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Combining Long Division of Polynomials and Exponential Shift Law to Solve Differential Equations

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Abstract Inspired by the method of undetermined coefficients, this paper presents an alternative method to solve linear differential equations with constant coefficients, using the technique of polynomial long division. Expanding this technique with the exponential shift law enables to solve all types of non-homogeneous differential equations, of where the undetermined coefficients can be applied.

Keywords: *undetermined coefficients, long division, exponential shift law*

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1. Introduction

The method of undetermined coefficients is a technique [1] for determining a particular solution y_p to linear, constant coefficient differential equations of order n

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x) \qquad (1)$$

for certain types of non-homogeneous terms $g(x)$. Based on the form of the function $g(x)$ on the right hand side of equation (1), a guess is made about the appropriate form of the solution $y_p(x)$, which then tested by differentiating the resulting equation [2,3]. To demonstrate the method let's solve the following initial-value problem:

$$y'' + 3y' + 2y = x + 1, \;\; y(0) = -1, \;\; y'(0) = 1 . \qquad (2)$$

The general solution of (2) has the form

$$y = y_c + y_p \qquad (3)$$

where y_c is the complementary solution to the homogeneous differential equation (hereafter, DE)

$$y'' + 3y' + 2y = 0$$

and y_p is a particular solution to the non-homogenous DE (2). From the auxiliary equation

$$r^2 + 3r + 2r = (r+1)(r+2) = 0$$

we find $r = -1$ and $r = -2$, and thus the complementary solution is

$$y_c(x) = c_1 e^{-x} + c_2 e^{-2x}.$$

To find y_p, we take a hint from the term to the right of DE (2), and make the guess that a particular solution might have the form

$$y_p(x) = Ax + B .$$

Testing the solution y_p, we have: $y_p' = A$ and $y_p'' = 0$. Substituting the expressions for y_p, y_p' and y_p'' into the DE (2), we obtain

$$3A + 2(Ax + B) = x + 1$$

or

$$2Ax + (3A + 2B) = x + 1. \qquad (4)$$

Comparing the coefficients of x and the constant terms on both sides of (4), we conclude that: $2A = 1$ and $3A + 2B = 1$, or $A = 1/2$, and $B = -1/4$. So, a solution of DE (2) is

$$y_p(x) = \frac{1}{2}x - \frac{1}{4},$$

as easily can be verified.

Now, (3) gives the general solution of DE (2):

$$y(x) = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{2}x - \frac{1}{4}$$

and

$$y'(x) = -c_1 e^{-x} - 2c_2 e^{-2x} + \frac{1}{2}.$$

The initial conditions imply

$$y(0) = c_1 + c_2 - \frac{1}{4} = -1 \;\text{ or }\; c_1 + c_2 = -\frac{3}{4}$$

$$y'(0) = -c_1 - 2c_2 + \frac{1}{2} = 1 \;\text{ or }\; -c_1 - 2c_2 = \frac{1}{2}.$$

Thus $c_1 = -1$ and $c_2 = \frac{1}{4}$, while the general solution of the DE (2) is

$$y(x) = -e^{-x} + \frac{1}{4}e^{-2x} + \frac{1}{2}x - \frac{1}{4}. \quad (5)$$

The method of undetermined coefficients can be used only if $g(x)$ consists of a sum of terms each of which has a finite number of linearly independent derivatives [2,4]. This restriction implies that $g(x)$ can only contain terms such as a, x^k, e^{ax}, $\sin ax$, $\cos ax$, and combinations of such terms, where a is a constant and k is a positive integer. Although the method does not require any integrations, is based on "good guesses" [2] to find a particular solution, so most textbooks on the topic give tables of trial particular solutions y_p for different forms of $g(x)$. For complex forms of $g(x)$ or for high order differential equations, the method of undetermined coefficient may require many calculations [5,6] in differentiating and solving systems of linear equations.

Many efforts have been made to improve the method and create alternatives that require less computation and are easiest to understand and implement. Gupta [6] proposed a recursive algorithm, i.e. a *n*-times differentiation and back-substitution process, to solve *n*-th order DE with constant coefficients. Using a complicated methodology, Krohn, Marino-Johnson and Ouyang [7] derived formulas that should be part of a computer program to give the coefficients. Gollwitzer [8] combined linear operators, matrix multiplication and successive differentiations over a characteristic identify to describe a strategy that replaces the method of undetermined coefficients. Oliveira [5] tried to overcome the computation complexity of both the undetermined coefficients and the annihilator methods and came up with a method that involves differentiations of the characteristic polynomial, back-substitution to reduce the order of the DE, solving systems of equations and integration to reach the particular solution of the DE.

