1.0000	1.0000	
**	KSTS	0.3321	0.6997	0.8542	0.8403	0.9740	0.9944	0.9772	0.9994	0.9997	
	LZTT	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
***	LZTS	0.8718	0.9643	0.9857	0.9913	0.9987	0.9996	0.9996	1.0000	1.0000	
	MRTT	0.0000	0.7837	0.9497	0.7523	0.9867	0.9984	0.9775	0.9992	0.9999	
***	MRTS	0.8219	0.9403	0.9719	0.9732	0.9946	0.9990	0.9976	0.9996	1.0000	
****	PSTS	0.9252	0.9874	0.9960	0.9957	0.9998	1.0000	1.0000	1.0000	1.0000	
	RCTT	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
***	RCTS	0.5618	0.7800	0.8790	0.8682	0.9749	0.9900	0.9800	0.9975	0.9993	
	SWTS	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
****	WETS	0.9276	0.9888	0.9971	0.9986	0.9999	1.0000	1.0000	1.0000	1.0000	
****	WWTS	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0006	⊗
****	ADTS	0.9655	0.9961	0.9997	0.9995	1.0000	1.0000	1.0000	1.0000	1.0000	

⊗ on the right panel indicates very low power, robust against the alternative considered.
 ★ on the left panel indicate large power and the more ★s the larger the power.

Table 9. Rejection Proportions for Gamma (0.5,1) Data

	Tests	n = 12			n = 20			n = 28			
		$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	
	AHTT	0.0093	0.0138	0.0229	0.0013	0.0353	0.1422	0.0116	0.2281	0.4145	
	AHTS	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
*	AKTT	0.1923	0.3170	0.4038	0.3483	0.5092	0.5975	0.4900	0.6596	0.7304	
*	AKTS	0.1743	0.3368	0.4086	0.3040	0.4993	0.5638	0.4242	0.6306	0.6848	
***	COTT	0.3914	0.5269	0.6016	0.5919	0.7352	0.7843	0.7435	0.8507	0.8841	
***	COTS	0.3384	0.5555	0.6315	0.5494	0.7392	0.7996	0.6762	0.8497	0.8890	
**	CMTS	0.2879	0.5430	0.6333	0.4761	0.6975	0.7801	0.6010	0.8081	0.8705	
*	DPTT	0.1461	0.4081	0.4952	0.2695	0.5751	0.6793	0.4283	0.7028	0.8048	
*	DPTS	0.2167	0.4044	0.4924	0.3973	0.6140	0.7117	0.5604	0.7596	0.8206	
*	EPTT	0.1916	0.3313	0.4247	0.3453	0.5184	0.6102	0.4819	0.6664	0.7404	
*	EPTS	0.1847	0.3861	0.4860	0.3441	0.5479	0.6555	0.4436	0.6810	0.7601	
	ESTT	0.1502	0.3127	0.4243	0.1960	0.3889	0.4976	0.2315	0.4414	0.5645	
	ESTS	0.1133	0.2421	0.3392	0.1745	0.3224	0.3982	0.1891	0.3814	0.4796	
*	FRTS	0.1972	0.3445	0.4394	0.3447	0.5307	0.6214	0.4592	0.6808	0.7492	
*	GSTT	0.1778	0.3719	0.4880	0.3243	0.5450	0.6523	0.4650	0.6825	0.7669	
*	GSTS	0.1966	0.3866	0.4846	0.3552	0.5516	0.6532	0.4564	0.6852	0.7605	
	GNTT	0.0007	0.0067	0.0128	0.0001	0.0028	0.0047	0.0001	0.0004	0.0017	⊗
	HGTT	0.0207	0.0688	0.1243	0.0158	0.0652	0.1081	0.0157	0.0624	0.1030	⊗
	HFTS	0.0000	0.0004	0.0006	0.0000	0.0000	0.0000	0.0000	0.0002	0.0001	⊗
	HRTS	0.0006	0.0035	0.0067	0.0004	0.0020	0.0021	0.0001	0.0001	0.0018	⊗
	HPTT	0.0000	0.0000	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
*	HPTS	0.1946	0.3935	0.5127	0.4018	0.6161	0.7034	0.5595	0.7588	0.8166	
**	KMTS	0.3115	0.5557	0.6518	0.4977	0.7222	0.8029	0.6429	0.8302	0.8792	
	KCTT	0.0025	0.0305	0.0731	0.0268	0.0873	0.1558	0.0900	0.2187	0.2987	
*	KCTS	0.1388	0.3272	0.4214	0.2545	0.4663	0.5721	0.3538	0.5816	0.6851	
*	KSTS	0.2174	0.4404	0.5385	0.3666	0.5771	0.6835	0.4763	0.7015	0.7871	
	LZTT	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
**	LZTS	0.2451	0.4493	0.5177	0.4105	0.6130	0.7066	0.5355	0.7545	0.7966	
***	MRTT	0.4622	0.5819	0.6507	0.6569	0.7731	0.8172	0.7940	0.8797	0.9063	
***	MRTS	0.3671	0.5671	0.6655	0.5872	0.7609	0.8232	0.7142	0.8679	0.9042	
	PSTS	0.0001	0.0024	0.0049	0.0000	0.0003	0.0016	0.0000	0.0002	0.0002	⊗
	RCTT	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
*	RCTS	0.1281	0.2688	0.3781	0.2945	0.5101	0.6224	0.4673	0.6862	0.7739	
*	SWTS	0.1044	0.2805	0.4160	0.1762	0.4044	0.5499	0.2436	0.5216	0.6653	
*	WETS	0.1018	0.2571	0.3644	0.1478	0.3588	0.4757	0.2120	0.4543	0.5780	
**	WWTS	0.2758	0.5236	0.6319	0.3040	0.6068	0.7364	0.3492	0.6749	0.7925	
	ADTS	0.0000	0.0025	0.0043	0.0000	0.0001	0.0012	0.0000	0.0001	0.0000	⊗

