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A Note on the Unique Solution of the Integral Equations in the Framework of Fixed Point Theorem on Partially Ordered Metric Space

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Abstract In this paper, we obtained the unique solution of the integral and coupled integral equation in the framework of fixed point theorem on partially ordered metric space. Our results unified some methods in studying the existence of unique solution for the integral equation. Moreover, all results are much more brief. In addition, the examples are given to illustrate the usability of the obtained results.

Keywords: *coupled integral equations, fixed point theorems, partially ordered metric spaces*

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1. Introduction

The fixed point theory centers on the process of solving the equation of the form $T(x)=x$. One of the most widely used theory is Banach fixed point theory and its several extensions in generalized metric spaces. Therefore, fixed point theory on partially ordered sets has been studied recently in [1,3,8,10,11,13]. For example, fixed point theorems for nonlinear and semi-linear operators on order intervals [1], coupled fixed point theorems [9] and extended the theoretical results to fixed points in partially sets [10] etc. On the other hand, given non-empty subsets A and B of the partially ordered set X and a non-self mapping S from A to B, one can perceive that the equation $S(x)=x$ is improbable to have a solution. Naturally, best proximity point theorems on partially ordered set are also be studied in [4,5,6,7].

It is well-known that those abstract results can be applied to obtain an abundance of concrete results for some special problems, for instance, (a) differential and difference equation; (b) integral equation; (c) periodic boundary value problems. The purpose of this paper is to obtain the existence of solution of the integral equation for mixed monotone, contractions in the setting of partially ordered sets endowed with metrics. It is remarked that the unique solution of integral equations in this paper are established in the setting of ordered metric spaces whereas the fixed point theorems in [1,3,9,12] are elicited in the framework of fixed point theorems on partially ordered metric space.

2. Fixed Point Theorems in Partially Ordered Metric Spaces

Definition 2.1 [2] Let $(X,\leq)$ be a partially ordered set, T: $X\to X$ be a mapping. If $x\leq y\Rightarrow\ T(x)\leq T(y)$, then T is said to have the monotone increasing property.

Let $(X,\leq)$ be a partially ordered set and suppose (X,d) is a complete metric space. Let T: $X\to X$ be an increasing and continuous mapping. The following Theorems establish the fact that the contractive nature of T is not restricted to the entire set X but only restricted to the comparable elements of $(X,\leq)$.

Definition 2.2 [14] Let X be a set and let $s\geq 1$ be a given real number. A functional $d: X\times X\to R_+$ is said to be a b-metric if the following conditions are satisfied:

1. $d(x,y)=0$ if and only if $x=y$;
2. $d(x,y)=d(y,x)$, for all $x,y\in X$;
3. $d(x,z)\leq s[d(x,y)+d(y,z)]$.

A pair (X,d) is called a b-metric space.

Theorem 2.1 [10] If there exists $\lambda<1$ with $d(T(x),T(y))\leq\lambda d(x,y)$, whenever $y\leq x$ and there exists $x_0\in X$, with $x_0\leq T(x_0)$, then T has a fixed point.

Theorem 2.2 [11] Assume that there exist upper and lower bounds of the pair $\{x,y\}$ for any $x,y\in X$. If there exists $\lambda<1$ with $d(T(x),T(y))\leq\lambda d(x,y)$, whenever $y\leq x$ and there exists $x_0\in X$, with $x_0\leq T(x_0)$ or $x_0\geq T(x_0)$, then T has a unique fixed point u. Moreover, for any $y\in X$, the orbit $\{T^n(y)\}$ converges to the fixed point u.

Let $I=[a,b]$, $X=C(I,R)$, we define the following order relation in $X: x,y\in C(I,R)$, $x\leq y\Leftrightarrow x(t)\leq y(t)$, for $\forall t \in I$, then $(X, \leq)$ is a partially ordered set. Define the metric on X as the follow:

$$d(x,y) = \sup_{t\in I}\left|x(t) - y(t)\right|, \left(\forall x, y \in C(I,R)\right)$$

then (X,d) is a complete metric space.

Next, we consider the existence of solutions for the following integral equation for an unknown function u (see [3]):

$$u(t) = v(t) + \lambda \int_a^b G(t,z) f(z,u(z))dz, (t \in I), \qquad (1)$$

where $f : I \times R \to R$, $G : I \times I \to [0,+\infty]$, $v : I \to R$ are given continuous functions.