In one way or the other, all these approaches seek from the start for solutions in the form $y_p(x) = e^{ax} p_n(x)$ and recursively reduce the order of the DE by integration. In this paper we give a simple, straightforward and intuitive approach to find the particular solution of the *n*-th order linear DE with constant coefficients. The main idea is to use a linear polynomial operator L to write the DE in the form $Ly = g(x)$ and then simply solve for y by dividing $g(x)$ by L, using the long division method. We start with $g(x)$ being a polynomial and proceed to solve the cases with $g(x)$ involving exponential and trigonometric functions. The only change needed in order to handle the latter cases is to "shift" the operator L before doing the long division. In short, in order to find a particular solution of the non-homogeneous linear DE with constant coefficients, there is no need to look for solutions of a specific form or to execute back-substitutions and integrations or solve any system of equations. Long division algorithm and basic algebra is all needed.

2. The Long Division Operator

The differential equation (1) can be rewritten in the form

$$\left(a_n D^n + a_{n-1} D^{(n-1)} + \cdots + a_1 D + a_0\right) y(x) = g(x) \quad (6)$$

where with the letter D we denote the differential operators:

$$Dy(x) = \frac{d}{dx} y(x),\ D^2 y(x) = \frac{d^2}{dx^2} y(x),$$
$$\cdots,\ D^n y(x) = \frac{d^n}{dx^n} y(x). \quad (7)$$

Equation (6) is a representation of the differential equation (1) in operational form

$$L(D) y(x) = g(x) \quad (8)$$

where the operator L is a linear combination of differential operators of orders 0 to n

$$L(D) = a_0 + a_1 D + a_2 D^2 + \cdots + a_n D^n,\ a_n \neq 0 \quad (9)$$

Thus, regarding differentiation as an operator, the solution of the operational equation (8) is given by

$$y(x) = [L(D)]^{-1} g(x), \quad (10)$$

where L^{-1} is an inverse operator which has to be suitably manipulated and interpreted [9]. In the case where $g(x)$ is a polynomial function of x, we define the action of the inverse operator L^{-1} on $g(x)$ as the long division of polynomial $g(x)$ by the polynomial operator $L(D)$,

$$y(x) = \frac{g(x)}{L(D)} = \frac{g(x)}{a_0 + a_1 D + a_2 D^2 + \cdots + a_n D^n}. \quad (11)$$

2.1. Implementing the Long Division

For example, equation (2), $y'' + 3y' + 2y = x + 1$, can be written as

$$\left(D^2 + 3D + 2\right) y(x) = x + 1$$

and a partial solution can be found by (10) performing the long division

$$y_p(x) = \frac{x+1}{2 + 3D + D^2}$$

which goes as follows:

$$\begin{array}{r|l} & \frac{1}{2}x - \frac{1}{4} \\ 2 + 3D + D^2 & \overline{\ x + 1} \\ & -x - \frac{3}{2} \\ \hline & -\frac{1}{2} \\ & +\frac{1}{2} \\ \hline & 0 \end{array}$$

yielding the particular solution $y_p(x) = \frac{1}{2}x - \frac{1}{4}$.

Analytic explanation of the process: As it is known from the long division algorithm of polynomials, in each step of the process, the first term of the dividend is divided by the first term of the divisor and the result is multiplied from the divisor and subtracted from the dividend. In the example above, *x*, the first term of the dividend, is divided by 2, the first term of the divisor. Thus, the first term of the quotient is $\frac{1}{2}x$ and multiplied by each term of the divisor:

$$2\frac{1}{2}x = x,$$

$$3D(\frac{1}{2}x) = \frac{3}{2}D(x) = \frac{3}{2}\frac{d}{dx}x = \frac{3}{2},$$

$$D^2(\frac{1}{2}x) = \frac{1}{2}\frac{d^2}{dx^2}x = 0.$$

The result, $x + \frac{3}{2}$, is subtracted from the dividend

$$\begin{array}{r} x+1 \\ -x-\frac{3}{2} \\ \hline -\frac{1}{2} \end{array}$$

and the same process repeats for the (interim) remainder, $-\frac{1}{2}$, until the remainder is zero.

