American Journal of Applied Mathematics and Statistics, 2015, Vol. 3, No. 6, 226-232
Available online at http://pubs.sciepub.com/ajams/3/6/3

DOI:10.12691/ajams-3-6-3

A Fixed Point Approach to Hyers-Ulam-Rassias Stability of Nonlinear Differential Equations

Maher Nazmi Qarawani*

Department of Mathematics, Al-Quds Open University, Salfit, West-Bank, Palestine
*Corresponding author: mkerawani@qou.edu

Abstract In this paper we use the fixed point approach to obtain sufficient conditions for Hyers-Ulam-Rassias stability of nonlinear differential. Some illustrative examples are given.

Keywords: *hyers-ulam-rassias stability, fixed point, nonlinear differential equations*

Cite This Article: Maher Nazmi Qarawani, "A Fixed Point Approach to Hyers-Ulam-Rassias Stability of Nonlinear Differential Equations." *American Journal of Applied Mathematics and Statistics*, vol. 3, no. 6 (2015): 226-232. doi: 10.12691/ajams-3-6-3.

1. Introduction

The objective of this article is to investigate the Hyers-Ulam-Rassias Stability for the nonlinear differential equation

$$y''(t)+2f(t)y'+y+g(t,y)=0,\ t\in R^{+} \tag{1}$$

and the perturbed nonlinear differential equation of second order

$$y''(t)+2f(t)y'+y+g(t,y)=h(t) \tag{2}$$

by fixed point method under assumptions: $f(t)>0$, $g(t,y)$ are continuous, and that

$$\int_0^t |f(s)|ds\to\infty \text{ as } t\to\infty,, \tag{3}$$

$$\int_0^t e^{-2\int_s^t f(u)du}(t-s)ds\le\frac{\alpha}{2} \tag{4}$$

where $\alpha<1,\ t\ge 0.$

Suppose that there is $L>0$ such that if $|x|,|y|\le L,$ then

$$|g(t,x)-g(t,y)|\le Ld(t)|x-y|,\ t\ge 0, \tag{5}$$

where $d(t)>0,\ d(t)\to 0$ as $t\to\infty,$ and $g(t,0)=0.$

Furthermore, we assume that there is a positive constant A such that $A<L,$ and $h(t):[0,\infty)\to R$ with

$$\int_0^t (t-s)e^{-\int_s^t f(u)du}|h(s)|ds\le A,\ t\ge 0 \tag{6}$$

In 1940, Ulam [1] posed the stability problem of functional equations. In the talk, Ulam discussed a problem concerning the stability of homomorphisms. A significant breakthrough came in 1941, when Hyers [2] gave a partial solution to Ulam's problem. During the last two decades very important contributions to the stability problems of functional equations were given by many mathematicians (see [3-11]). More than twenty years ago, a generalization of Ulam's problem was proposed by replacing functional equations with differential equations: The differential equation $F(t,y(t),y'(t),...,y^{(n)}(t))=0$ has the Hyers-Ulam stability if for given $\varepsilon>0$ and y a function such that

$$\left|F(t,y(t),y'(t),...,y^{(n)}(t))\right|\le\varepsilon$$

there exists a solution y_0 of the differential equation such that

$$|y(t)-y_0(t)|\le K(\varepsilon)$$

and $\lim_{\varepsilon\to 0} K(\varepsilon)=0.$

The first step in the direction of investigating the Hyers-Ulam stability of differential equations was taken by Obloza (see [12,13]). Thereafter, Alsina and Ger [14] have studied the Hyers-Ulam stability of the linear differential equation $y'(t)=y(t)$. The Hyers-Ulam stability problems of linear differential equations of first order and second order with constant coefficients were studied in the papers ([15,16]) by using the method of integral factors. The results given in [17,18,19] have been generalized by Popa and Rasa [20,21] for the linear differential equations of nth order with constant coefficients. In addition to above-mentioned studies, several authors have studied the Hyers-Ulam stability for differential equations of first and second order (see 22-26). The Hyers-Ulam-Rassias Stability by Fixed Point Technique for Half-linear Differential Equations with Unbounded Delay has been established by Qarawani [27]. Burton in [28] has used fixed point theory to establish Liapunov stability for functional differential equations. Some researchers have used the fixed point approach to investigate the Hyers-Ulam stability for differential equations [e.g. [29,30]].

