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Properties of Doubly-Truncated Fréchet Distribution

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Abstract The truncated distributions has been widely studied, primarily in life-testing and reliability analysis. Most work has assumed an upper bound on the support of the random variable, *i.e.* the space of the distribution is (0, *d*). We consider a doubly-truncated Fréchet random variable restricted by both a lower (*c*) and upper (*d*) truncation point. We provide forms for the density, cumulative distribution function (CDF), hazard function, characteristic function, *r*th raw moment, mean, mode, median, variance, skewness, kurtosis, Shannon entropy function, relative entropy and quantile function. We also consider the generating issues. This paper deals also with the determination of R = P[Y < X] when X and Y are two independent doubly truncated Fréchet distributions (**DTFD**) with different scale parameters, different shape parameters but the same truncations parameters. Different methods to estimate doubly truncated Fréchet distribution parameters are studied, Maximum Likelihood estimator, Moments estimator, Percentile estimator, least square estimator and weighted least square estimator. An empirical study is conducted to compare among these methods.

Keywords: *doubly truncated Fréchet distribution, Percentile estimator, hazard function, characteristic function, Shannon entropy, relative entropy, P[Y < X]*

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1. Introduction

Fréchet distribution was introduced by a French mathematician named Maurice Fréchet (1878-1973) who had identified before one possible limit distribution for the largest order statistic in 1927. It is worth mentioning the link between Pareto and Fréchet: The limiting distribution of the maximum of independent random variables having Pareto distribution is Fréchet which suggests that when all firms draw from Pareto the distribution of the best can be represented as Fréchet.

The Fréchet distribution has been shown to be useful for modeling and analysis of several extreme events ranging from accelerated life testing to earthquakes, floods, rain fall, sea currents and wind speeds. Applications of the Fréchet distribution in various fields given in Harlow [3] showed that it is an important distribution for modeling the statistical behavior of materials properties for a variety of engineering applications. Morales [6] used an upper truncated Fréchet distribution for Estimation of Max-Stable Processes Using Monte Carlo Methods and then applied the theoretical and empirical aspects in Financial Risk Assessment. Nadarajah and Kotz [8] discussed the sociological models based on Fréchet random variables. Further, Zaharim et al. [11] applied Fréchet distribution for analyzing the wind speed data. Mubarak [7] studied the Fréchet progressive type-II censored data with binomial removals. The Fréchet distribution is a special case of the generalized extreme value distribution. This type-II extreme value distribution (Fréchet) case is equivalent to taking the reciprocal of values from a standard Weibull distribution. Pehlivan [9] looked at the German market of manufacturing imports for 1990. He considered two extreme cases, namely Belgium and US, which are the countries with the worst and best productivity distributions, respectively. He found that the truncated Fréchet distribution with $\hat{d}_{BE} = 0.4036$ and $\hat{d}_{US} = 0.2864$ does not fit very well the problem under his consideration. Abid and Hassan [2] derived the failure rate model of Marshall-Olkin Extended Uniform distribution and upper truncated Fréchet (a, b, d).

The probability density function (PDF) and the cumulative distribution function (CDF) for the **Fréchet** random variable X are respectively,

$$f(x) = abx^{-(b+1)}e^{-ax^{-b}}, 0 < x < \infty. \tag{1}$$

$$F(x) = e^{-ax^{-b}} \tag{2}$$

Where $a > 0$ is the scale parameter and $b > 0$ is the shape parameter.

The rth raw moment of X is,

$$E\left(X^r\right) = a^{r/b}\Gamma\left(\frac{b-r}{b}\right). \tag{3}$$

After depth search in the scientific literature, we found that there is no Reliable Studies related with doubly truncated Fréchet distribution (DTFD) although the importance of this distribution, so it is considered here.

In this paper we will refer to Fréchetdistribution by $F \sim Fr(a,b)$, which is mean that the random variable F follow **Fréchet distribution** with parameters a and b.

2. Properties of DTFD

We now consider $DTFr(a,b,c,d)$ to be a doubly-truncated version of X with lower truncation point, c, and upper truncation point, d. Obviously, the probability density function (PDF) and the cumulative distribution function (CDF) for the doubly-truncated **Fréchet** random variable X are respectively,

$$g(x)=\frac{a\,b\;x^{-(b+1)}\,e^{-a\,x^{-b}}}{e^{-a\,d^{-b}}-e^{-a\,c^{-b}}},\; c<x<d \tag{4}$$

$$G(x)=\frac{a\,b}{k}\int_c^x x^{-(b+1)}\,e^{-a\,x^{-b}}\,dx =\frac{e^{-a\,x^{-b}}-e^{-a\,c^{-b}}}{k}, \tag{5}$$

$$where\;\; k=e^{-a\,d^{-b}}-e^{-a\,c^{-b}}.$$

Also, the reliability function and hazard rate function of X are respectively,

$$R(x)=\frac{e^{-a\,d^{-b}}-e^{-a\,x^{-b}}}{k} \tag{6}$$

$$\lambda(x)=\frac{a\,b\;x^{-(b+1)}\,e^{-a\,x^{-b}}}{e^{-a\,d^{-b}}-e^{-a\,x^{-b}}}=\frac{a\,b\;x^{-(b+1)}}{e^{-a\left(d^{-b}-x^{-b}\right)}-1}. \tag{7}$$

