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Coincidence Points and Common Fixed Points for Four Self-Mappings via Weakly Compatible Mappings in Cone Metric Spaces

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Abstract In this paper, we obtain coincidence points and common fixed point theorem for four self-mappings via weakly compatible mappings in cone metric spaces, where the cone is not necessarily normal. These results are improved and generalized several well- known comparable results existing in the references.

Keywords: *cone metric space, common fixed point, coincide point*

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1. Introduction

Huang and Zhang [8] introduced the concept of a cone metric space, they replaced set of real numbers by an ordered Banach space and proved some fixed point theorems for single mapping for contractive type conditions in cone metric spaces, using the normality condition. Later on many authors have [1,2,3,4,6,7,9,10,11] generalized and extended the results of Huang and Zhang [8] and obtained the common fixed points for two or more self-mappings for different types of contractive conditions in cone metric spaces with and without using the normality condition. In this paper, we obtained coincidence points and common fixed points for four self-mappings in cone metric spaces without the normality condition. Our results are extended and improved the results of Arshad, Azam and Vetro [5].

The following definitions are due to Huang and Zhang [8].

Definition 1.1. [8]

Let B be a real Banach space and P be a subset of B. The set P is called a cone if and only if:

(a). P is closed, non–empty and $P \neq \{0\}$;

(b). $a,b \in \mathbb{R}$, $a, b \geq 0$, $x,y \in P \Rightarrow ax+by \in P$;

(c). $P \cap (-P) = \{0\}$.

Definition 1.2. [8]

Let P be a cone in a Banach space B, define partial ordering '$\leq$' with respect to P by $x \leq y$ if and only if $y-x \in P$. We shall write $x<y$ to indicate $x \leq y$ but $x \neq y$ while $x<<y$ will stand for $y-x \in$ int P, where, int P denotes the interior of the set P. This cone P is called an order cone.

Definition 1.3. [8] Let B be a Banach space and $P \subset B$ be an order cone. The order cone P is called normal if there exists $M > 0$ such that for all $x, y \in B$,

$$0 \leqq x \leq y \Rightarrow \| x \| \leq M \| y \|.$$

The least positive number M satisfying the above inequality is called the normal constant of P.

Definition 1.4. [8] Let X be a nonempty set of B Suppose that the map $d: X \times X \rightarrow B$ satisfies:

(d1). $0 < d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = 0$ if and only if $x = y$;

(d2). $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3). $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

It is clear that the concept of a cone metric space is more general than that of a metric space.

Example 1.5. [8] Let $B = \mathbb{R}^2$, $P = \{(x, y) \in B$ such that: $x, y \geq 0\} \subseteq \mathbb{R}^2$, $X = \mathbb{R}$ and $d: X \times X \rightarrow B$ such that $d(x , y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.6 [8] Let (X, d) be a cone metric space .We say that $\{x_n\}$ is said to be

(i) a convergent sequence if for any $c>>0$, there is a natural number N such that for all $n>N$, $d(x_n, x) <<c$, for some fixed x in X. We denote this $x_n \rightarrow x$ (as $n \rightarrow \infty$).

(ii) a Cauchy sequence if for every c in E with $c>>0$, there is a natural number N such that for all $n, m>N$, $d(x_n, x_m) << c$.

(iii) a cone metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

Definition 1.7. [5] A pair (S, T) of self-mappings on X is said to be weakly compatible if they commute at their coincidence point (i. e., $STx = TSx$), whenever $Sx = Tx$).

2. Main Results

In this section, we prove a result on points of coincidence and common fixed points for self-mappings and then show that this result generalizes some of the recent results existing in the literature.

Lemma 2.1. Let X be a nonempty set and the mappings S, T, I and J: X→ X point of coincidence p in X. If {S, I} and {T, J} are weakly compatible if they commute at their coincidence p in X. If {S, I} and {T, J} are weakly compatibles, then S, T, I and J have a unique common fixed point.

Proof. Since, p is a coincidence point of S, T, I and J. Therefore p = Iq = Jq = Sq = Tq for some q∈ X. By weakly compatibility of {S, I} and {T, J} we have Sp = SIq = ISq = Ip, Tp = TJq = JTq = Jp.