Definition 1 Let

$$S = \{\phi : R^+ \to R \mid \phi(0) = y_0, \|\phi\| \le L\}$$

on R^+, $\phi \in C$ }, where $R^+ = [0,\infty)$. We say that equation (1.2) (or (1.1) with $h(t) \equiv 0$) has the Hyers-Ulam-Rassias (HUR) stability with respect to φ if there exists a positive constant $k > 0$ with the following property: For each $y(t) \in S$, if

$$\left| \begin{array}{l} y''(t) + 2f(t)y' \\ + y + g(t,y) - h(t) \end{array} \right| \le \varphi(t), \tag{7}$$

then there exists some $y_0(t)$ of the equation (4) such that $|y(t) - y_0(t)| \le k\varphi(t)$.

Theorem 1 The Contraction Mapping Principle.

Let (S,ρ) be a complete metric space and let $P : S \to S$. If there is a constant $\alpha < 1$ such that for each pair $\phi_1, \phi_2 \in S$ we have $\rho(P\phi_1, P\phi_2) \le \alpha\rho(\phi_1, \phi_2)$, then there is one and only one point $\phi \in S$ with $P\phi = \phi$.

2. Main Results On Hyers-Ulam-Rassias Stability

Theorem 2 Suppose that $y(t) \in S$ satisfies the inequality (1) with small initial condition $y(0) = y_0$. Let $\varphi(t) : [0,\infty) \to (0,\infty)$ be a continuous function such that

$$\int_0^t \varphi(s) e^{-\int_s^t f(u)du} (t-s)ds \le C\varphi(t), \quad \forall t \ge 0. \tag{8}$$

If (3)-(6) hold, then the solution of (1) is stable in the sense of Hyers-Ulam-Rassias.

Proof. Let C be the space of all continuous functions from $R^+ \to R$ and define the set S by

$$S = \left\{ \begin{array}{l} \phi : R^+ \to R \mid \phi(0) = y_0, \\ \|\phi\| \le L, \, on \; R^+, \phi \in C \end{array} \right\}$$

Then, equipped with the supremum metric $(\|\cdot\|, s)$, is a complete metric space. Now suppose that (3) holds. For L and α, find appropriate constants δ, a and B such that

$$(1+a)\delta + \frac{L\alpha}{2} + \frac{LB\alpha}{2} \le L.$$

Multiplying both sides of (1) by $e^{2\int_0^t f(s)ds}$, and then integrating once with respect to t yields

$$e^{2\int_0^t f(s)ds} y' = y'(0) - \int_0^t y(s) e^{2\int_0^t f(u)du} ds - \int_0^t g(s,y(s)) e^{2\int_0^t f(u)du} ds \tag{9}$$

Now, we multiply Eq. (9) by $e^{-2\int_0^t f(s)ds}$, and integrate with respect to t to obtain

$$\begin{aligned} y(t) = y(0) + y'(0)\int_0^t e^{-2\int_0^s f(u)du} ds \\ - \int_0^t (t-s) y(s) e^{-2\int_s^t f(u)du} ds \\ - \int_0^t (t-s) g(s,y(s)) e^{-2\int_s^t f(u)du} ds. \end{aligned}$$

Define $P : S \to S$ by

$$\begin{aligned} (P\phi)(t) = y(0) + y'(0)\int_0^t e^{-2\int_0^s f(u)du} ds \\ - \int_0^t (t-s) y(s) e^{-2\int_s^t f(u)du} ds \\ - \int_0^t (t-s) g(s,y(s)) e^{-2\int_s^t f(u)du} ds \end{aligned} \tag{10}$$

It is clear that for $\phi \in S$, $P\phi$ is continuous. Let $\phi(t) \in S$ with $\|\phi\| \le L$, for some positive constant L . Then there is a $\delta > 0$ with $|\phi(0)| \le \delta.$ Since $\int_0^t |f(s)| ds \to \infty$, as $t \to \infty$, then we can find a constant $a > 0$ such that

$$\left| y'(0) \int_0^t e^{-2\int_0^s f(u)du} ds \right| < a\delta.$$

Then using (3),(4) in the definition of $(P\phi)(t)$, we have

$$\|P\phi\| \le |y(0)| + \left| y'(0)\int_0^t e^{-2\int_0^s f(u)du} ds \right|$$

$$+\int_0^t |(t-s)y(s)| e^{-2\int_s^t f(u)du} ds$$

$$+\left|\int_0^t (t-s)g(s,y(s))\right| e^{-2\int_s^t f(u)du} ds$$

$$\le (1+a)\delta + L\int_0^t (t-s)e^{-2\int_s^t f(u)du} ds$$

$$+LB\int_0^t (t-s)e^{-2\int_s^t f(u)du} d(s)ds$$

Since $d(t) \to 0,$ as $t \to \infty,$ we can choose a number B sufficiently small such that $0 < d(t) \le B,$ on R^+ and with