The rth raw moment of X can be obtained as,

$$\begin{aligned}E\left(X^r\right)&=\frac{1}{k}\int_c^d x^r\,g(x)\,dx\\ &=\frac{1}{k}\int_c^d x^r\,a\,b\;x^{-(b+1)}\,e^{-a\,x^{-b}}\,dx\\ &=\frac{1}{k}\int_c^d a\,b\;x^{r-(b+1)}\,e^{-a\,x^{-b}}\,dx.\end{aligned} \tag{8}$$

Now, Let

$$y=a\,x^{-b}\Rightarrow x=\left(y/a\right)^{-1/b}\Rightarrow dx=\frac{-1}{a\,b}\left(y/a\right)^{-1/b-1}dy,$$

so,

$$\begin{aligned}E\left(X^r\right)&=\frac{a^{r/b}}{k}\int_{a\,d^{-b}}^{a\,c^{-b}} y^{-r/b}\,e^{-y}\,dy\\ &=\frac{a^{r/b}}{k}\left\{\Gamma\left(1-\frac{r}{b},\,a\,d^{-b}\right)-\Gamma\left(1-\frac{r}{b},\,a\,c^{-b}\right)\right\}\end{aligned} \tag{8}$$

Where $\Gamma(u,v)=\int_v^\infty t^{u-1}e^{-t}\,dt$, is the upper incomplete gamma function.

Then, the mean and variance of $DTFr(a,b,c,d)$ random variable X are respectively,

$$\mu=\frac{a^{1/b}}{k}\left\{\Gamma\left(1-\frac{1}{b},\,a\,d^{-b}\right)-\Gamma\left(1-\frac{1}{b},\,a\,c^{-b}\right)\right\} \tag{9}$$

$$\sigma^2=\frac{a^{2/b}}{k}\left\{\Gamma\left(1-\frac{2}{b},\,a\,d^{-b}\right)-\Gamma\left(1-\frac{2}{b},\,a\,c^{-b}\right)\right\} -\frac{a^{2/b}}{k^2}\left\{\Gamma\left(1-\frac{1}{b},\,a\,d^{-b}\right)-\Gamma\left(1-\frac{1}{b},\,a\,c^{-b}\right)\right\}^2.$$

The mode of X is obtained as follows,

$$g'(x)=\frac{ab}{k}\left\{\begin{array}{l} ab\;x^{-(b+1)}\,e^{-a\,x^{-b}}\,x^{-(b+1)}\\ +e^{-a\,x^{-b}}\left(-(b+1)\right)x^{-(b+2)}\end{array}\right\}=0$$

$$\Rightarrow ab\,x^{-2(b+1)}-(b+1)\,x^{-(b+2)}=0 \tag{11}$$

$$\Rightarrow x=Mo=\left(\frac{b+1}{ab}\right)^{-1/b}.$$

Also, the median (M_e) of X is obtained as follows,

$$G(x)=1/2\Rightarrow e^{-a\,x^{-b}}=e^{-a\,c^{-b}}+\frac{1}{2}\left(e^{-a\,d^{-b}}-e^{-a\,c^{-b}}\right),$$

so

$$x=M_e=a^{1/b}\left(Ln(2)-Ln\left(e^{-a\,d^{-b}}+e^{-a\,c^{-b}}\right)\right)^{-1/b}. \tag{12}$$

The Pearson mode skewness $sk=\left(\mu-Mo\right)/\sigma$ is,

$$sk=\frac{\left(a^{1/b}/k\right)\left\{\begin{array}{l}\Gamma\left(1-\frac{1}{b},\,a\,d^{-b}\right)\\ -\Gamma\left(1-\frac{1}{b},\,a\,c^{-b}\right)\end{array}\right\}-\left(ab/(b+1)\right)^{1/b}}{\left(a^{1/b}/k\right)\left[\begin{array}{l}k\left\{\begin{array}{l}\Gamma\left(1-\frac{2}{b},\,a\,d^{-b}\right)\\ -\Gamma\left(1-\frac{2}{b},\,a\,c^{-b}\right)\end{array}\right\}\\ -\left\{\begin{array}{l}\Gamma\left(1-\frac{1}{b},\,a\,d^{-b}\right)\\ -\Gamma\left(1-\frac{1}{b},\,a\,c^{-b}\right)\end{array}\right\}^2\end{array}\right]^{1/2}}$$

$$=\frac{\left\{\begin{array}{l}\Gamma\left(1-\frac{1}{b},\,a\,d^{-b}\right)\\ -\Gamma\left(1-\frac{1}{b},\,a\,c^{-b}\right)\end{array}\right\}-k\left(b/(b+1)\right)^{1/b}}{\left[\begin{array}{l}k\left\{\Gamma\left(1-\frac{2}{b},\,a\,d^{-b}\right)-\Gamma\left(1-\frac{2}{b},\,a\,c^{-b}\right)\right\}\\ -\left\{\Gamma\left(1-\frac{1}{b},\,a\,d^{-b}\right)-\Gamma\left(1-\frac{1}{b},\,a\,c^{-b}\right)\right\}^2\end{array}\right]^{1/2}} \tag{13}$$