2.2. Comparison with Power Series Method

Let's solve the DE

$$y'' + 3y' + 2y = x + 1, \;\; y(0) = -1, \;\; y'(0) = 1, \qquad (2)$$

using power series solutions in order to illustrate that the long division approach we propose, gives the same solution with the solution obtained via series methods, much faster. Assuming that DE (2) has a solution of the form $y = c_0 + c_1x + c_2x^2 + c_3x^3 \cdots = \sum_{n=0}^{\infty} c_n x^n$ and by taking derivatives term by term we have

$$y' = c_1 + 2c_2x + 3c_3x^2 \cdots = \sum_{n=1}^{\infty} nc_n x^{n-1}$$

$$y'' = 2c_2 + 2\cdot 3c_3x^2 + \cdots = \sum_{n=1}^{\infty} n(n-1)c_n x^{n-2}.$$

In order to compare the expressions for y, y' and y'' more easily, we rewrite the last two as follows:

$$y' = \sum_{n=0}^{\infty} (n+1)c_{n+1}x^n$$

$$y'' = \sum_{n=0}^{\infty} (n+1)(n+2)c_{n+2}x^n.$$

Substituting the expressions in DE (2), we obtain

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)c_{n+2} + 3(n+1)c_{n+1} + 2c_n\right]x^n = x+1.$$

If two power series are equal, then the corresponding coefficients must be equal. Therefore, the coefficients of the constant term 1 (obtained for $n = 0$) and x (obtained for $n = 1$) must be 1 in both cases:

$$2c_2 + 3c_1 + 2c_0 = 1$$

$$6c_3 + 6c_2 + 2c_1 = 1.$$

Taking into account the initial conditions, $y(0) = -1$ yields $c_0 = -1$, and $y'(0) = 1$ yields $c_1 = 1$.

Therefore, $c_2 = 0$ and $c_3 = -1/6$.

All the other coefficients for powers equal and higher than x^2 ($n \geq 2$) should be zero:

$$(n+1)(n+2)c_{n+2} + 3(n+1)c_{n+1} + 2c_n = 0.$$

So, the coefficients c_j, for $j \geq 4$, can be calculated using the recurrence relation

$$c_{n+2} = \frac{-3(n+1)c_{n+1} - 2c_n}{(n+1)(n+2)}, \; n \geq 2.$$

So, the general solution to DE (2) is

$$y(x) = -1 + x - \frac{1}{6}x^3 + \cdots$$

Since it is not obvious that this series solution matches its analytic counterpart (5), let's take (5)

$$y(x) = -e^{-x} + \frac{1}{4}e^{-2x} + \frac{1}{2}x - \frac{1}{4}$$

and expand the exponents into power series:

$$y(x) = -\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right) + \frac{1}{4}\left(1 - 2x + \frac{4x^2}{2} - \frac{8x^3}{6} + \cdots\right) + \frac{1}{2}x - \frac{1}{4} = -1 + x - \frac{1}{6}x^3 + \cdots$$

Although power series is an important tool to confront differential equations that can't be solved explicitly in terms of finite combinations of simple familiar functions [4], in the case of linear DEs with constant coefficients it's an arduous, unnecessary alternative.

2.3. In the Case Where $a_0 = 0$

To solve for equation (11), the constant term in the denominator has to be $a_0 \neq 0$. Let's try, for example, to find a particular solution which satisfies the differential equation

$$y'' + 3y' = 3x^2 + 2x + 3. \qquad (12)$$

Now $a_0 = 0$. To handle this case, we use an auxiliary function $u(x)$ and write (12) as

$$\begin{cases} Dy(x) + 3y(x) = u(x) \\ Du(x) = 3x^2 + 2x + 3 \end{cases}. \qquad (13)$$

Integrating the second equation in (13), we find $u(x)$

$$u(x) = \int (3x^2 + 2x + 3)dx = x^3 + x^2 + 3x,$$

and substituting in the first equation in (13), we have to find a particular solution for the equation

$$(D+3)\,y(x) = x^3 + x^2 + 3x.$$

Implementing (11),

$$y_p(x) = \frac{x^3 + x^2 + 3x}{3+D}$$

and performing the long division

$$\begin{array}{rl}
 & \tfrac{1}{3}x^3 \qquad + x \; - \tfrac{1}{3} \\
3+D & \big)\; x^3 + x^2 + 3x \\
 & \underline{-x^3 - x^2} \\
 & \qquad\quad 3x \\
 & \qquad\quad \underline{-3x - 1} \\
 & \qquad\qquad\quad -1 \\
 & \qquad\qquad\quad \underline{\quad 1} \\
 & \qquad\qquad\quad\; 0
\end{array}$$

we find the particular solution $y_p(x) = \dfrac{1}{3}x^3 + x - \dfrac{1}{3}$.