$$LB < 1 \tag{11}$$

Then from (4) we obtain

$$\|P\phi\| \le (1+a)\delta + \frac{L\alpha}{2} + \frac{LB\alpha}{2}$$

which implies that $\|P\phi\| \le L.$

To see that P is a contraction under the supremum metric, let $\phi, \eta \in S,$ then

$$\|(P\phi)(t) - (P\eta)(t)\|$$

$$\le \int_0^t (t-s)|\phi(s)-\eta(s)| e^{-2\int_s^t f(u)du} ds$$

$$+\int_0^t |g(s,\phi(s)) - g(s,\eta(s))|(t-s)e^{-2\int_s^t f(u)du} ds$$

$$\le \int_0^t (t-s)e^{-2\int_s^t f(u)du} \|\phi-\eta\| ds$$

$$+LB\int_0^t (t-s)e^{-2\int_s^t f(u)du} \|\phi-\eta\| ds$$

From this and in view of (4) and (11) we get the estimate

$$\|(P\phi)(t) - (P\eta)(t)\| \le \alpha\|\phi-\eta\|, \text{ with } \alpha < 1.$$

Thus, by the contraction mapping principle, P has a unique fixed point, say y_0 in S which solves (1) and is bounded.

Next we show that the solution y_0 is stable in Hyers-Ulam-Rassias. From the inequality (7) we get

$$-\varphi(t) \le y''(t) + 2f(t)y' + y + g(t,y) \le \varphi(t) \tag{12}$$

Multiplying the inequality (12) by $e^{2\int_0^t f(u)du}$, we obtain

$$-\varphi(t)e^{2\int_0^t f(u)du}$$

$$\le e^{2\int_0^t f(u)du} y''(t) + 2f(t)y'(t)e^{2\int_0^t f(u)du}$$

$$+y(s)e^{2\int_0^t f(u)du} + g(s,y(s))e^{2\int_0^t f(u)du}$$

$$\le \varphi(t)e^{2\int_0^t f(u)du}$$

Or equivalently, we have

$$-\varphi(t)e^{2\int_0^t f(u)du}$$

$$\le \left(e^{2\int_0^t f(u)du} y'(t) \right)' + y(s)e^{2\int_0^t f(u)du}$$

$$+g(s,y(s))e^{2\int_0^t f(u)du} \le \varphi(t)e^{2\int_0^t f(u)du}$$

Integrate the last inequality from 0 to $t,$ and then multiply the obtained inequality by $e^{-2\int_0^t f(s)ds}$ to get

$$-\int_0^t \varphi(s)e^{-2\int_s^t f(u)du} ds$$

$$\le y' - y'(0)e^{-2\int_0^t f(s)ds}$$

$$+\int_0^t y(s)e^{-2\int_s^t f(u)du} ds + \int_0^t g(s,y(s))e^{-2\int_s^t f(u)du} ds$$

$$\le \int_0^t \varphi(s)e^{-2\int_s^t f(u)du} ds$$

Integrating again with respect to $t,$ we have

$$-\int_0^t (t-s)\varphi(s)e^{-2\int_s^t f(u)du}ds$$
$$\le y(t)-y(0)$$
$$-y'(0)\int_0^t e^{-2\int_0^s f(u)du}ds+\int_0^t (t-s)y(s)e^{-2\int_s^t f(u)du}ds$$
$$+\int_0^t (t-s)g(s,y(s))e^{-2\int_s^t f(u)du}ds$$
$$\le \int_0^t (t-s)\varphi(s)e^{-2\int_s^t f(u)du}ds$$

Hence from (8), (20) we infer that $\|Py-y\|\le C\varphi.$ To show that y_0 is stable we estimate the difference

$$\|y(t)-y_0(t)\|\le \|Py-y\|+\|Py-Py_0\|$$
$$\le C\varphi+\alpha\|y-y_0\|$$

Thus

$$\|y(t)-y_0(t)\|\le \frac{C\varphi}{1-\alpha}$$

which means that (7) holds true (with $h(t)\equiv 0$) for all $t\ge 0$.