Let X be the set $C[a,b]$ of real continuous functions on $[a,b], d(u,v) = \sup_{t\in I}|u(t) - v(t)| = \max_{t\in I}|u(t) - v(t)|$. It is easy to check that (X,d) is a complete metric space. Define a mapping $T : X \to X$ by

$$T(u(t)) = v(t) + \lambda \int_a^b G(t,z) f(z,u(z))dz, t \in I. \qquad (2)$$

Then $u(t)$ is a solution of (1) if and only if it is a fixed point of T.

Theorem 2.3 Consider the integral equation (1) under the following assumptions:

(H_1) $0 < \lambda < 1$;

(H_2) for all $x \in I$, if $u_2(t) \leq u_1(t)$, then $0 \leq f(x,u_1(t)) - f(x,u_2(t)) \leq |u_1(t) - u_2(t)|$;

(H_3) $0 \leq G(t,z) \leq \frac{1}{b-a}, \forall (t,z) \in I \times I$;

(H_4) $\exists x_0 \in C(I,R), x_0 \leq T(x_0)$ or $x_0 \geq T(x_0)$.

Then (1) has a unique solution u. Moreover, for any $y \in X$, the orbit $\{T^n(y)\}$ converges to the solution u.

Proof. Let $u_2(t) \leq u_1(t)$, then

$$\begin{aligned}
&T(u_1(t)) - T(u_2(t)) \\
&= \left(v(t) + \lambda \int_a^b G(t,z) f(z,u_1(z))dz\right) \\
&- \left(v(t) + \lambda \int_a^b G(t,z) f(z,u_2(z))dz\right) \\
&= \lambda \int_a^b G(t,z)\left(f(z,u_1(z)) - f(z,u_2(z))\right)dz \geq 0.
\end{aligned}$$

It implies that $T(u_1(t)) \leq T(u_2(t))$. So T is an increasing and continuous mapping.

$$\begin{aligned}
d(Tu_1, Tu_2) &= \max_{t\in I}\left|T(u_1(t)) - T(u_2(t))\right| \\
&= \max_{t\in I}\left|\begin{array}{l}\left(v(t) + \lambda \int_a^b G(t,z) f(z,u_1(z))dz\right) \\ -\left(v(t) + \lambda \int_a^b G(t,z) f(z,u_2(z))dz\right)\end{array}\right| \\
&= \max_{t\in I}\left|\lambda \int_a^b G(t,z)\left(f(z,u_1(z)) - f(z,u_2(z))\right)dz\right| \\
&\leq \lambda \frac{1}{b-a}\int_a^b \left|f(z,u_1(z)) - f(z,u_2(z))\right| dz \\
&\leq \lambda \frac{1}{b-a}\int_a^b \left|u_1(z) - u_2(z)\right| dz \\
&\leq \lambda \max_{t\in I}\left|u_1(z) - u_2(z)\right| \\
&= d(u_1,u_2).
\end{aligned}$$

Obviously, there exist upper and lower bounds of the pair $\{x,y\}$ for any $x, y \in C[a,b]$. Hence, all conditions of Theorem 2.2 are fulfilled. This means that (1) has a unique solution u.

Moreover, for any $y \in X$, the orbit $\{T^n(y)\}$ converges to the solution u.

Next we present an example as follows.

Example 2.1 In the integral equation (1), let $v(t) = t$, $\lambda = \frac{1}{2}$, $G(s,t) \equiv 1$, $a = 0$, $b = 1$, $f(s,t) = s + t$. Then (1) become

$$u(t) = t + \frac{1}{2}\int_0^1 (z + u(z))dz, t \in I \qquad (3)$$

Let $x_0 = t$, then $x_0 \leq T(x_0)$. Now, all conditions of Theorem 2.3 are satisfied. On the other hand, we can easy to solve the integral equation (3) and the unique solution is $u(t) = t + 1$.