It implies that Sp = Tp = Ip = Jp = r(say), then r is a point of coincidence of S,T, I and J. Therefore, p = r by uniqueness. Thus p is a unique common fixed point of S, T, I and J.

We adopt the technique which was used in [5].

Let (X, d) be a cone metric space and S, T, I and J be self-mappings on X such that S(X) ⊆ J(X), T(X) ⊆ I(X). Suppose $x_0 \in X$ and $x_1, x_2 \in X$ is chosen such that $S(x_0) = J(x_1)$, $T(x_1) = I(x_2)$ continuing this process we can define $\{x_n\}$ by

$$Jx_{2n+1} = Sx_{2n},\ Ix_{2n+2} = Tx_{2n+1},\ n = 0,1,2\ldots$$

Denote,

$$y_{2n} = Jx_{2n+1} = Sx_{2n}, y_{2n+1} = Ix_{2n+2} = Tx_{2n+1},\quad n = 0,1,2\ldots \tag{1}$$

The sequence $\{y_n\}$ is called an S-T-sequence with initial point x_0.

The following result is an extension of Proposition 3.2,. in [5].

Proposition 2.2. Let (X, d) be a cone metric space and P be an order cone. Let S, T, I, and J:X→X be such that S(X) ⊆ J(X), T(X) ⊆ I(X). Assume that the following conditions hold:

(i) $d(Sx, Ty) \le \alpha\, d(Ix, Sx) + \beta\, d(Jy, Ty) + \gamma\, d(Ix, Jy)$, for all x, y∈X with x ≠ y, where α, β, γ are non-negative real numbers with $\alpha + \beta + \gamma < 1$;

(ii) $d(Sx, Tx) < d(Ix, Sx) + d(Jx, Tx)$, for all x∈X, whenever Sx ≠ Tx.

Then every S-T-sequence with initial point $x_0 \in X$ is a Cauchy sequence.

Proof. Let x_0 be an arbitrary point in X and $\{y_n\}$ be an S-T-sequence with initial point x_0 . First we assume that $y_{2n} \ne y_{2n+1}$ for all n. Then

$$\begin{aligned}
&d(y_{2n},\ y_{2n+1}) = d(Sx_{2n},\ Tx_{2n+1}),\\
&\le \alpha d(Ix_{2n}, Sx_{2n}) + \beta d(Jx_{2n+1}, Tx_{2n+1})\\
&\quad + \gamma d(Ix_{2n},\ Jx_{2n+1}),\\
&\le \alpha d(y_{2n-1}, y_{2n}) + \beta d(y_{2n}, y_{2n+1})\\
&\quad + \gamma d(y_{2n-1}, y_{2n}),\\
&\le (\alpha+\gamma) d(y_{2n-1}, y_{2n}) + \beta d(y_{2n},\ y_{2n+1}).\\
&\Rightarrow (1-\beta) d(y_{2n}, y_{2n+1}) \le (\alpha+\lambda) d(y_{2n-1}, y_{2n})\\
&\Rightarrow \beta(y_{2n}, y_{2n+1}) \le (\alpha+\lambda)/(1-\beta) d(y_{2n-1}, y_{2n}).
\end{aligned} \tag{2}$$

Similarly, we obtain that

$$d(y_{2n+1}, y_{2n+2}) \le (\beta+\lambda)/(1-\alpha) d(y_{2n}, y_{2n+1}). \tag{3}$$

Now from (2) and (3), we deduce that

$$\begin{aligned}
&d(y_{2n+2}, y_{2n+1}) \le (\beta+\lambda)/(1-\alpha) d(y_{2n+1}, y_{2n})\\
&\le [(\alpha+\gamma)/(1-\beta)][(\beta+\lambda)/(1-\alpha)] d(y_{2n-1}, y_{2n})\\
&\le \ldots \le [(\alpha+\gamma)/(1-\beta)]\\
&\qquad \bullet([(\beta+\lambda)/(1-\alpha)][(\alpha+\gamma)/(1-\beta)])^{n} d(y_1, y_0).
\end{aligned} \tag{4}$$