3. The Exponential Shift Law for Polynomial Operators

Up to now, we used the familiar method of long division of polynomials to find a particular solution of linear differential equations with constant coefficients, when the non-homogeneous term $g(x)$ (the right side of equation) is a polynomial function of x. To extend the usability of this method in cases where $g(x)$ is a more complex function of x, we need to use the exponential shift law (theorem):

If $L(D) = a_0 + a_1 D + a_2 D^2 + \cdots + a_n D^n$, $a_n \neq 0$, is a polynomial operator with constant coefficients and $u(x)$ is an nth order differentiable function of x, then

$$L(D)\left(ue^{mx}\right) = e^{mx} L(D+m)u \qquad (14)$$

where m is a constant [2,10].

The proof of the theorem is straightforward and can be found in the Appendix. As can be seen from (14), when the polynomial operator $L(D)$ operates on the product ue^{mx} , the exponential e^{mx} shifts out and the new operator $L(D+m)$ acts only on function u. Utilizing the exponential shift law, we can solve linear non-homogeneous differential equations with constant coefficients, where the non-homogeneous term has the form $g(x) = f(x)e^{mx}$ and $f(x)$ is a polynomial function of x. Starting with the equation (8), $L(D)y(x) = g(x)$, we search for a particular solution in the form

$$y_p(x) = u(x)e^{mx}. \qquad (15)$$

By substitution of $y_p(x)$ and $g(x)$ into (8), we obtain

$$L(D)(u(x)e^{mx}) = f(x)e^{mx}$$

or

$$e^{mx} L(D+m)u(x) = f(x)e^{mx} \qquad (16)$$

as a result of applying the exponential shift law (equation (14)). By eliminating e^{mx} we obtain

$$L(D+m)u(x) = f(x)$$

which returns us to the basic case (equation (11)),

$$u(x) = \frac{f(x)}{L(D+m)} = \frac{f(x)}{a_0 + a_1(D+m) + a_2(D+m)^2 + \cdots + a_n(D+m)^n} \qquad (17)$$

Now, the particular solution is simply

$$y_p(x) = e^{mx}\frac{f(x)}{L(D+m)} = e^{mx}\frac{f(x)}{a_0 + a_1(D+m) + a_2(D+m)^2 + \cdots + a_n(D+m)^n}. \qquad (18)$$

Let’s try for example to find a particular solution of the equation

$$y'' + y' + y = x^2 e^{3x},$$

or, in operational notation,

$$\left(D^2 + D + 1\right)y(x) = x^2 e^{3x}.$$

Since the non-homogeneous term $g(x)$ has the form $f(x)e^{mx}$, a particular solution is given by (18)

$$y_p(x) = e^{3x}\frac{x^2}{1 + (D+3) + (D+3)^2} = e^{3x}\frac{x^2}{13 + 7D + D^2}. \qquad (19)$$

We perform the long division of polynomials

$$\begin{array}{rl}
 & \frac{1}{13}x^2 - \frac{14}{169}x \; + \frac{72}{2197} \\
13 + 7D + D^2 & \big)\; x^2 \\
 & \underline{-x^2 \quad -\frac{14}{13}x \quad -\frac{2}{13}} \\
 & \qquad\quad -\frac{14}{13}x \quad -\frac{2}{13} \\
 & \qquad\quad \underline{\frac{14}{13}x \quad\;\; \frac{98}{169}} \\
 & \qquad\qquad\qquad \frac{72}{169} \\
 & \qquad\qquad\quad \underline{-\frac{72}{169}} \\
 & \qquad\qquad\qquad 0
\end{array}$$

and from (19) we obtain the particular solution

$$y_p(x) = e^{3x}\left(\frac{1}{13}x^2 - \frac{14}{169}x + \frac{72}{2197}\right).$$