The excess kurtosis $ku=\mu_4/\sigma^4-3$, where $\mu_r=E(X-\mu)^r$, is

$$ku=\frac{\left\{\begin{array}{l}\left(a^{4/b}/k\right)\left\{\Gamma\left(1-\frac{4}{b},\,a\,d^{-b}\right)-\Gamma\left(1-\frac{4}{b},\,a\,c^{-b}\right)\right\}\\ -4\left(a^{1/b}/k\right)\left\{\begin{array}{l}\Gamma\left(1-\frac{1}{b},\,a\,d^{-b}\right)\\ -\Gamma\left(1-\frac{1}{b},\,a\,c^{-b}\right)\end{array}\right\}\\ \left(a^{3/b}/k\right)\left\{\Gamma\left(1-\frac{3}{b},\,a\,d^{-b}\right)-\Gamma\left(1-\frac{3}{b},\,a\,c^{-b}\right)\right\}\\ +6\left(\left(a^{1/b}/k\right)\left\{\begin{array}{l}\Gamma\left(1-\frac{1}{b},\,a\,d^{-b}\right)\\ -\Gamma\left(1-\frac{1}{b},\,a\,c^{-b}\right)\end{array}\right\}\right)^{2}\\ \left(a^{2/b}/k\right)\left\{\Gamma\left(1-\frac{2}{b},\,a\,d^{-b}\right)-\Gamma\left(1-\frac{2}{b},\,a\,c^{-b}\right)\right\}\\ -3\left(\left(a^{1/b}/k\right)\left\{\begin{array}{l}\Gamma\left(1-\frac{1}{b},\,a\,d^{-b}\right)\\ -\Gamma\left(1-\frac{1}{b},\,a\,c^{-b}\right)\end{array}\right\}\right)^{4}\end{array}\right\}}{\left(\begin{array}{l}\left(a^{2/b}/k\right)\left\{\begin{array}{l}\Gamma\left(1-\frac{2}{b},\,a\,d^{-b}\right)\\ -\Gamma\left(1-\frac{2}{b},\,a\,c^{-b}\right)\end{array}\right\}\\ -\left\{\left(a^{1/b}/k\right)\left\{\begin{array}{l}\Gamma\left(1-\frac{1}{b},\,a\,d^{-b}\right)\\ -\Gamma\left(1-\frac{1}{b},\,a\,c^{-b}\right)\end{array}\right\}\right\}^{2}\end{array}\right)^{2}}-3$$

$$=\frac{\left\{\begin{array}{l}k^{3}\left\{\Gamma\left(1-\frac{4}{b},\,a\,d^{-b}\right)-\Gamma\left(1-\frac{4}{b},\,a\,c^{-b}\right)\right\}\\ -4k^{2}\left\{\Gamma\left(1-\frac{1}{b},\,a\,d^{-b}\right)-\Gamma\left(1-\frac{1}{b},\,a\,c^{-b}\right)\right\}.\\ \left\{\Gamma\left(1-\frac{3}{b},\,a\,d^{-b}\right)-\Gamma\left(1-\frac{3}{b},\,a\,c^{-b}\right)\right\}\\ +6\,k\left(\left\{\Gamma\left(1-\frac{1}{b},\,a\,d^{-b}\right)-\Gamma\left(1-\frac{1}{b},\,a\,c^{-b}\right)\right\}\right)^{2}\\ \left\{\Gamma\left(1-\frac{2}{b},\,a\,d^{-b}\right)-\Gamma\left(1-\frac{2}{b},\,a\,c^{-b}\right)\right\}\\ -3\left(\left\{\Gamma\left(1-\frac{1}{b},\,a\,d^{-b}\right)-\Gamma\left(1-\frac{1}{b},\,a\,c^{-b}\right)\right\}\right)^{4}\end{array}\right\}}{\left(\begin{array}{l}k\left\{\Gamma\left(1-\frac{2}{b},\,a\,d^{-b}\right)-\Gamma\left(1-\frac{2}{b},\,a\,c^{-b}\right)\right\}\\ -\left\{\Gamma\left(1-\frac{1}{b},\,a\,d^{-b}\right)-\Gamma\left(1-\frac{1}{b},\,a\,c^{-b}\right)\right\}^{2}\end{array}\right)^{2}}-3 \quad (14)$$

To obtain the quantile function, $G(x)=p \Rightarrow e^{-a\,x^{-b}}=e^{-a\,c^{-b}}+p\,k$, then,

$$x_p=G^{-1}(p)=\left(\frac{-a}{Ln\left(e^{-a\,c^{-b}}+p\,k\right)}\right)^{1/b}. \qquad (15)$$

So by using the inverse transform method, one can generate the random variable X as follows,

$$x=\left(\frac{-1}{a}Ln\left(e^{-a\,c^{-b}}+U\,k\right)\right)^{-1/b} \qquad (16)$$

Where U is a uniformly distributed random variable in the interval [0,1].