And

$$\begin{aligned}
&d(y_{2n+3},\ y_{2n+2}) \le (\beta+\lambda)/(1-\alpha) d(y_{2n+2}, y_{2n+1})\\
&\le ([(\beta+\lambda)/(1-\alpha)][(\alpha+\lambda)/(1-\beta)])^{m+1} d(y_1,\ y_0)
\end{aligned} \tag{5}$$

Let M = (α + γ) /(1-β), N = (β + γ) /(1-α).

Then MN < 1. Now for any n > m, we have

$$\begin{aligned}
&d(y_{2n+1}, y_{2m+1})\\
&\le d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1}) + \ldots\\
&\quad + d(y_{2m+2}, y_{2m+1}),\\
&\le \left[\sum_{i=m+1}^{n} (MN)^i + N\sum_{i=m}^{n-1} (MN)^i\right] d(y_1, y_0),\\
&\le \left[(MN)^{m+1}/1-MN + N\ (MN)^{m}/1-MN\right] d(y_1, y_0).
\end{aligned}$$

In analogous way, we again

$$d(y_{2n}, y_{2m+1}) \le (M+1) N (MN)^{m}/1-MN\ d(y_1, y_0),$$

$$d(y_{2n}, y_{2m}) \le (N+1)(MN)^{m+1}/1-MN\ d(y_1, y_0).$$

And $d(y_{2n+1}, y_{2m}) \le (N+1)(MN)^{m+1}/1\text{-}MN\ d(y_1, y_0)$.

Thus, for n > m > 0,

$$\begin{aligned}
&d(y_n, y_m)\\
&\le \text{Max}\begin{Bmatrix}(N+1)(MN)^{m+1}/1-MN,\\ (M+1)N(MN)^{m}/1-MN\end{Bmatrix} d(y_1, y_0)\\
&= b_m d(y_1, y_0), \text{where } b_m \to 0 \text{ as } m \to \infty.
\end{aligned}$$

For each c >> 0, choose δ > 0 such that c – x∈int P, where ||x||<< δ, that is, x << c, for this δ, we can choose a natural number N_1 such that $\|b_m\, d(y_1, y_0)\| < \delta$, for $m > N_1$.

Thus we get that $d(y_n, y_m) \le b_m\, d(y_1, y_0) \ll c$, for all $n > m > N_1$. Therefore $\{y_n\}$ is a Cauchy sequence in (X, d). Now we suppose that $y_{2n} = y_{2n+1}$ for some $n \in N_1$. If $x_{2n} = x_{2n+1}$, by (ii) we have

$$\begin{aligned}
&d(y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1}),\\
&< d(Ix_{2n}, Sx_{2n}) + d(Jx_{2n}, Tx_{2n})\\
&< d(Ix_{2n+1}, Sx_{2n}) + d(Jx_{2n+1}, Tx_{2n+1})\\
&= d(y_{2n}, y_{2n}) + d(y_{2n}, y_{2n+1})\\
&= d(y_{2n}, y_{2n+1}),
\end{aligned}$$

which implies that $y_{2n} = y_{2n+1}$.

If $x_{2n} \ne x_{2n+1}$, we use (i) to obtain $y_{2n} = y_{2n+1}$.

Similarly, we deduce that $y_{2n+1} = y_{2n+2}$ and so $y_n = y_m$ for every n ≥ m. Hence, $\{y_n\}$ is a Cauchy sequence.

Theorem 2. 3. Let (X, d) be a cone metric space and P be an order cone, let S, T, I and J: X → X be such that

$S(X) \subseteq J(X)$, $T(X) \subseteq I(X)$. Assume that the following conditions are hold.

(i) $(Sx, Ty) \le \alpha\, d(Ix, Sx) + \beta\, d(Jy, Ty) + \gamma\, d(Ix, Jy)$, for all $x, y \in X$ with $x \ne y$, where α, β, γ are non-negative real numbers with $\alpha + \beta + \gamma < 1$;

(ii) $d(Sx, Tx) < d(Ix, Sx) + d(Jx, Tx)$, for all $x \in X$, whenever $Sx \ne Tx$.