4. Manipulating More Complex Situations

The principle of superposition [11] is a mathematical tool that is used in many areas of science and technology for solving complex problems. The principle of superposition applies to linear systems governed by linear differential equations [12] and states that the response of a system to different stimuli is the sum of the responses which would have been caused by each stimulus individually. Applying the principle of superposition, in place of the linear differential equation

$$L(D)y(x) = g_1(x) + g_2(x) + \cdots + g_n(x), \qquad (20)$$

where $L(D)$ is a polynomial operator (see equation (9)), we write the n equations

$$L(D)y(x) = g_1(x),\ L(D)y(x) = g_2(x),\ L(D)y(x) = g_n(x). \qquad (21)$$

Let $y_{1p}(x)$, $y_{2p}(x)$, $\cdots$, $y_{np}(x)$ be respective particular solutions of the n equations of (21). Therefore

$$L(D)y_{1p}(x) = g_1(x),\ L(D)y_{2p}(x) = g_2(x),\ L(D)y_{np}(x) = g_n(x). \qquad (22)$$

Adding all the equations in (22), there results

$$L(D)\left(y_{1p}(x) + \cdots + y_{np}(x)\right) = g_1(x) + g_2(x) + \cdots + g_n(x)$$

which implies that

$$y_p(x) = y_{1p}(x) + y_{2p}(x) + \cdots + y_{np}(x) \qquad (23)$$

is a solution of (20). Thus, using the principle of superposition, a particular solution $y_p(x)$ of a linear non-homogeneous differential equation (20), can be obtained by summing the particular solutions $y_{1p}(x)$, $y_{2p}(x)$, $\cdots$, $y_{np}(x)$ of the n equations of (21).

Let's try to find a particular solution of the equation

$$(D^2+1)y(x) = x^2 + xe^{2x} + 5e^{-3x}. \qquad (24)$$

According to the principle of superposition, a particular solution $y_p(x)$ of (24) is the sum of the particular solutions of each of the following equations,

$$(D^2+1)y(x) = x^2,\ (D^2+1)y(x) = xe^{2x},\ (D^2+1)y(x) = 5e^{-3x}. \qquad (25)$$

Particular solutions of each of these equations are respectively (see equation (18))

$$y_{1p}(x) = \frac{x^2}{1+D^2}$$

$$y_{2p}(x) = e^{2x}\frac{x}{1+(D+2)^2} = e^{2x}\frac{x}{5+4D+D^2}$$

$$y_{3p}(x) = e^{-3x}\frac{5}{1+(D-3)^2} = e^{-3x}\frac{5}{10-6D+D^2}.$$

Now we perform the long divisions:

$$\begin{array}{r|l} & x^2-2 \\ \hline 1+D^2 & x^2 \\ & \underline{-x^2-2} \\ & -2 \\ & \underline{2} \\ & 0 \end{array}$$

$$\begin{array}{r|l} & \frac{1}{5}x - \frac{4}{25} \\ \hline 5+4D+D^2 & x \\ & \underline{-x - \frac{4}{5}} \\ & -\frac{4}{5} \\ & \underline{\frac{4}{5}} \\ & 0 \end{array}$$

$$\begin{array}{r|l} & \frac{1}{2} \\ \hline 10-6D+D^2 & 5 \\ & \underline{-5} \\ & 0 \end{array}$$

Hence a particular solution of (24) is

$$y_p(x) = y_{1p}(x) + y_{2p}(x) + y_{3p}(x) = \\ = x^2 - 2 + e^{2x}\left(\frac{1}{5}x - \frac{4}{25}\right) + \frac{1}{2}e^{-3x}.$$

Being able to handle exponential functions which are parts of the non-homogeneous term $g(x)$, enable us [2,12] to manipulate trigonometric functions, by simply exploiting the richness of the complex exponential. It's easy to see that, if $g(x)$ is a complex-valued function (i.e., a function which can take on complex values) and $y_p(x)$ is a partial solution of the equation $L(D)y(x) = g(x)$, then the real (or imaginary) part of $y_p(x)$ is a solution of the equation with $g(x)$ replaced by its real (or imaginary) part.