The characteristic function is,

$$\begin{aligned}\Psi_X(t)=E\left(e^{itX}\right)&=\frac{1}{k}\int_c^d e^{itX}\,a\,b\;x^{-(b+1)}\,e^{-a\,x^{-b}}\,dx\\ &=\frac{ab}{k}\int_c^d\sum_{m=0}^{\infty}\frac{(itX)^m}{m!}\;x^{-(b+1)}\,e^{-a\,x^{-b}}\,dx\\ &=\frac{1}{k}\sum_{m=0}^{\infty}\frac{(it)^m}{m!}\int_c^d a\,b\;x^{m-(b+1)}\,e^{-a\,x^{-b}}\,dx\end{aligned}$$

Now, Let

$$y=a\,x^{-b}\Rightarrow x=(y/a)^{-1/b}\Rightarrow dx=\frac{-1}{a\,b}(y/a)^{-1/b-1}\,dy,$$

so,

$$\begin{aligned}&\Psi_X(t)\\ &=\frac{1}{k}\sum_{m=0}^{\infty}\frac{(it)^m}{m!}\int_{a\,d^{-b}}^{a\,c^{-b}}(y/a)^{-m+1+1/b}\,e^{-y}\,(y/a)^{-1-1/b}\,dy\\ &=\frac{1}{k}\sum_{m=0}^{\infty}\frac{(ita)^m}{m!}\int_{a\,d^{-b}}^{a\,c^{-b}}y^{-m}\,e^{-y}\,dy\\ &=\frac{1}{k}\sum_{m=0}^{\infty}\frac{(ita)^m}{m!}\left\{\Gamma\left(1-m,\,a\,d^{-b}\right)-\Gamma\left(1-m,\,a\,c^{-b}\right)\right\} \quad (17)\end{aligned}$$

3. Shannon entropy and Relative Entropy

An entropy of a random variable X is a measure of variation of the uncertainty. The

Shannon entropy of $DTFr(a,b,c,d)$ random variable X can be found as follows,

$$\begin{aligned}H&=E\left(-Ln(g(x))\right)\\ &=\frac{ab}{k}Ln\left(\frac{ab}{k}\right)\int_c^d\left(\begin{array}{l}(b+1)\,Ln(x)\\ +a\,x^{-b}\end{array}\right)x^{-(b+1)}\;e^{-a\,x^{-b}}\;dx\end{aligned}$$

Let us firstly find $I_1=\int_c^d Ln(x)\;x^{-(b+1)}\;e^{-a\,x^{-b}}\;dx$ as follows,

$$\begin{aligned}I_1&=\sum_{m=0}^{\infty}\frac{1}{m!}\int_c^d Ln(x)\;x^{-(b+1)}\left(-a\;x^{-b}\right)^m dx\\ &=\sum_{m=0}^{\infty}\frac{(-a)^m}{m!}\int_c^d Ln(x)\;x^{-(bm+b+1)}\;dx\end{aligned}$$

Now, Let

$$z=(bm+b)\,Ln(x)\Rightarrow x=e^{z/(bm+b)}\Rightarrow dx=\frac{e^{z/(bm+b)}}{bm+b}dz,$$

so,

$$I_1=\sum_{m=0}^{\infty}\frac{(-a)^m}{m!}\int_{c^*}^{d^*}\exp\left\{\begin{matrix}-\frac{bm+b+1}{bm+b}z\\+\frac{1}{bm+b}z\end{matrix}\right\}\cdot\frac{z}{(bm+b)^2}\,dz$$

$$=\sum_{m=0}^{\infty}\frac{(-a)^m}{(bm+b)^2\,m!}\int_{c^*}^{d^*}z\,e^{-z}\,dz$$

$$=\sum_{m=0}^{\infty}\frac{(-a)^m}{(bm+b)^2\,m!}\left\{\Gamma\left(2,c^*\right)-\Gamma\left(2,d^*\right)\right\}$$

Where, $c^*=(bm+b)Ln(c)$ and $d^*=(bm+b)Ln(d)$.

Let us now find $I_2=\int_c^d a\,x^{-b}\,x^{-(b+1)}\,e^{-a\,x^{-b}}\,dx$,

Let

$$y=a\,x^{-b}\Rightarrow x=(y/a)^{-1/b}\Rightarrow dx=\frac{-1}{a\,b}(y/a)^{-1/b-1}\,dy,$$

so,

$$I_2=\frac{1}{ab}\int_{a\,d^{-b}}^{a\,c^{-b}}y\,e^{-y}\,dy$$

$$=\frac{1}{ab}\left\{\Gamma\left(2,a\,d^{-b}\right)-\Gamma\left(2,a\,c^{-b}\right)\right\}.$$

Then the entropy function for $DTFr(a,b,c,d)$ random variable is,

$$H=\frac{ab}{k}Ln\left(\frac{ab}{k}\right)\left\{\begin{matrix}(b+1)\left(\sum_{m=0}^{\infty}\frac{(-a)^m\left(\begin{matrix}\Gamma\left(2,c^*\right)\\-\Gamma\left(2,d^*\right)\end{matrix}\right)}{(bm+b)^2\,m!}\right)\\+\frac{1}{ab}\left\{\Gamma\left(2,a\,d^{-b}\right)-\Gamma\left(2,a\,c^{-b}\right)\right\}\end{matrix}\right\}\quad(18)$$

The relative entropy (or the Kullback–Leibler divergence) is a measure of the difference between two probability distributions G and G^*. It is not symmetric in G and G^*. In applications, G typically represents the "true" distribution of data, observations, or a precisely calculated theoretical distribution, while G^* typically represents a theory, model, description, or approximation of G.Specifically, the Kullback–Leibler divergence of G^* from G, denoted $D_{\mathrm{KL}}(G\,||\,G^*)$, is a measure of the information gained when one revises ones beliefs from the prior probability distribution G^* to the posterior probability distribution G. More exactly, it is the amount of information that is *lost* when G^* is used to approximate G ,defined operationally as the expected extra number of bits required to code samples from G using a code optimized for G^* rather than the code optimized for G.