⊗ on the right panel indicates very low power, robust against the alternative considered.

★ on the left panel indicates large power and the more ★s the larger the power.

Table 10. Rejection Proportions for Log Normal (0, 0.8) Data

	Tests	n = 12			n = 20			n = 28			
		$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	
***	AHTT	0.4678	0.6555	0.7576	0.5727	0.7795	0.8705	0.6913	0.8575	0.9170	
	AHTS	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
	AKTT	0.0061	0.0762	0.2269	0.0179	0.1807	0.3447	0.0493	0.2524	0.4142	
	AKTS	0.0066	0.0793	0.2031	0.0087	0.1325	0.2805	0.0278	0.1873	0.3313	
	COTT	0.0021	0.1001	0.2888	0.0225	0.2600	0.4606	0.0870	0.3799	0.5615	
	COTS	0.0701	0.2193	0.3297	0.1126	0.3352	0.4907	0.1873	0.4508	0.5703	
	CMTS	0.0049	0.0308	0.1101	0.0080	0.1332	0.2977	0.0450	0.2775	0.4925	
**	DPTT	0.2477	0.3902	0.4468	0.3213	0.5154	0.6381	0.4527	0.6423	0.7389	
*	DPTS	0.0712	0.2100	0.3383	0.1200	0.3546	0.5188	0.1994	0.5079	0.6573	
	EPTT	0.0105	0.1425	0.2926	0.0445	0.2364	0.3851	0.0935	0.2962	0.4408	
	EPTS	0.0639	0.1855	0.2793	0.0940	0.2493	0.3741	0.1289	0.3012	0.4245	
	ESTT	0.0141	0.0627	0.1238	0.0208	0.0869	0.1601	0.0267	0.1100	0.2017	⊗
	ESTS	0.0113	0.0519	0.1017	0.0150	0.0616	0.1191	0.0186	0.0768	0.1369	⊗
*	FRTS	0.0645	0.2076	0.3383	0.0936	0.3308	0.5040	0.1736	0.4429	0.6203	
	GSTT	0.0489	0.1639	0.2693	0.0888	0.2466	0.3620	0.1353	0.3044	0.4266	
	GSTS	0.0559	0.1666	0.2607	0.0817	0.2333	0.3640	0.1198	0.2951	0.4221	
	GNTT	0.0342	0.1273	0.2325	0.0413	0.1699	0.2860	0.0596	0.2009	0.3300	
	HGTT	0.0462	0.1798	0.2989	0.0846	0.2638	0.4141	0.1315	0.3612	0.5178	
*	HFTS	0.1331	0.3025	0.4295	0.2052	0.4235	0.5677	0.2772	0.5440	0.6912	
	HRTS	0.1059	0.2238	0.3260	0.1435	0.2878	0.4178	0.1963	0.3689	0.5078	
	HPTT	0.0000	0.0112	0.1197	0.0000	0.0012	0.0610	0.0000	0.0000	0.0343	⊗
	HPTS	0.0606	0.1760	0.2723	0.0929	0.2962	0.4656	0.1603	0.4343	0.5968	
**	KMTS	0.0070	0.0321	0.0811	0.0095	0.0541	0.1117	0.0166	0.0712	0.1394	⊗
	KCTT	0.1674	0.4708	0.6369	0.1806	0.4580	0.6111	0.2019	0.4463	0.5816	
	KCTS	0.0461	0.1356	0.2188	0.0606	0.1785	0.2828	0.0846	0.2078	0.3117	
	KSTS	0.0163	0.0809	0.1701	0.0360	0.1803	0.3135	0.0819	0.2855	0.4541	
	LZTT	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
	LZTS	0.0487	0.1557	0.2595	0.0795	0.2432	0.3976	0.1417	0.3575	0.4945	
	MRTT	0.0000	0.0556	0.2598	0.0091	0.2504	0.4951	0.0653	0.4229	0.6465	
*	MRTS	0.0771	0.2373	0.3462	0.1337	0.3930	0.5564	0.2130	0.5452	0.6758	
	PSTS	0.0826	0.2397	0.3563	0.1142	0.3142	0.4735	0.1618	0.3802	0.5147	
	RCTT	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
*	RCTS	0.0650	0.2307	0.3676	0.1596	0.4400	0.6011	0.2712	0.6272	0.7734	
	SWTS	0.0293	0.0838	0.1431	0.0391	0.0938	0.1531	0.0492	0.1103	0.1685	⊗
	WETS	0.0639	0.1618	0.2550	0.0759	0.2053	0.3142	0.1070	0.2301	0.3360	⊗
	WWTS	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	⊗
	ADTS	0.0784	0.2447	0.3583	0.1070	0.3090	0.4460	0.1538	0.3552	0.4913	⊗