To verify this, assume that $g(x)$ is a complex-valued function, $g(x) = \mathrm{Re}(g(x)) + i\,\mathrm{Im}(g(x))$, and $y_p(x)$ is a partial (complex-valued) solution, $y_p(x) = v(x) + iu(x)$, of the equation $L(D)y(x) = g(x)$. Then,

$$L(D)\left(v(x) + iu(x)\right) = \mathrm{Re}(g(x)) + i\,\mathrm{Im}(g(x))$$

$$L(D)v(x) + iL(D)u(x) = \mathrm{Re}(g(x)) + i\,\mathrm{Im}(g(x))$$

and equating the real and imaginary parts on both sides, we take

$$L(D)v(x) = \text{Re}(g(x)) \textit{ and } L(D)u(x) = \text{Im}(g(x)).$$

From Euler's formula we know that, for any real number x, $e^{iax} = \cos ax + i \sin ax$, or $\cos ax = \text{Re}\left(e^{iax}\right)$, $\sin ax = \text{Im}\left(e^{iax}\right)$. Therefore, if $g(x)$ contains the trigonometric functions $\cos rx$ and/or $\sin sx$, we can replace them with e^{irx} and/or e^{isx}, solve the new equation to find a partial solution $z_p(x)$ and finally, extract the real or the imaginary part to find $y_p(x)$: $y_p(x) = \text{Re}\left(z_p(x)\right)$ or $y_p(x) = \text{Im}\left(z_p(x)\right)$.

Let's try for example to find a particular solution of the equation

$$(D^2+2)y(x) = 4x\sin 2x + (x^2-2x)e^{-x}\cos x \qquad (26)$$

Again, a particular solution $y_p(x)$ of (26) is the sum of the particular solutions of each of the following equations,

$$\begin{aligned} (D^2+2)y(x) &= 4x\sin 2x \\ (D^2+2)y(x) &= (x^2-2x)e^{-x}\cos x \end{aligned} \qquad (27)$$

Since the non-homogeneous term $g(x)$ in equations (27) contains trigonometric functions, we transform these equations by substitution of the trigonometric terms $\sin 2x$ and $\cos x$ with the corresponding complex-valued exponentials e^{i2x} and e^{ix}

$$\begin{aligned} &(D^2+2)z(x) = 4xe^{i2x}, \\ &(D^2+2)z(x) = (x^2-2x)e^{-x}e^{ix} = (x^2-2x)e^{(-1+i)x}. \end{aligned} \qquad (28)$$

Particular solutions of each of these equations are respectively

$$\begin{aligned} z_{1p}(x) &= e^{i2x}\frac{4x}{2+(D+2i)^2} \\ &= e^{i2x}\frac{4x}{-2+4iD+D^2} \\ z_{2p}(x) &= e^{(-1+i)x}\frac{x^2-2x}{2+(D+i-1)^2} = \\ &= e^{(-1+i)x}\frac{x^2-2x}{(2-2i)-(2-2i)D+D^2} \end{aligned}$$

after implementing equation (18), since $g(x)$ in (28) has the form $f(x)e^{mx}$. Performing the long divisions we have:

$$\begin{array}{r|l} & -2x-4i \\ \hline -2+4iD+D^2 & 4x \\ & \underline{-4x+8i} \\ & 8i \\ & \underline{-8i} \\ & 0 \end{array}$$

$$\begin{array}{r|l} & \frac{1}{2-2i}x^2 \quad -\frac{2}{(2-2i)^2} \\ \hline (2-2i)-(2-2i)D+D^2 & x^2-2x \\ & \underline{-x^2+2x-\frac{2}{2-2i}} \\ & -\frac{2}{2-2i} \\ & \underline{+\frac{2}{2-2i}} \\ & 0 \end{array}$$

Hence, the particular solutions $z_{1p}(x)$ and $z_{2p}(x)$ are

$$\begin{aligned} z_{1p}(x) &= e^{i2x}(-2x-4i) = (\cos 2x + i\sin 2x)(-2x-4i) \\ &= (-2x\cos 2x + 4\sin 2x) + i(-4\cos 2x - 2x\sin 2x) \end{aligned}$$

and

$$\begin{aligned} z_{2p}(x) &= e^{(-1+i)x}\left(\frac{1}{2-2i}x^2 - \frac{2}{(2-2i)^2}\right) = \\ &= e^{-x}(\cos x + i\sin x)\left(\frac{x^2}{4} + i\frac{x^2-1}{4}\right) \\ &= e^{-x}\left(\frac{x^2}{4}\cos x - \frac{x^2-1}{4}\sin x\right) \\ &\quad + ie^{-x}\left(\frac{x^2-1}{4}\cos x + \frac{x^2}{4}\sin x\right) \end{aligned}$$

respectively. Now, the particular solution $y_{1p}(x)$ of $(D^2+2)y(x) = 4x\sin 2x$, is the imaginary part of $z_{1p}(x)$, because $\sin 2x = \text{Im}\left(e^{i2x}\right)$.