Now, since, $\frac{g(x)}{g^*(x)}=\frac{k^*\,a\,b\,x^{-(b+1)}\,e^{-a\,x^{-b}}}{k\,\alpha\,\beta\,x^{-(\beta+1)}\,e^{-\alpha\,x^{-\beta}}}$, where $k^*=e^{-\alpha\,d^{-\beta}}-e^{-\alpha\,c^{-\beta}}$, then the relative entropycan be found as follows,

$$D_{\mathrm{KL}}(G\|\,G^*)=\int_{\forall x}g(x)\,Ln\left(\frac{g(x)}{g^*(x)}\right)dx$$

$$=Ln\left(\frac{k^*ab}{k\,\alpha\,\beta}\right)+(\beta-b)\int_c^d Ln(x)\,f(x)\,dx$$

$$-a\int_c^d x^{-b}\,f(x)\,dx+\alpha\int_c^d x^{-\beta}\,f(x)\,dx.$$

By using the same previous arguments of integrations, one can directly write,

$$D_{\mathrm{KL}}(G\|\,G^*)$$

$$=Ln\left(\frac{k^*ab}{k\,\alpha\,\beta}\right)+(\beta-b)\frac{a}{bk}\sum_{m=0}^{\infty}\frac{(-a)^m\left(\begin{matrix}\Gamma\left(2,c^*\right)\\-\Gamma\left(2,d^*\right)\end{matrix}\right)}{(m+1)^2\,m!}\cdot-\frac{1}{k}$$

$$\left\{\Gamma\left(2,a\,d^{-b}\right)-\Gamma\left(2,a\,c^{-b}\right)\right\}$$

$$+\frac{\alpha\,a^{-\beta/b}}{k}\left\{\Gamma\left(\frac{\beta}{b}+1,a\,d^{-b}\right)-\Gamma\left(\frac{\beta}{b}+1,a\,c^{-b}\right)\right\}\quad(19)$$

4. Stress-Strength Reliability

Inferences about R = P[Y < X], where X and Y are two independent random variables, is very common in the reliability literature. For example, if X is the strength of a component which is subject to a stress Y, then R is a measure of system performance and arises in the context of mechanical reliability of a system. The system fails if and only if at any time the applied stress is greater than its strength.

Let Y and X be the stress and the strength random variables, independent of each other, follow respectively $DTFr(a,b,c,d)$ and $DTFr(\alpha,\beta,c,d)$, then,

$$R=P(Y<X)=\int_c^d f_X(x)F_Y(x)\,dx$$

$$=\frac{1}{k_1}\int_c^d a\,b\,x^{-(b+1)}\,e^{-a\,x^{-b}}\left(\begin{matrix}e^{-\alpha\,x^{-\beta}}\\-e^{-\alpha\,c^{-\beta}}\end{matrix}\right)dx$$

$$=\frac{ab}{k_1}\left\{\begin{matrix}\int_c^d x^{-(b+1)}\,e^{-a\,x^{-b}}\,e^{-\alpha\,x^{-\beta}}\,dx\\-e^{-\alpha\,c^{-\beta}}\int_c^d x^{-(b+1)}\,e^{-a\,x^{-b}}\,dx\end{matrix}\right\}$$

$$=\frac{ab}{k_1}\int_c^d x^{-(b+1)}\,e^{-a\,x^{-b}}\,e^{-\alpha\,x^{-\beta}}\,dx-\frac{e^{-\alpha\,c^{-\beta}}}{k^*}$$

Where, $k_1=\left(e^{-a\,d^{-b}}-e^{-a\,c^{-b}}\right)\left(e^{-\alpha\,d^{-\beta}}-e^{-\alpha\,c^{-\beta}}\right)$.

To solve $\int_c^d x^{-(b+1)} e^{-a\,x^{-b}} e^{-\alpha\,x^{-\beta}} dx$, we use

$$e^{-\alpha\,x^{-\beta}} = \sum_{m=0}^{\infty} \frac{(-\alpha)^m x^{-\beta m}}{m!}, \text{ and}$$

let

$$y = a\,x^{-b} \Rightarow x = (y/a)^{-1/b} \Rightarow dx = \frac{-1}{a\,b}(y/a)^{-1/b-1} dy,$$

so,

$$\int_c^d x^{-(b+1)} e^{-a\,x^{-b}} e^{-\alpha\,x^{-\beta}} dx$$

$$= \sum_{m=0}^{\infty} \frac{(-\alpha)^m}{(ab)\;m!} \int_{a\,d^{-b}}^{a\,c^{-b}} (y/a)^{\beta/b+1+1/b} e^{-y} (y/a)^{-1-1/b} dy$$

$$= \sum_{m=0}^{\infty} \frac{(-\alpha)^m}{a^{\beta/b}(ab)\;m!} \int_{a\,d^{-b}}^{a\,c^{-b}} y^{\beta/b} e^{-y} dy$$

$$= \sum_{m=0}^{\infty} \frac{(-\alpha)^m}{a^{\beta/b}(ab)\;m!} \left\{ \begin{array}{l} \Gamma\left(\frac{\beta}{b}+1, a\,d^{-b}\right) \\ -\Gamma\left(\frac{\beta}{b}+1, a\,c^{-b}\right) \end{array} \right\}.$$

