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The Moment Approximation of the First–Passage Time for the Birth–Death Diffusion Process with Immigraton to a Moving Linear Barrier

Basel M. Al-Eideh*

Department of Quantitative Methods and Information System, Kuwait University, College of Business Administration, Safat, Kuwait
*Corresponding author:basel@cba.edu.kw

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Abstract Today, the the development of a mathematical models for population growth of great importance in many fields. The growth and decline of real populations can in many cases be well approximated by the solutions of a stochastic differential equations. However, there are many solutions in which the essentially random nature of population growth should be taken into account. In this paper, we approximating the moments of the first – passage time for the birth and death diffusion process with immigration to a moving linear barriers. This was done by approximating the differential equations by an equivalent difference equations. A simulation study is considered and applied to some values of parameters which showed the capability of the technique.

Keywords: *first passage time, birth-death diffusion process, immigration, difference equations*

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1. Introduction

First – passage time play an important rule in the area of applied probability theory especially in stochastic modeling. Several examples of such problems are the extinction time of a branching process, or the cycle lengths of a certain vehicle actuated traffic signals. Actually the the first – passage times to a moving barriers for diffusion and other markov processes arises im biological modeling (Cf. Ewens [8]), in statistics (Cf. Darling and Siegert [6] and Durbin [7]) and in engineering (Cf. Blake and Lindsey [4]).

Many important results related to the first – passage time have been studied from different points of view of different authors. For example, McNeil [13] has derived the distribution of the integral functional

$$Wx = \int_0^{Tx} g\{X(t)\}dt,$$

where T_x is the first – passage time to the origin in a general birth – death process with X(0) = x and g(.) is an arbitrary function. Also, Iglehart [10], McNeil and Schach [14] have been shown a number of classical birth and death processes upon taking diffusion limits to asympotically approach the Ornstein – Uhlenbeck (O.U.).

Many properties such as a first – passage time to a barrier, absorbing or reflecting, located some distance from an initial starting point of the O.U. process and the related diffusion process and the related diffusion process such as the case of the first passage time of a Wiener process to a linear barrier is a closed form expression for the density available is discussed in Cox and Miller [5]. Also, others such as, Karlin and Taylor [11], Thomas [15], Ferebee [9], Tuckwell and Wan [16], Alawneh and Al-Eideh [1], Al-Eideh [2,3] etc. have been discussed the first passage time from different points of view.

In particular, Thomas (1975) describes some mean first – passage time approximation for the Ornstein – Uhlendeck process. Tuckwell and Wan [16] have studied the first-passage time of a Markov process to a moving barriers as a first-exit time for a vector whose components include the process and the barrier.

Alawneh and Al-Eideh [1] have discussed the problem of finding the moments of the first passage time distribution for the Ornstien-Uhlenbeck process with a single absorbing barrier using the method of approximating the differential equations by difference equations.

Also, Al-Eideh [2,3] has discussed the problem of finding the moments of the first passage time distribution for the birth-death diffusion and the Wright-Fisher diffusion processes to a moving linear barriers and to a single absorbing barrier, respectively using the method of approximating the differential equations by difference equations.

In this paper, we consider the birth and death diffusion process with immigration and study the first – passage time for such a process to a moving linear barrier. More specifically, the moment approximations are derived using the method of difference equations used in Al-Eideh [3] considering the immigration rate ε .

2. First – Passage Time Moment Approximations

Consider the birth and death diffusion Process with immigration $\{X(t):t\geq 0\}$ with infinitesimal mean $bx+\varepsilon$ and variance $2ax$ starting at some x_0 > 0, where b and a are the drift and the diffusion coefficients respectively and ε is the constant immigration rate. Also, $\{X(t):t\geq 0\}$ is a Markov process with state space $S=[0,\infty)$ and satisfies the Ito stochastic differential equation

$$dX(t)=(bX(t)+\varepsilon)dt+\sqrt{2aX(t)}dW(t) \quad (1)$$

Where $\{W(t):t\geq 0\}$ is a standard Wiener process with zero mean and variance t. Assume that the existence and uniqueness conditions are satisfied (Cf. Gihman and Skorohod). Let $\{Y(t):t\geq 0\}$ be a moving linear barrier equation such that $Y(t)=ct+k$, with $Y(0)=k$. Or equivalently $\frac{dY(t)}{dt}=c$.