3. Coupled Fixed Point Theorems in Partially Ordered Metric Spaces

Now, we endow the product space $X \times X$ with the partial order as the following:

$$(u,v) \leq (x,y) \Leftrightarrow u \leq x, y \leq v, \textit{ for } (u,v),(x,y) \in X \times X.$$

Definition 3.1 [2] Let $(X, \leq)$ be a partially ordered set, $F : X \times X \to X$ be a mapping. If $F(x,y)$ is monotone increasing in x and is monotone decreasing in y, that is, for any $x, y \in X$, if $x_1, x_2 \in X$ and $x_1 \leq x_2 \Rightarrow F(x_1,y) \leq F(x_2,y)$ and if $y_1, y_2 \in X$ and $y_1 \leq y_2$ imply

$$F(x,y_2) \leq F(x,y_1).$$

Thus we say that F has the mixed monotone property.

Definition 3.2 [2] We call an element $(x,y) \in X \times X$ a coupled fixed point of the mapping F, if $F(x,y) = x, F(y,x) = y$.

Theorem 3.1 [2] Let $F : X \times X \to X$ be a continuous mapping satisfy the mixed monotone property on X. Assume that there exists $k \in [0,1)$ with

$$d\left(F(x,y),F(u,v)\right)\le\frac{k}{2}\left[d(x,u)+d(y,v)\right],$$
$$for\,(u,v)\le(x,y).$$

If there exists $x_0,y_0\in X$ such that $x_0\le F(x_0,y_0)$ and $F(y_0,x_0)\le y_0$.

Then, there exist $x,y\in X$ such that $F(x,y)=x$ and $F(y,x)=y$.

Theorem 3.2 [2] In addition to the hypothesis of Theorem 3.1, suppose that ever pair of elements of X has an upper bound or a lower bound in X, then $x=y$.

We assume that T and F are related by the relation $T(x)=F(x,x)$.

Next, we will study the existence of a unique solution to the integral equation (1), as an application to the fixed pointed Theorem 3.2.

Let $X\times X=C(I,R)\times C(I,R)$, then $X\times X$ is a partially ordered set if we define the following order relation in $X\times X$: $(x,y)\le(u,v)\Leftrightarrow x(t)\le u(t)$ and $v(t)\le y(t)$, for (x,y), $(u,v)\in X\times X$ and for all $t\in I$.

Consider the integral equation (1) under the following assumptions:

$(H_1^{'})$ $\lambda>0$;

$(H_2^{'})$ there exists $u>0$, for all $x\in I$, if $u_2(t)\le u_1(t)$, then $0\le f(x,u_1(t))-f(x,u_2(t))\le\mu\left|u_1(t)-u_2(t)\right|$.

Let

$$K_1(t,z)=\frac{G(t,z)+\left|G(t,z)\right|}{2},$$
$$K_2(t,z)=\frac{G(t,z)-\left|G(t,z)\right|}{2},$$

then $G(t,z)=K_1(t,z)+K_2(t,z)$ and $K_1(t,z)\ge 0$, $K_2(t,z)\le 0$.

Define $F:X\times X\to X$ by

$$\begin{aligned}F(x,y)(t)&=v(t)+\lambda\int_a^b k_1(t,z)\,f(z,x(z))dz\\&+\lambda\int_a^b k_2(t,z)\,f(z,y(z))dz,t\in I.\end{aligned}\tag{4}$$

Now, we will show that F has the mixed monotone property. Indeed, for $x_1\le x_2$, that is $x_1(t)\le x_2(t)$, for all $t\in I$, we have

$$\begin{aligned}&F(x_1,y)(t)-F(x_2,y)(t)\\&=v(t)+\lambda\int_a^b k_1(t,z)\,f(z,x_1(z))dz\\&+\lambda\int_a^b k_2(t,z)\,f(z,y(z))dz\\&-\begin{pmatrix}v(t)+\lambda\int_a^b k_1(t,z)\,f(z,x_2(z))dz\\+\lambda\int_a^b k_2(t,z)\,f(z,y(z))dz\end{pmatrix}\\&=\lambda\int_a^b k_1(t,z)\left(f(z,x_1(z))-f(z,x_2(z))\right)dz\le 0.\end{aligned}$$