⊗ on the right panel indicates very low power, robust against the alternative considered.

★ on the left panel indicates large power and the more ★s the larger the power.

In Table 3, when samples are taken from the standard exponential distribution (null distribution), the levels of significances are closely estimated for the tests AKTS, COTS, CMTS, DPTS, EPTT, EPTS, ESTT, ESTS, FRSTS, GSTT, GSTS, GNTT, HGTT, HFTS, HRTS, HPTS, KMTS, KCTS, KSTS, LZTS, MRTS, SWTS, WETS, WWTS, and ADTS. The tests PSTS and RCTS are also closely estimating the levels of significances with exceptions for smaller samples. Tests AHTT, AKTT, DPTT, KCTT, and MRTT are not estimating the levels of significances closely. Tests AHTS, HPTT, LZTT, and RCTT are wrongly estimating the levels of significances always as zeros or very close to zeros.

In Table 4, when samples are taken from standard uniform distribution, the worst performing tests are, AHTS, HFTS, HRTS, HPTT, LZTT, RCTT, and SWTS, their rejection proportions are zeros or close to zeros. Tests AKTT, AKTS, COTT, COTS, DPTT, DPTS, EPTT, EPTS, FRSTS, GSTT, GSTS, GNTT, HPTS, KMTS, PSTS, and WETS, have some decent powers at least for higher samples. Tests AHTT, KCTT, and ADTS have high powers, among them KCTT has highest powers for all sample sizes and for all levels considered.

In Table 5, when samples are taken from standard half normal distribution, the worst performing tests are, AHTS, HFTS, HRTS, LZTT, RCTT, and SWTS, their rejection proportions are zeros. Tests AHTT, and KCTT have high powers, between them KCTT has highest powers for higher sample sizes and for higher levels of significances.