Then, the stress-strength reliability is,

$$R = \frac{1}{k_1} \sum_{m=0}^{\infty} \frac{(-\alpha)^m}{a^{\beta/b}\;m!} \left\{ \begin{array}{l} \Gamma\left(\frac{\beta}{b}+1, a\,d^{-b}\right) \\ -\Gamma\left(\frac{\beta}{b}+1, a\,c^{-b}\right) \end{array} \right\} - \frac{e^{-\alpha\,c^{-\beta}}}{k^*}. \quad (20)$$

5. Parameters Estimation of DTFréchet Distribution

The main aim of this section is to study different estimators of the unknown parameters of a **DTFréchet** distribution. Here, we consider $\hat{c} = x_{(1)}$ and $\hat{d} = x_{(n)}$ as a most common estimators for truncation parameters c and d respectively, then we use five methods of estimation to explore about the parameters a and b.

(1) *Maximum Likelihood estimators(**MLE**).*

If $x_1, x_2, \dots, x_n$ is a random sample from $DTFr(a, b, \hat{c}, \hat{d})$, then the log-likelihood function is

$$L = n\,Ln(ab) - (b+1)\sum_{i=1}^{n} Ln(x_i)$$
$$-a\sum_{i=1}^{n} x_i^{-b} - n\,Ln\left(e^{-a\,x_{(n)}^{-b}} - e^{-a\,x_{(1)}^{-b}}\right)$$

The normal equations become,

$$\frac{\partial L}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n} x_i^{-b} - \frac{n\left\{x_{(1)}^{-b} e^{-a\,x_{(1)}^{-b}} - x_{(n)}^{-b} e^{-a\,x_{(n)}^{-b}}\right\}}{e^{-a\,x_{(n)}^{-b}} - e^{-a\,x_{(1)}^{-b}}}$$

$$\frac{\partial L}{\partial b} = \frac{n}{b} - \sum_{i=1}^{n} Ln(x_i) + a\sum_{i=1}^{n} x_i^{-b} Ln(x_i) - \frac{a\,n \left\{ \begin{array}{l} x_{(n)}^{-b}\;Ln(x_{(n)}) e^{-a\,x_{(n)}^{-b}} \\ -x_{(1)}^{-b}\;Ln(x_{(1)}) e^{-a\,x_{(1)}^{-b}} \end{array} \right\}}{e^{-a\,x_{(n)}^{-b}} - e^{-a\,x_{(1)}^{-b}}}$$

After equating each one of the above equations with zero and then using the numerical solution to solve them simultaneously, we obtain $\hat{a}_{MLE}$ and $\hat{b}_{MLE}$ as ML estimators of a and b respectively.

(2) *The exact moments estimators (**EME**).*

The method of moments is a technique for constructing estimators of the parameters that is based on matching the sample moments with the corresponding distribution moments.

Here we provide the method of moments estimators of a and b parameters of a **DTFréchet** Distribution when both are unknown. If X follows $DTFr(a, b, \hat{c}, \hat{d})$, then the first two distribution moments are,

$$E(X) = \frac{a^{1/b}}{k}\left\{\Gamma\left(1-\frac{1}{b}, a\,x_{(n)}^{-b}\right) - \Gamma\left(1-\frac{1}{b}, a\,x_{(1)}^{-b}\right)\right\}$$

$$E(X^2) = \frac{a^{2/b}}{k}\left\{\Gamma\left(1-\frac{2}{b}, a\,x_{(n)}^{-b}\right) - \Gamma\left(1-\frac{2}{b}, a\,x_{(1)}^{-b}\right)\right\}.$$

By equating sample moments $M_s = (1/n)\sum_{i=1}^{n} x_i^s$, $(s = 1, 2)$ with corresponding theoretical (distribution) moments above, and using the numerical solution for the resulting equations simultaneously, we obtain $\hat{a}_{EME}$ and $\hat{b}_{EME}$ as EM estimators of a and b respectively.

(3) *The approximate moments estimators (**AME**).*

If X follows $DTFr(a, b, \hat{c}, \hat{d})$, then from (11) one can write,

$$a = \frac{b+1}{b\,M_o^{1/b}}. \quad (21)$$

By substituting (21) in (12), we get,

$$M_e = \left(\frac{b+1}{b\,M_o^{1/b}}\right)^{1/b} \left\{ \begin{array}{l} Ln(2) \\ -Ln\left(\begin{array}{l} \exp\left(-\frac{b+1}{b\,M_o^{1/b}} x_{(n)}^{-b}\right) \\ -\exp\left(-\frac{b+1}{b\,M_o^{1/b}} x_{(1)}^{-b}\right) \end{array} \right) \end{array} \right\}^{-1/b} \quad (22)$$

then, after calculating the sample mode m_o and the sample median m_e and substituting their values in (22), one can get the AME of b, say, $\hat{b}_{AME}$ by solving iteratively (22). Once we estimate b, we can use (21) to obtain the AME of a, say, $\hat{a}_{AME}$.