Hence, $F(x_1,y)(t)\le F(x_2,y)(t)$ for $\forall t\in I$, that is, $F(x_1,y)(t)\le F(x_2,y)(t)$ Similarly, if $y_2\le y_1$, that is $y_2(t)\le y_1(t)$, for all $t\in I$, we have

$$\begin{aligned}&F(x,y_1)(t)-F(x,y_2)(t)\\&=v(t)+\lambda\int_a^b k_1(t,z)\,f(z,x(z))dz\\&+\lambda\int_a^b k_2(t,z)\,f(z,y_1(z))dz\\&-\begin{pmatrix}v(t)+\lambda\int_a^b k_1(t,z)\,f(z,x(z))dz\\+\lambda\int_a^b k_2(t,z)\,f(z,y_2(z))dz\end{pmatrix}\\&=\lambda\int_a^b k_2(t,z)\left(f(z,y_1(z))-f(z,y_2(z))\right)dz\le 0.\end{aligned}$$

Hence, $F(x,y_1)(t)\le F(x,y_2)(t)$ for $t\in I$, that is, $F(x,y_1)(t)\le F(x,y_2)(t)$.

Thus $F(x,y)$ is monotone increasing in x and is monotone decreasing in y.

Now, for $(u',v')\le(x,y)$, that is, $x(t)\ge u'(t)$, $y(t)\le v'(t)$ for all $t\in I$, we have

$$\begin{aligned}&d\left(F(x,y),F(u',v')\right)=\max_{t\in I}\left|F(x,y)(t),F(u',v')(t)\right|\\&=\max_{t\in I}\left|\begin{matrix}v(t)+\lambda\int_a^b k_1(t,z)\,f(z,x(z))dz\\+\lambda\int_a^b k_2(t,z)\,f(z,y(z))dz\end{matrix}\right.\\&\left.-\begin{pmatrix}v(t)+\lambda\int_a^b k_1(t,z)\,f(z,u'(z))dz\\+\lambda\int_a^b k_2(t,z)\,f(z,v'(z))dz\end{pmatrix}\right.\\&=\max_{t\in I}\left|\begin{matrix}\lambda\int_a^b k_1(t,z)\left(f(z,x(z))-f(z,u'(z))\right)dz\\+\lambda\int_a^b k_2(t,z)\left(f(z,y(z))-f(z,v'(z))\right)dz\end{matrix}\right|\\&\le\max_{t\in I}\left|\begin{matrix}\lambda\int_a^b k_1(t,z)\mu\left|x(z)-u'(z)\right|dz\\+\lambda\int_a^b k_2(t,z)\mu\left|y(z)-v'(z)\right|dz\end{matrix}\right|\\&\le\max_{t\in I}\left|\begin{matrix}\lambda\int_a^b\left|G(t,z)\right|\mu\left|x(z)-u'(z)\right|dz\\+\lambda\int_a^b\left|G(t,z)\right|\mu\left|y(z)-v'(z)\right|dz\end{matrix}\right|\\&\le\max_{t\in I}\lambda\mu\int_a^b\left|G(t,z)\right|dz\begin{pmatrix}\max_{t\in I}\left|x(z)-u'(z)\right|\\+\max_{t\in I}\left|y(z)-v'(z)\right|\end{pmatrix}\\&=\max_{t\in I}\lambda\mu\int_a^b\left|G(t,z)\right|dz\left(d(x,u')+d(v',y)\right).\end{aligned}$$

Assume that

$(H_3^{'})$ $\max_{t\in I}\lambda\mu\int_a^b\left|G(t,z)\right|dz<\frac{1}{2}$

(H_4') there exists $x_0, y_0 \in X$ such that $x_0 \le F(x_0, y_0)$ and $F(y_0, x_0) \le y_0$.

Theorem 3.3 Suppose the integral equation (1) satisfy $(H_1')-(H_4')$, then (1) has a unique solution u.

Proof. From the above analysis and Theorem 3.2, we can immediately obtain the result.