In Table 6, when samples are taken from Weibull (1.0,1.4) distribution, the worst performing tests are, AHTS, HFTS, HRTS, HPTT, LZTT, RCTT, SWTS, and WWTS, their rejection proportions are zeros or close to zeros. Tests AKTT, COTT, COTS, DPTT, DPTS, EPTT, EPTS, FRSTS, GSTT, GSTS, GNTT, HPTS, KCTS, LZTS, MRTT, MRTS, PSTS, WETS, and ADTS have some decent powers at least for higher samples and higher levels of significances. Tests AHTT, and KCTT have high powers, between them KCTT has highest powers for higher sample sizes and for higher levels of significances.

In Table 7, when samples are taken from Gamma (2.0,1.0) distribution, the worst performing tests are, AHTS, ESTT, ESTS, HPTT, LZTT, RCTT, SWTS, and WWTS, their rejection proportions are zeros or close to zeros. Tests AKTT, AKTS, COTT, COTS, DPTS, EPTT, EPTS, FRSTS, GSTT, GSTS, GNTT, HPTS, KCTS, LZTS, MRTT, MRTS, RCTS, and WETS have some decent powers at least for higher samples and higher levels of significances. Tests DPTT, KCTT, PSTS, and ADTS have some good powers at least for higher samples and higher levels of significances. Tests AHTT, HFTS, and HRTS have high powers, among them HFTS has highest powers for higher sample sizes and for higher levels of significances.

In Table 8, when samples are taken from Beta(2.0,1.0) distribution, the worst performing tests are, AHTS, HFTS, HRTS, LZTT, RCTT, SWTS, and WWTS, their rejection proportions are zeros or close to zeros. Tests HGTT and HPTT have some decent powers at least for higher samples and higher levels of significances. Tests AKTT, AKTS, COTT, CMTS, ESTT, ESTS, KMTS, and KSTS, have high powers at least for higher samples and higher levels of significances. Tests AHTT, COTS, DPTT, DPTS, EPTT, EPTS, FRSTS, GSTT, GSTS, GNTT, HPTS, KCTT,

KCTS, LZTS, MRTS, PSTS, RCTS, WETS, and ADTS have very high powers, among them AHTT, DPTT, EPTS, FRSTS, GSTT, GSTS, HPTS, KCTT, PSTS, WETS, and ADTS have highest powers irrespective of sample sizes and levels of significances.

In Table 9, when samples are taken from Gamma (0.5,1.0) distribution, the worst performing tests are, AHTS, GNTT, HGTT, HFTS, HRTS, HPTT, LZTT, PSTS, RCTT, and ADTS, their rejection proportions are zeros or close to zeros. Tests AKTT, AKTS, DPTT, DPTS, EPTT, EPTS, FRSTS, GSTT, GSTS, HPTS, KCTS, KSTS, RCTS, SWTS, and WETS have some good powers at least for higher samples and higher levels of significances. Tests CMTS, KMTS, LZTS, and WWTS have good powers at least for higher samples and higher levels of significances. Tests COTT, COTS, MRTT, and MRTS, have high powers, among them MRTT and MRTS have highest powers irrespective of sample sizes and levels of significances.

In Table 10, when samples are taken from Log Normal(0,0.8) distribution, the worst performing tests are, AHTS, ESTT, ESTS, HPTT, KMTS, LZTT, RCTT, SWTS, and WWTS, their rejection proportions are zeros or close to zeros. Tests DPTS, FRSTS, HFTS, MRTS, and RCTS have some decent powers at least for higher samples and higher levels of significances. Tests DPTT and KCTT have good powers at least for higher samples and higher levels of significances. Only AHTT have some high powers irrespective of sample sizes and levels of significances.

4. Concluding Remarks

Overall worst performing tests are Ahsanullah characterization test using simulation (AHTS), Hegazy-Green test using simulation (HFTS), Hegazy-Green alternative test using simulation (HRTS), Lorenz test using normal approximation (LZTT), Rossberg characterization test using normal approximation (RCTT), Shapiro-Wilk test using simulation (SWTS), and Wong and Wong test using simulation (WWTS). These tests do not estimate the levels of significances correctly and have very low powers against the alternatives considered here.

Ahsanullah characterization test using normal approximation (AHTT) and Kochar test using normal approximation (KCTT) have high powers except for Gamma distribution with parameters 0.5 and 1.0. A few other tests also show higher powers but for some selected alternatives.

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