If one of S(X), T(X), I(X) and J(X) is a complete subspace of X, then S, T and f have a unique point of coincidence. Moreover, if {S, I} and {T, J} are weakly compatible, then S, T, I and J have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. By Proposition 2.2 every S-T-sequence $\{y_n\}$ with initial point x_0 is a Cauchy sequence. Suppose J(X) is complete there exists $p \in J(X)$ such that $y_{2n} = Sx_{2n} = Jx_{2n+1} \to p$ as $n \to \infty$. We can find a $u \in X$ such that $Ju = p$ (If S(X) is complete, there exists $p \in S(X) \subseteq J(X)$, then the conclusions remain the same). Now we show that Tu = p. By (i), we have

$$\begin{aligned}
d(Sv,p) &\le d(Sv,Tx_{2n+1}) + d(Tx_{2n+1},p)\\
&\le \alpha d(Iv,Sv) + \beta d(Jx_{2n+1},Tx_{2n+1})\\
&\quad + \gamma d(Iv,Jx_{2n+1}) + d(Tx_{2n+1},p)\\
&\le \alpha\ d(p\ ,Sv) + (\beta+\gamma) d(y_{2n},y_{2n+1})\\
&\quad + (\gamma+1) d(y_{2n+1},p)\\
&\Rightarrow (1-\alpha) d(p,Sv)\\
&\le (\beta+\gamma) d(y_{2n},y_{2n+1}) + (\gamma+1) d(y_{2n+1},p)\\
&\Rightarrow d(p\ ,Sv) \le (\beta\ +\ \gamma)/(1-\alpha) d(y_{2n},y_{2n+1})\\
&\qquad + (\gamma+1)/(1-\alpha) d(y_{2n+1},p).
\end{aligned}$$

Fix $0 \ll c$ and choose $n_1 \in N$ be such that $d(y_{2n-1}, y_{2n}) \ll (1-\alpha)/(\beta+\gamma)\, c/2$, and $d(y_{2n}, p) \ll (1-\alpha)/(\gamma+1)\, c/2$ for all $n \ge n_1$.

Consequently, $d(p, Sv) \ll c/2 + c/2 = c$.

$\Rightarrow d(p, Sv) \ll c$ and hence $d(p, Sv) \ll c/m$ for every $m \in N$. From $c/m - d(p, Sv) \in \text{int } P$, being P is closed as $m \to \infty$, we deduce $-d(p, Sv) \in P$ and so $d(v, Sp) = 0$.

$\Rightarrow p = Sv$.

Therefore, $Sv = p = Iv$.

Therefore, $Tu = Ju = Sv = Iv$ (= p).

$\Rightarrow$ p is a point of coincidence of S, T, I and J have a unique point of coincidence. For his we assume that there exists another point p^* in X such that $p^* = Tu^* = Tu^* = Sv^* = Iv^*$, then

$$\begin{aligned}
d(p,p^*) &= d(Su,Tu^*)\\
&\le \alpha d(Iu,Su) + \beta d(Ju^*,Tu^*) + \gamma d(Iu,Ju^*)\\
&\le \alpha d(p,p) + \beta d(p^*,p^*) + \gamma d(p,p^*)\\
&\le \gamma d(p,p^*).
\end{aligned}$$

We deduce $p = p^*$. Since, {S, I} and {T, J} are weakly compatible, then p is the unique point of coincidence of S, T, I and J then by the above Proposition we get that p is the unique common fixed point of S, T, I and J.

Remark 2.4.

(i) If we choose I = J = f in the above Theorem 2.3., we can get the Theorem 3.3., of [5].

The above Theorem 2.3., generalizes the Theorem 3.3., of [5].

(ii) If we choose S = T and I = J = f in the above Theorem 2.3., we can get the Theorem 3.4., of [5].

The above Theorem 2.3., generalizes the Theorem 3.4., of [5].

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