Example 3.1 In integral equation (1), let

$$\begin{aligned} &v(t)=\frac{t^3}{8}+\frac{t^2}{4}-\frac{15t}{4}+\frac{19}{2}, \lambda=\frac{3}{4}, \\ &G(t,z)=t-z, a=1, b=2, \\ &f(z,u)=z+\frac{zu}{4}, z\in[1,2], \end{aligned}$$

then (1) become

$$\begin{aligned} u(t)&=\frac{t^3}{8}+\frac{t^2}{4}-\frac{15t}{4}+\frac{19}{2} \\ &+\frac{3}{4}\int_1^2 (t-z)(z+zu(z))dz, t\in[1,2]. \end{aligned} \tag{5}$$

Let $\mu=\frac{1}{2}$, then $f(z,u)$ satisfy (H_2').

$$k_1(t,z)=\begin{cases} t-z, t\ge z \\ 0, \ t<z \end{cases}, k_2(t,z)=\begin{cases} 0, \ t\ge z \\ t-z, t<z \end{cases}$$

$$\begin{aligned} f(x,y)(t)&=\frac{t^3}{8}+\frac{t^2}{4}-\frac{15t}{4}+\frac{19}{2} \\ &+\frac{3}{4}\int_1^2 k_1(t,z) f(z,x(z))dz \\ &+\frac{3}{4}\int_1^2 k_2(t,z) f(z,y(z))dz \\ &=\frac{t^3}{8}+\frac{t^2}{4}-\frac{15t}{4}+\frac{19}{2}+\frac{3}{4}\int_1^t (t-z)\left(z+\frac{zx}{4}\right)dz \\ &+\frac{3}{4}\int_t^2 (t-z)\left(z+\frac{zy}{4}\right)dz. \end{aligned}$$

Then

$$\max_{t\in I} \lambda\mu\int_a^b |G(t,z)|dz \le \frac{3}{16}<\frac{1}{2}.$$

Let $x_0=4, y_0=8$, then $x_0 \le F(x_0,y_0)=\frac{t^2}{4}+4$ and $F(x_0,y_0)=\frac{t^3}{4}+\frac{t^2}{4}-\frac{15t}{8}+\frac{25}{4}\le y_0$.

So, all conditions of Theorem 3.3 are fulfilled. This means that (5) has a unique solution u.

Next, we will study the existence of a unique solution to the following system of integral equation as another application of the fixed pointed Theorem 3.2.

$$\begin{cases} x(t)=g(t)+\int_0^T G(s,t) f(s,x(s),y(s))ds \\ y(t)=g(t)+\int_0^T G(s,t) f(s,y(s),x(s))ds, \end{cases} \tag{6}$$

where $t\in[0,T]$.

A solution of the above system is a pair $(x,y)\in[0,T]\times[0,T]$ satisfying the above relations for all $t\in[0,T]$.

We consider $X\in[0,T]$ endowed with the partial order relation:

$$x\le y \Leftrightarrow x(t)\le y(t) \ \textit{for all } t\in[0,T].$$

We will also consider the following metric on X:

$$d(x,y):=\max_{t\in[0,T]}|x(t)-y(t)|.$$

Notice that d is a metric and d can be represented by using the supermum type norm

$$d(x,y):=\|x(t)-y(t)\|_c.$$

Then we have the following existence and uniqueness result.

Theorem 3.4 Consider the integral system (6) under the following assumptions:

(1) $g:[0,T]\to R$ and $f:[0,T]\times R^2\to R$ are continuous and $G:[0,T]\times[0,T]\to R^+$ is integrable with respect to the first variable.

(2) $f(s,\cdot,\cdot)$ has the generalized mixed monotone property with respect to the last two variables for all $s\in[0,T]$.

(3) There exist $\alpha,\beta:[0,T]\to R^+$ in $L^1[0,T]$ such that for each $x_1,x_2,y_1,y_2\in R$ with $x_1\le y_1$ and $y_2\le x_2$ (or reversely), we have

$$|f(s,x_1,x_2)-f(s,y_1,y_2)|\le\alpha(s)|x_1-y_1|+\beta(s)|x_2-y_2|$$

for each $s\in[0,T]$.

(4) $\max_{t\in[0,T]}\left(\int_0^T G(s,t)\alpha(s)\right)<\frac{1}{2}$,

$$\max_{t\in[0,T]}\left(\int_0^T G(s,t)\beta(s)\right)<\frac{1}{2}.$$

(5) There exist $x_0, y_0\in[0,T]$ such that

$$\begin{cases} x(t)\le g(t)+\int_0^T G(s,t) f(s,x(s),y(s))ds \\ y(t)\ge g(t)+\int_0^T G(s,t) f(s,y(s),x(s))ds \end{cases}$$

or

$$\begin{cases} x(t)\ge g(t)+\int_0^T G(s,t) f(s,x(s),y(s))ds \\ y(t)\le g(t)+\int_0^T G(s,t) f(s,y(s),x(s))ds \end{cases}$$

for all $t\in[0,T]$.