Now, denote the first – passage time of a process $X(t)$ to a moving linear barrier $Y(t)=ct+k$ by the random variables

$$T_Y=\inf\{t\geq 0:X(t)\geq ct+k\} \quad (2)$$

with probability density function

$$g(t;x_0)=-\frac{d}{dt}\int_{-\infty}^{ct+k}p(x_0,x;t)dx$$

Here p (x_0 , x ; t) is the probability density function of X (t) conditional on X (0) = x_0

Let $M_n(x_0,Y;t)$; n = 1,2,3,……, be the n-th moment of the first – passage time T_Y , i.e.

$$M_n(x_0,Y;t)=E(T_Y^n),n=1,2,3,..., \quad (3)$$

It follows from the forward Kolmogorov equation that the n-th moment of T_Y must satisfy the ordinary differential equation

$$\begin{aligned}&axM_n''(x_0,Y;t)+(bx+\varepsilon)M_n'(x_0,Y;t)\\&+cM_n'(x_0,Y;t)=-nM_{n-1}(x_0,Y;t)\end{aligned} \quad (4)$$

Or equivalently

$$\begin{aligned}&M_n''(x_0,Y;t)+\frac{bx+\varepsilon}{ax}M_n'(x_0,Y;t)\\&+\frac{c}{ax}M_n'(x_0,Y;t)=-\frac{n}{ax}M_{n-1}(x_0,Y;t)\end{aligned} \quad (5)$$

Where $M_n'(x_0,Y;t)$ and $M_n''(x_0,Y;t)$ are the first derivatives of $M_n(x_0,Y;t)$ with respect to x $(x_0\leq x\leq Y)$, with appropriate boundary conditions for n=1,2,3,……. Note that $M_0(x_0,Y;t)=1$.

Now, rewrite the equation in (5), we obtain

$$\begin{aligned}&M_n''(x_0,Y;t)\\&=-\frac{n}{ax}M_{n-1}(x_0,Y;t)-\left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right)M_n'(x_0,Y;t)\end{aligned} \quad (6)$$

Let Δ be the difference operator. Then we defined the first order difference of $M_n(x_0,Y;t)$ as follows:

$$\Delta M_n(x_0,Y;t)=M_{n+1}(x_0,Y;t)-M_n(x_0,Y;t) \quad (7)$$

(Cf. Kelly and Peterson [12]).

Note that equation (6) can be approximated by

$$\begin{aligned}&M_n''(x_0,Y;t)=-\frac{n}{ax}M_{n-1}(x_0,Y;t)\\&-\left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right)\Delta M_n(x_0,Y;t)\end{aligned} \quad (8)$$

By applying equation (7) to equation (8) we get :

$$\begin{aligned}M_n''(x_0Y;t)=&-\frac{n}{ax}M_{n-1}(x_0,Y;t)\\&+\left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right)M_n(x_0,Y;t)\\&-\left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right)M_{n+1}(x_0,Y;t)\end{aligned} \quad (9)$$

Now, we will use the matrix theory to solve the differential equation defined in equation (9). If we let

$$\vec{M}(x_0,Y;t)=[M_1(x_0,Y;t),M_2(x_0,Y;t),\cdots]'$$

Then we get

$$\frac{d^2\vec{M}(x_0,Y;t)}{dx^2}=A\vec{M}(x_0,Y;t) \quad (10)$$

Where

$$A=\begin{bmatrix}\left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right) & -\left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right) & 0 & 0 & \cdots\\ -\frac{2}{ax} & \left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right) & -\left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right) & 0 & \cdots\\ 0 & -\frac{3}{ax} & \left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right) & -\left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right) & \cdots\\ 0 & 0 & -\frac{4}{ax} & \left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right) & \cdots\\ \vdots & \vdots & \vdots & \ddots & \ddots\end{bmatrix}$$

Now let

$$\frac{d\vec{M}(x_0,Y;t)}{dx}=\vec{R}(x_0,Y;t) \quad (11)$$

This imply

$$\frac{d^2\vec{M}(x_0,Y;t)}{dx^2}=\frac{d\vec{R}(x_0,Y;t)}{dx} \quad (12)$$

Apply to equation (10), we get

$$\frac{d}{dx}\begin{bmatrix}\vec{R}(x_0,Y;t)\\ \vec{M}(x_0,Y;t)\end{bmatrix}=\begin{bmatrix}0 & \mathrm{A}\\ \mathrm{I} & 0\end{bmatrix}\cdot\begin{bmatrix}\vec{\mathrm{R}}(x_0,Y;t)\\ \vec{M}(x,Y;t_0)\end{bmatrix} \quad (13)$$

Where **I** is the identity matrix and 0 is the zero matrix.