$$y_{1p}(x) = \text{Im}\left(z_{1p}(x)\right) = -4\cos 2x - 2x\sin 2x.$$

The particular solution $y_{2p}(x)$ of

$$(D^2+2)y(x) = (x^2-2x)e^{-x}\cos x,$$

is the real part of $z_{2p}(x)$, since $\cos x = \text{Re}\left(e^{ix}\right)$.

$$y_{2p}(x) = \text{Re}\left(z_{2p}(x)\right) = e^{-x}\left(\frac{x^2}{4}\cos x - \frac{x^2-1}{4}\sin x\right).$$

Finally, a particular solution of (26) is the sum

$$\begin{aligned} y_p(x) &= y_{1p}(x) + y_{2p}(x) \\ &= -4\cos 2x - 2x\sin 2x \\ &\quad + e^{-x}\left(\frac{x^2}{4}\cos x - \frac{x^2-1}{4}\sin x\right). \end{aligned}$$

5. Conclusion

In this note, an alternative way to solve linear non-homogeneous differential equations with constant coefficients, was illustrated. Having in mind the certain types of non-homogeneous terms for which the method of undetermined coefficients can be used, the familiar technique of long division of polynomials was exploited to solve the basic case (i.e. the non-homogeneous term is a polynomial) along with the exponential shift law to handle the more complex cases (i.e. the non-homogeneous term includes exponentials or trigonometric functions). This technique is intuitive, straightforward and overcomes the burden of solving systems of equations to find the undetermined coefficients.

Appendix

Exponential Shift Theorem.

If $L(D)=a_0+a_1D+a_2D^2+\cdots+a_nD^n$, $a_n \neq 0$, is a polynomial operator with constant coefficients and $u(x)$ is an nth order differentiable function of x , then

$$L(D)\left(ue^{mx}\right)=e^{mx}L(D+m)u \qquad \text{(a)}$$

where m is a constant.

Proof. We shall first prove by induction that

$$D^k\left(ue^{mx}\right)=e^{mx}(D+m)^k u, \qquad \text{(b)}$$

is true for every k.

For $k=1$,

$$D\left(ue^{mx}\right)=mue^{mx}+e^{mx}u'=e^{mx}\left(u'+mu\right)$$
$$=e^{mx}\left(D+m\right)u.$$

Now, we assume that (b) is true for $k=n$, i.e., that the equality

$$D^n\left(ue^{mx}\right)=e^{mx}(D+m)^n u \qquad \text{(c)}$$

is valid.

We must then prove that (b) is true for $k=n+1$. The left side of (b) becomes

$$D^{n+1}\left(ue^{mx}\right)=D\left[D^n\left(ue^{mx}\right)\right]=D\left[e^{mx}(D+m)^n u\right]=$$
$$=e^{mx}D(D+m)^n u+me^{mx}(D+m)^n u=$$
$$=e^{mx}\left[D(D+m)^n+m(D+m)^n\right]u=$$
$$=e^{mx}\left[(D+m)^n(D+m)\right]u=$$
$$=e^{mx}(D+m)^{n+1}u$$

where in the second step we used equality (c). Having proved (b) we proceed to prove the theorem (a). Starting from the left side of equation (a) we obtain

$$L(D)\left(ue^{mx}\right)=(a_0+a_1D+a_2D^2+\cdots+a_nD^n)\left(ue^{mx}\right)=$$
$$=a_0ue^{mx}+a_1D\left(ue^{mx}\right)+\cdots+a_nD^n\left(ue^{mx}\right)=$$
$$=e^{mx}\left[a_0+a_1(D+m)+\cdots+a_n(D+m)^n\right]u=$$
$$=e^{mx}L(D+m)u$$

where in the third step we used equality (b).

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