Example 1 Consider the differential equation

$$y''(t)+\left(4+2\sin t\right)y'+y+\frac{\sin y}{(1+t)^2}=0.$$

Let us estimate the integrals

$$\int_0^t |f(s)|ds=\int_0^t \left(2+\sin t\right)ds\ge \int_0^t ds\ge t\to\infty,$$
as $t\to\infty,$

and for all $t>0$ we obtain

$$\int_0^t e^{-2\int_s^t f(u)du}(t-s)ds$$
$$=\int_0^t e^{-\int_s^t \left(4+2\sin u\right)du}(t-s)ds$$
$$\le \int_0^t e^{-2\int_s^t du}(t-s)ds$$
$$\le \int_0^t e^{-2(t-s)}(t-s)ds$$
$$=\frac{1}{4}\left(1-e^{-2t}-2te^{-2t}\right)<\frac{1}{4},$$

Since $g(t,y(t)))=\dfrac{\sin y}{(1+t)^2}$, then $|g(t,x)-g(t,y)|$

$$=\left|\frac{\sin x}{(1+t)^2}-\frac{\sin y}{(1+t)^2}\right|\le \frac{1}{(1+t)^2}|x-y|.$$

Therefore, we take $d(t)=\dfrac{1}{(1+t)^2}$, which tends to zero as $t\to\infty.$

Now, if we set $\varphi(t)=e^t$, then we have

$$\int_0^t (t-s)\varphi(s)e^{-\int_s^t a(u)du}ds$$
$$=\int_0^t e^s e^{-\int_s^t \left(4+2\sin u\right)du}(t-s)ds$$
$$\le \frac{e^t}{9}\left(1-e^{-3t}-3te^{-3t}\right)<\frac{e^t}{9}\le C\varphi(t),$$
with $C\ge \frac{1}{9},\forall t\ge 0.$

Let us take $L=1,\ \alpha=\frac{1}{2}, B=0.1$. Then for the corresponding coefficients by (1.3), we can choose small positive constants a,δ such that

$$\left(1+a\right)\delta+\frac{L\alpha}{2}+\frac{L\alpha B}{2}\le L$$

and so

$$\left(1+a\right)\delta\le \frac{29}{40}.$$

Thus, all the conditions of Theorem (3.1) are satisfied, hence the Eq. (3.6) is HUR stable for $t\ge 0.$

Theorem 3 Suppose that $y(t)\in S$ satisfies the inequality (7) with small initial condition $y(0)=y_0$. Let $\varphi(t):[0,\infty)\to(0,\infty)$ be a continuous function such that

$$\int_0^t \varphi(s)e^{-\int_s^t f(u)du}(t-s)ds\le C\varphi(t),\ \forall t\ge 0. \quad (13)$$

If (3)-(7) hold, then the solution of (2) is stable in the sense of Hyers-Ulam-Rassias.

Proof. Define $S=\left\{\phi\ :\ R^+\to R\,|\,\phi(0)=y_0,\|\phi\|\le L,\ on\ R^+,\phi\in C\right\}$ where $\|\cdot\|$ is the supremum metric. Then $(S,\|\cdot\|)$ is a complete metric space.

Now suppose that (3) holds. For L,A and α we find constants δ,a and B so that $\left(1+a\right)\delta+\dfrac{L\alpha}{2}+\dfrac{L\alpha B}{2}+A\le L.$

Applying the same approach used in Theorem 1 we define $P\ :\ S\to S$ by

$$(P\phi)(t) = y(0) + y'(0)\int_0^t e^{-2\int_0^s f(u)du} ds$$

$$-\int_0^t (t-s)y(s)e^{-2\int_s^t f(u)du} ds$$

$$-\int_0^t (t-s)g(s,y(s))e^{-2\int_s^t f(u)du} ds$$

$$+\int_0^t (t-s)h(s)e^{-2\int_s^t f(u)du} ds$$

Then from (4) we obtain

$$\|P\phi\| \le (1+a)\delta + \frac{L\alpha}{2} + \frac{L\alpha B}{2} + A$$

which implies that $\|P\phi\| \le L.$

To see that P is a contraction under the supremum metric, let $\phi,\eta \in S,$ then

$$\|(P\phi)(t) - (P\eta)(t)\|$$

$$\le \int_0^t (t-s)\left|\phi(s)-\eta(s)\right| e^{-2\int_s^t f(u)du} ds$$

$$+\int_0^t (t-s)\left|g(s,\phi(s)) - g(s,\eta(s))\right| e^{-2\int_s^t f(u)du} ds$$

$$\le \int_0^t (t-s)e^{-2\int_s^t f(u)du} \|\phi-\eta\| ds$$

$$+LB\int_0^t (t-s)e^{-2\int_s^t f(u)du} \|\phi-\eta\| ds$$

From this and using (4) and (11) we get the estimate

$$\|(P\phi)(t) - (P\eta)(t)\| \le \alpha\|\phi-\eta\|, \text{ with } \alpha < 1.$$

Thus, by the contraction mapping principle, P has a unique fixed point, say y_0 in S which solves (1) and is bounded.