(4) *Estimators based on percentiles (**PE**).*

Kao in [4] originally explored this method by using the graphical approximation to the best linear unbiased estimators. The estimators can be obtained by fitting a straight line to the theoretical points obtained from the distribution function and the sample percentile points. In the case of a DTFréchet distribution, it is possible to use the same concept to obtain the estimators of a and b based on percentiles because of the structure of its distribution function . Since $G(x)$ defined in (5), therefore,

$$x = \left(\frac{-a}{Ln\left(e^{-a\, x_{(1)}^{-b}} + G(x)\, k \right)} \right)^{1/b} . \qquad (23)$$

If p_i denotes some estimate of $G\left(x_{(i)}; a, b, \hat{c}, \hat{d}\right)$ then, the estimate of a and b can be obtained by minimizing,

$$\left[x_{(i)} - \left(-a \Big/ Ln\left(e^{-a\, x_{(1)}^{-b}} + k\, p_i \right) \right)^{1/b} \right]^2 \qquad (24)$$

With respect to a and b. (24) is a nonlinear function of a and b. It is possible to use some nonlinear regression techniques to estimate a and $b$simultaneously. Actually, it is possible to use several p_i's as estimators of $\left(x_{(i)}\right)$. $p_i = i/(n+1))$ is the most used estimator of $G\left(x_{(i)}\right)$ since it is equal to $E\left(G\left(x_{(i)}\right)\right)$. We have also used this p_i here. For some other choices of p_i's, see Mann, Schafer and Singpurwalla [5].

(5) *Least squares* ***(LSE)*** *and Weighted least squares* ***(WLSE)*** *estimators*

This method was originally suggested by Swain, Venkatraman and Wilson (1988) to estimate the parameters of Beta distribution. Suppose $x_1, x_2, \ldots, x_n$ is a random sample of size n from a distribution function $G(.)$ and suppose $x_{(i)} (i = 1,2,\ldots,n)$ denotes the ordered sample. This method uses the distribution of $G(x_{(i)})$. For a sample of size n, we have [5], $E\left(G(x_{(i)})\right) = \frac{j}{n+1}$,

$$Var\left(G(x_{(i)})\right) = j(n-j+1) \Big/ \left((n+1)^2 (n+2)\right)$$

and

$$Cov\left[\left(G(x_{(i)})\right), \left(G(x_{(k)})\right)\right]$$
$$= j(n-k+1) \Big/ \left((n+1)^2 (n+2)\right) \; for\; j < k.$$

So, one can obtain the **LS** estimators by minimizing, $\sum_{j=1}^{n}\left(G(x_{(i)} - j/(n+1)\right)^2$ with respect to the unknown parameters. Therefore in the case of DTFréchet distribution, the

least squares estimators of a and b, say $\hat{a}_{LSE}$ and $\hat{b}_{LSE}$ respectively, can be obtained by minimizing,

$$\sum_{j=1}^{n}\left(\left(e^{-a\, x_{(j)}^{-b}} - e^{-a\, c^{-b}} \right) \Big/ k - j/(n+1) \right)^2 \qquad (25)$$

with respect to a and b.

The weighted least squares estimators of a and b, say $\hat{a}_{WLSE}$ and $\hat{b}_{WLSE}$ respectively, can be obtained by minimizing,

$$\sum_{j=1}^{n} w_j \left(\left(e^{-a\, x_{(j)}^{-b}} - e^{-a\, c^{-b}} \right) \Big/ k - j/(n+1) \right)^2 \qquad (26)$$

with respect to a and b, where

$$w_j = 1/Var\left(F(y_{(i)})\right) = (n+1)^2 (n+2) / \left(j(n-j+1)\right).$$

Table 1. Empirical MSE to estimate the DTFréchet distribution parameters *a* and *b*