Then there exists a unique solution (x^*,y^*) of the system (6).

Proof. We can prove that all the assumptions of Theorem 3.2 are satisfied. We define $F: X \times X \to X$ by

$$F(x,y)(t) = g(t) + \int_0^T G(s,t)\, f(s,x(s),y(s))\,ds$$

for each $t \in [0,T]$.

Then system (6) can be written as a couple fixed point problem for F:

$$\begin{cases} x = F(x,y) \\ y = F(y,x) \end{cases}$$

First, we will show that F has the mixed monotone property. Indeed, for $x_1 \le x_2$, that is $x_1(t) \le x_2(t)$, for all $t \in [0,T]$, we have

$$\begin{aligned} &F(x_1,y)(t) - F(x_2,y)(t) \\ &= \int_0^T G(s,t)\begin{bmatrix} f(s,x_1(s),y(s)) \\ -f(s,x_2(s),y(s)) \end{bmatrix} ds \le 0. \end{aligned}$$

Hence, $F(x_1,y)(t) \le F(x_2,y)(t)$ for all $t \in [0,T]$, that is, $F(x_1,y)(t) \le F(x_2,y)(t)$.

Similarly, if $y_2 \ge y_1$, that is $y_2(t) \ge y_1(t)$, for all $t \in [0,T]$, we have $F(x,y_1)(t) \ge F(x,y_2)(t)$ for all $t \in [0,T]$, that is, $F(x,y_1)(t) \ge F(x,y_2)(t)$.

Then, for all $x \ge u$ and $y \le v$ or ($x \le u$ and $y \ge v$), we have

$$\begin{aligned} &d(F(x,y)(t), F(u,v)(t)) \\ &= \max_{t\in[0,T]} \left| F(x,y)(t) - F(u,v)(t) \right| \\ &\le \int_0^T G(s,t) \begin{vmatrix} f(s,x(s),y(s)) \\ -f(s,u(s),v(s)) \end{vmatrix} ds \\ &\le \int_0^T G(s,t) \begin{pmatrix} \alpha(s)\left|x(s)-u(s)\right| \\ +\beta(s)\left|y(s)-v(s)\right| \end{pmatrix} ds \\ &= \|x-u\| \int_0^T G(s,t)\alpha(s)\,ds \\ &+ \|y-v\| \int_0^T G(s,t)\beta(s)\,ds \\ &< \frac{1}{2}\left(d(x,y) + d(u,v)\right). \end{aligned}$$

We see that all the assumptions of Theorem 3.2 are satisfied and the conclusion follows.

Next, we conclude our work by an example.

Example 3.2 In integral equations system (6), let $G(s,t) = \frac{1}{6}s + \frac{t}{3}, g(t) = t + \frac{11}{9}$, $T = 1, \alpha(s) = s, \beta(s) = s$, $f(s,x(s),y(s)) = \alpha(s)x(s) - \beta(s)y(s) = sx(s) - sy(s)$.

Then (6) become

$$\begin{cases} x(t) = t + \frac{11}{9} + \int_0^1 (\frac{1}{6}s + \frac{t}{3})(sx(s) - sy(s))\,ds \\ y(t) = t + \frac{11}{9} + \int_0^1 (\frac{1}{6}s + \frac{t}{3})(sy(s) - sx(s))\,ds \end{cases} \quad (7)$$

$$F(x,y)(t) = t + \frac{11}{9} + \int_0^1 \left(\frac{1}{6}s + \frac{t}{3}\right)(sx(s) - sy(s))\,ds.$$

Then

$$\max_{t\in[0,T]} \left(\int_0^T G(s,t)\alpha(s)\right) = \max_{t\in[0,T]} \left(\int_0^T G(s,t)\beta(s)\right) = \frac{2}{9} < \frac{1}{2}.$$

Let $x_0 = 1, y_0 = 3$, then $x_0 \le F(x_0,y_0) = \frac{2}{3}t + \frac{10}{9}$ and $y_0 \le F(y_0,x_0) = \frac{4}{3}t + \frac{4}{3}$.