Next we show that the solution y_0 is stable in Hyers-Ulam-Rassias. From the inequality (7) we get

$$\begin{aligned} -\varphi(t) \le y''(t) + 2f(t)y' + y \\ + g(t,y) - h(t) \le \varphi(t) \end{aligned} \tag{14}$$

Multiplying the inequality (14) by $e^{2\int_0^t f(u)du}$, we obtain

$$-\varphi(t)e^{2\int_0^t f(u)du}$$

$$\le e^{2\int_0^t f(u)du} y''(t) + 2f(t)y'(t)e^{2\int_0^t f(u)du}$$

$$+y(s)e^{2\int_0^t f(u)du} + g(s,y(s))e^{2\int_0^t f(u)du}$$

$$-h(s)e^{2\int_0^t f(u)du} \le \varphi(t)e^{2\int_0^t f(u)du}$$

Or equivalently, we have

$$-\varphi(t)e^{2\int_0^t f(u)du} \le \left(e^{2\int_0^t f(u)du} y'(t)\right)'$$

$$+y(s)e^{2\int_0^t f(u)du}$$

$$-h(s)e^{2\int_0^t f(u)du} + g(s,y(s))e^{2\int_0^t f(u)du}$$

$$\le \varphi(t)e^{2\int_0^t f(u)du}$$

Integrating the last inequality from 0 to $t,$ and then multiplying the obtained inequality by $e^{-2\int_0^t f(s)ds}$ we get

$$-\int_0^t \varphi(s)e^{-2\int_s^t f(u)du} ds$$

$$\le y' - y'(0)e^{-2\int_0^t f(s)ds}$$

$$+\int_0^t y(s)e^{-2\int_s^t f(u)du} ds + \int_0^t g(s,y(s))e^{-2\int_s^t f(u)du} ds$$

$$-\int_0^t h(s)e^{-2\int_s^t f(u)du} ds \le \int_0^t \varphi(s)e^{-2\int_s^t f(u)du} ds$$

Integrating again with respect to $t,$ we have

$$-\int_0^t (t-s)\varphi(s)e^{-2\int_s^t f(u)du} ds$$

$$\le y(t) - y(0) + \int_0^t (t-s)g(s,y(s))e^{-2\int_s^t f(u)du} ds$$

$$-\int_0^t h(s)e^{-2\int_s^t f(u)du}ds \le \int_0^t (t-s)\varphi(s)e^{-2\int_s^t f(u)du}ds$$

From the definition of Py and in view of (20), we infer that $\|Py-y\| \le C\varphi.$ Now, to show that y_0 is stable we estimate the difference

$$\begin{aligned}\|y(t)-y_0(t)\| &\le \|Py-y\|+\|Py-Py_0\| \\ &\le C\varphi+\alpha\|y-y_0\|\end{aligned}$$

Thus

$$\|y(t)-y_0(t)\| \le \frac{C\varphi}{1-\alpha}$$

which completes the proof.

Example 2 Consider the nonlinear differential equation

$$y''(t)+(4+2\sin t)y'+y+\frac{\sin y}{(1+t)^2}=\frac{e^{-2t}\cos^2 t}{1+t}$$

One can similarly, as in Example 1 establish the validity of conditions (1.3)-(1.6). So, to establish the stability of this equation, it remains to estimate the integral

$$\begin{aligned}&\int_0^t (t-s)e^{-\int_s^t f(s)ds}|h(s)|ds \\ &=\int_0^t (t-s)e^{-\int_s^t (4+2\sin t)ds}\frac{e^{-2s}\cos^2 s}{1+s}ds \\ &\le \int_0^t (t-s)e^{-2(t-s)}e^{-2s}ds \\ &=\int_0^t (t-s)e^{-2t}ds \le \frac{t^2e^{-2t}}{2} \le \frac{1}{2e^2}, \quad \forall t \ge 0.\end{aligned}$$

Let us take $L=1,\ \alpha=\frac{1}{2}, A=\frac{1}{2e^2},$ and $B=0.1$.

Then for these coefficients by (3), we can choose small positive constants a,δ such that

$$(1+a)\delta+\frac{L\alpha}{2}+\frac{L\alpha B}{2}+A \le L$$

From which it follows that

$$(1+a)\delta \le \frac{29}{40}-\frac{1}{2e^2}<\frac{53}{80}.$$

Hence the conditions of Theorem 2 are satisfied.

3. Conclusion

We have obtained two theorems which provide the sufficient conditions for the Hyers-Ulam-Rassias Stability of solutions of two nonlinear differential equations. To illustrate the results we provided two examples satisfying the assumptions of the two proved theorems.

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