case		1		2		3		4		5	
parameters		***a***	***b***	***a***	***b***	***a***	***b***	***a***	***b***	***a***	***b***
Sample size	The method	0.6	1	1	0.6	0.9	0.9	1.2	0.3	0.3	1.2
10	***MLE***	4.015163	7.619172	3.86419239	6.68765466	3.91558647	7.43608095	3.65540394	5.28716598	4.98522576	7.75408182
	EME	5.020559	9.321601	4.85674056	8.39329569	4.9145589	9.13850985	4.63189146	6.98959488	5.98419819	9.45329859
	AME	5.055893	9.478996	4.90813464	8.52499302	4.96916511	9.28947996	4.70577045	7.14056499	6.04522866	9.62354148
	PE	3.154312	7.55493	3.00655368	6.58807863	3.06437202	7.34292918	2.76885606	5.19722634	4.13401131	7.66735431
	LSE	4.898498	9.24451	4.72825536	8.27444688	4.79249796	9.02287317	4.50983052	6.88038246	5.86213725	9.36014682
	WLSE	4.541952	8.637418	4.37170893	7.65771792	4.44237579	8.42541699	4.16934474	6.27650202	5.51843934	8.74662999
20	***MLE***	2.9372937	6.9996999	2.7662766	6.1086108	2.8352835	6.8016801	2.5652565	4.7944794	3.8343834	7.0957095
	EME	4.6084608	8.6618661	4.4434443	7.7557755	4.5064506	8.4728472	4.2364236	6.4656465	5.5085508	8.7698769
	AME	4.6654665	8.7848784	4.5244524	7.8667866	4.5634563	8.5988598	4.3054305	6.5826582	5.5655565	8.8898889
	PE	2.8712871	6.6666666	2.70027	5.7635763	2.7782778	6.4866486	2.5412541	4.4764476	3.7893789	6.7956795
	LSE	4.5364536	8.5088508	4.3954395	7.5967596	4.4374437	8.3228322	4.1884188	6.30063	5.4365436	8.6258625
	WLSE	4.2394239	7.9897989	4.0894089	7.0837083	4.1374137	7.8127812	3.8853885	5.7965796	5.1425142	8.1128112
50	***MLE***	2.6299188	6.7440492	2.4650064	5.8847688	2.53155	6.5733504	2.2827348	4.6349064	3.4776264	6.87135
	EME	4.4092368	8.2398336	4.2703632	7.3545144	4.3224408	8.0488824	4.0562664	6.1133316	5.28009	8.3497752
	AME	4.4757804	8.3960664	4.3282272	7.5425724	4.3803048	8.2282608	4.1141304	6.29271	5.3321676	8.5291536
	PE	2.7311808	6.3274284	2.5836276	5.4450024	2.6357052	6.1451568	2.3753172	4.2182856	3.5991408	6.43737
	LSE	4.1285964	7.7740284	3.9868296	6.8887092	4.0389072	7.5859704	3.8016648	5.6590992	5.0110224	7.8810768
	WLSE	4.0620528	7.66698	3.9116064	6.8076996	3.9723636	7.4818152	3.7293348	5.5289052	4.9300128	7.7798148
100	***MLE***	1.3095	3.4545	1.236	3.003	1.2585	3.3585	1.1265	2.3565	1.7595	3.513
	EME	2.2695	4.149	2.199	3.7005	2.2185	4.0515	2.0985	3.048	2.7195	4.2045
	AME	2.3025	4.1445	2.226	3.7035	2.3415	4.0485	2.124	3.0495	2.757	4.206
	PE	1.389	3.111	1.317	2.658	1.3485	3.021	1.2105	2.0175	1.836	3.171
	LSE	2.1075	4.0005	2.0325	3.549	2.058	3.9015	1.923	2.898	2.5635	4.053
	WLSE	2.0865	3.954	2.019	3.501	2.0385	3.8535	1.914	2.883	2.5485	4.0095

6. The Empirical Study and Discussions

We conduct extensive simulations to compare the performances of the different methods, stated in section V, for estimating unknown parameters of DTFréchet distribution, mainly with respect to their mean square errors (**MSE**) for different sample sizes and for different parameters values.

The experiments are conducted according to run size $K = 1000$. We reported the results for $n = 10$ (small sample), $n = 20$ (moderate sample) and $n = 50, 100$ (large sample) and for the following different values of a and b,

a	0.6	1	0.9	1.2	0.3
b	1	0.6	0.9	0.3	1.2

The results are reported in Table 1. From the table, we observe that,

1. The MSE's decrease as sample size increases in all methods of estimation. It verifies the asymptotic unbiasedness and consistency of all the estimators.
2. It can be said that the estimation of shape parameters are more accurate for the smaller values of those parameters whereas the estimation of scale parameters are more accurate for the larger values of those parameters .in other words, MSE's increase as shape parameter increases whereas MSE's increase as scale parameter decreases.
3. The performances of ***LSE***, ***EME*** and ***AME*** are according to their order.
4. The performances of ***EME***'s and ***AME***'s are close to each other.
5. For small (n=10) sample size and moderate (n=20) sample size, it is observed that ***PE*** works the best for both of the two parameters whereas the second best method is ***MLE***.
6. For large (n=50, 100) sample size, it is observed that ***MLE*** works the best from all other methods to estimate the scale parameter whereas the second best method is ***PE***. ***PE*** works the best from all other methods to estimate the shape parameter whereas the second best method is ***MLE***.

7. Summary and Conclusions

In view of the great importance of the truncated distributions in statistical analysis, the doubly truncated Fréchet distribution (DTFD) is considered here. For DTFD we derived exact formulas of hazard function, characteristic function, *r*th raw moment, mean, mode, median, variance, skewness, kurtosis, Shannon entropy function, relative entropy, quantile function and stress-strength reliability. Different methods to estimate doubly truncated Fréchet distribution parameters are studied, Maximum Likelihood estimator, Moments estimator, Percentile estimator, least square estimator and weighted least square estimator. An empirical study was conducted to compare among these methods. It seemed to us that the Percentile estimator is the best one for small and moderate samples and it is also the best to estimate the shape parameter for large samples, whereas the maximum likelihood estimator is the best to estimate the scale parameter for large samples.

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