Hence, all conditions of Theorem 3.4 are fulfilled. This means that (7) has a unique solution.

$$\begin{cases} x^* = F(x^*, y^*), \\ y^* = F(y^*, x^*), \end{cases} (x^*, y^*) \in C[0,1] \times C[0,1].$$

4. Coupled Fixed Point Theorems in *b*-metric Spaces

Theorem 4.1 [14] Let (X,d) be a complete b-metric spaces with $s \ge 1$ and $T: X \times X \to X$ be a continuous mapping with the mixed monotone property on $X \times X$. Suppose that the following conditions are satisfied:

(1) there exists $k \in \left[0, \frac{1}{s}\right)$ such that

$$\begin{aligned} &d(T(x,y), T(u,v)) \le \frac{k}{2}\left[d(x,u) + d(y,v)\right], \\ &\forall x \ge u, y \le v; \end{aligned}$$

(2) there exists $x_0, y_0 \in X$ such that $x_0 \le T(x_0,y_0)$ and $y_0 \ge T(y_0,x_0)$.

Then there exists $x, y \in X$ such that $x = T(x,y)$ and $y = T(y,x)$.

In this section, we present an existence Theorem for such a nonlinear coupled system

$$\begin{cases} x(t) = \varphi(t) + \int_a^b K(t,r,x(r),y(r))\,dr \\ y(t) = \varphi(t) + \int_a^b K(t,r,y(r),x(r))\,dr, \end{cases} \quad (8)$$

where $a, b \in R$ with $a < b$, $x, y \in C[a,b]$, $\varphi: [a,b] \to R$ and $K: [a,b] \times [a,b] \times R \times R \to R$ are given mapping.

Next, we consider the following b-metric on X

$$d(x,y) = \max_{t\in[a,b]} \left| x(t) - y(t) \right|^p.$$

It is note that (X,d) is a complete b-metric space with $p \ge 1$.

Theorem 4.2 Consider the nonlinear coupled system (1). Suppose that the following conditions hold:

(1) $K: [a,b] \times [a,b] \times R \times R \to R$ is continuous;

(2) $K(t,r,\cdot,\cdot)$ has the generalized mixed monotone property with respect to the last two variables for all $t \in [a,b]$;

(3) There exist continuous mappings

$$\alpha,\beta:[a,b]\times[a,b]\to R^+,$$

for each x_1, x_2, y_1, y_2 $\in R$ with $x_1 \le y_1$ and $y_2 \le x_2$ (or reversely), we have

$$\begin{aligned}&\left|K(t,r,x_1(r),x_2(r))-K(t,r,y_1(r),y_2(r))\right|^p\\&\le 2^{p-1}\left[\alpha(t,r)^p|x_1-y_1|^p+\beta(t,r)^p|x_2-y_2|^p\right];\end{aligned}$$

(4) $\max\limits_{t\in[a,b]} 2^{p-1}(b-a)^{\frac{p}{q}}\int_a^b \alpha(t,r)^p dr < \frac{1}{2s}$; $\max\limits_{t\in[a,b]} 2^{p-1}(b-a)^{\frac{p}{q}}\int_a^b \beta(t,r)^p dr < \frac{1}{2s}$, where $s = 2^{p-1}$.

(5) There exists x_o, $y_0 \in C[a,b]$ such that

$$\begin{cases}x_0(t)\le\varphi(t)+\int_a^b K(t,r,x(r),y(r))dr\\ y_0(t)\ge\varphi(t)+\int_a^b K(t,r,y(r),x(r))dr\end{cases}$$

for all $t \in [a,b]$.

Then, there exists a pair coupled solution (x,y) for system (8).

Proof. We can prove that all the assumptions of Theorem 4.1 are satisfied. Define $T: X\times X\to X$ by

$$T(x,y)(t)=\varphi(t)+\int_a^b K(t,r,x(r),y(r))dr,$$

for each $t\in[a,b]$. Then system (8) can be regarded as a couple fixed point question of T:

$$\begin{cases}x=T(x,y)\\ y=T(y,x).\end{cases}$$

In the first place, we will prove that T has the mixed monotone property. For $x_1 \le x_2$, we have

$$\begin{aligned}&T(x_1,y)-T(x_2,y)\\&=\int_a^b\begin{bmatrix}K(t,r,x_1(r),y(r))\\-K(t,r,x_2(r),y(r))\end{bmatrix}dr\le 0.\end{aligned}$$

Thus, $T(x_1,y)\le T(x_2,y)$ for every element $t\in[a,b]$. Similarly, we can know that

$$T(x,y_1)\ge T(x,y_2)\ \textit{for all}\ y_1\le y_2.$$

Then, for all $x\ge u$ and $y \le v$ or ($x\le u$ and $y\ge v$), we have

$$d(T(x,y),T(u,v))=\max_{t\in[a,b]}\begin{vmatrix}\int_a^b K(t,r,x(r),y(r))dr\\-\int_a^b K(t,r,u(r),v(r))dr\end{vmatrix}^p$$

$$\le\max_{t\in[a,b]}\left[\left(\int_a^b 1^q dr\right)^{\frac{1}{q}}\left(\int_a^b\begin{vmatrix}K(t,r,x(r),y(r))\\-K(t,r,u(r),v(r))\end{vmatrix}^p dr\right)^{\frac{1}{p}}\right]^p$$

$$=\max_{t\in[a,b]}(b-a)^{\frac{p}{q}}\left(\int_a^b\begin{vmatrix}K(t,r,x(r),y(r))\\-K(t,r,u(r),v(r))\end{vmatrix}^p dr\right)$$

$$\le\max_{t\in[a,b]}2^{p-1}(b-a)^{\frac{p}{q}}\left(\int_a^b\begin{bmatrix}\alpha(t,r)^p|x-u|^p\\+\beta(t,r)^p|y-v|^p\end{bmatrix}dr\right)$$

$$\le\max_{t\in[a,b]}2^{p-1}(b-a)^{\frac{p}{q}}\left(\begin{array}{l}\int_a^b\alpha(t,r)^p\,dr\,d(x,u)\\+\int_a^b\beta(t,r)^p\,dr\,d(y,v)\end{array}\right)$$

$$=k_1d(x,u)+k_2d(y,v)$$

$$<\frac{1}{2s}\left[d(x,u)+d(y,v)\right],$$

where

$$k_1=\max_{t\in[a,b]}2^{p-1}(b-a)^{\frac{p}{q}}\int_a^b\alpha(t,r)^p dr<\frac{1}{2s},$$

$$k_2=\max_{t\in[a,b]}2^{p-1}(b-a)^{\frac{p}{q}}\int_a^b\beta(t,r)^p dr<\frac{1}{2s},$$

$$\frac{1}{p}+\frac{1}{q}=1.$$

From the above proof, we find that all the assumptions of Theorem 4.1 are satisfied.

Example 4.1 For the integral equation (8), let $a=0, b=1, \varphi(t)=-t$, $\alpha(t,r)=\beta(t,r)=\frac{tr}{4}$,

$$K(t,r,x(r),y(r))=t+r+\alpha(t,r)\left[x(r)-y(r)\right].$$

Then (8) become

$$\begin{cases}x(t)=-t+\int_0^1\left[t+r+\frac{tr}{4}\left(x(r)-y(r)\right)\right]dr\\ y(t)=-t+\int_0^1\left[t+r+\frac{tr}{4}\left(y(r)-x(r)\right)\right]dr\end{cases},$$

$t\in[0,1]$.

Define $T(x,y)=-t+\int_0^1\left[t+r+\frac{tr}{4}\left(x(r)-y(r)\right)\right]dr.$

Obviously, $\max\limits_{t\in[0,1]}2^{p-1}\int_0^1\left[\alpha(t,r)\right]^p dr<\frac{1}{2s}$, Let $x_0=0$, $y_0=1$, we have $x_0\le T(x_0,y_0)=\frac{1}{2}-\frac{t}{8}$, and $y_0\ge T(y_0,x_0)=\frac{1}{2}+\frac{t}{8}$. Then, all the conditions of Theorem 4.1 are satisfied. It indicates that (8) has a pair coupled solution

$$\begin{cases} x^* = T\left(x^*, y^*\right), \\ y^* = T\left(y^*, x^*\right), \end{cases} \left(x^*, y^*\right) \in C[0,1] \times C[0,1].$$

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