Thus, the solution of the system of equation in (13) is then given by

$$\begin{bmatrix}\vec{R}(x_0,Y;t)\\ \vec{M}(x_0,Y;t)\end{bmatrix} = e^{\begin{bmatrix}0 & A^*\\ D & 0\end{bmatrix}} \cdot \begin{bmatrix}\vec{R}(x_0,Y;t)\\ \vec{M}(x,Y;t_0)\end{bmatrix} \qquad (14)$$

Where $D = [\, d_{ij} \,]$; $i,j \ge 1$ is the diagonal matrix with entries

$$d_{ij} = \begin{cases}(ct+k-x_0) & ;\mathrm{j=i}\\ 0 & ;\text{Otherwise}\end{cases} \qquad (15)$$

And $A^* = \left[a_{ij}^*\right]$; $i,j \ge 1$ is the matrix with entries

$$a_{ij}^* = \begin{cases} -\dfrac{i}{ax}\ln\left(\dfrac{ct+k}{x_0}\right) & ;\ j=i-1\\ \left(\dfrac{b}{a}+\dfrac{\varepsilon+c}{ax}\right)(ct+k-x_0) & ;\ j=i\\ -\left(\dfrac{b}{a}+\dfrac{\varepsilon+c}{ax}\right)(ct+k-x_0) & ;\ j=i+1\\ 0 & ;\mathit{Otherwise}\end{cases} \qquad (16)$$

Note that the matrix e^B where $B=\begin{bmatrix}0 & A^*\\ D & 0\end{bmatrix}$ is defined by

$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \ldots\ldots\ldots\ldots.$$

This series is convergent since it is a cauchy operator of equation (2.6) (Cf. Zeifman [17]).

3. Simmulation Study

In this section we will consider the simmulating the birth and death diffusion process with immigration $\{X(t): t \ge 0\}$ as considered in equation (1) as well as approximating the moments of the first-passage time for such a process using the first and the second order difference operators to the differential equation in (9).

For simulation of the process $\{X(t): t \ge 0\}$ we used the following discrete approximation.

For integer values $k = 1,2,3,\ldots$, and $n = 1,2,3,\ldots$, define

$$X_n^*\left(\frac{k+1}{n}\right) = X_n^*\left(\frac{k}{n}\right) + \frac{1}{n} b X_n^*\left(\frac{k}{n}\right) + \varepsilon + \frac{1}{n}\sqrt{2aX_n^*\left(\frac{k}{n}\right)} \cdot Z(k+1) \qquad (17)$$

where $\{Z(k)\}$ is an independent sequence of standard normal random variables.

For each set of positive integers $k, t_1, \ldots, t_k$, the sequence of random vecrors $(X_n^*(t_1),\ldots,X_n^*(t_k))'$ converges in distribution to $(X_n(t_1),\ldots,X_n(t_k))'$. As an example n is chosen to be 50, i.e. each unit of time is broken into 50 steps for the purpose of simulating $X_n(t_1), X_2(t_k),\ldots$, with $t_1 = 1, t_2 = 1, \ldots$. Therefore, the following graph, Figure 1, represents the simulated process when $X_0(1) = 50$ and the parameters are set to $a = 0.5$, $b = 0.02$ and $\varepsilon = 0.02$.

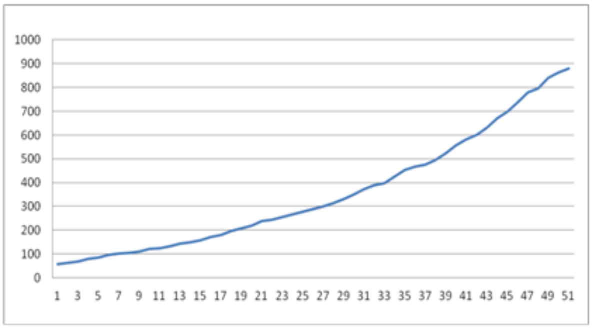

Figure 1. The Simulated Process X(t) when a=0.5, b=0.02 and ε=0.02

Now for approximating the moments of the first-passage time for such a process using the first and the second order difference operators to the differential equation in (9), we define the operators as follows:

Let Δ^2 be the second order difference operators. Then we defined the second order differences of $M_n(x_0,Y;t)$ and as follows:

$$\begin{aligned}\Delta^2 M_n(x_0,Y;t) &= M_{n+2}(x_0,Y;t)\\ &-2M_{n+1}(x_0,Y;t) + M_n(x_0,Y;t)\end{aligned} \qquad (18)$$

(Cf. Kelley and Peterson [12]).

Note that equation (9) can be approximated by

$$\begin{aligned}\Delta^2 M_n(x_0,Y;t) = -\frac{n}{ax} & M_{n-1}(x_0,Y;t)\\ &+\left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right)M_n(x_0,Y;t)\\ &-\left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right)M_{n+1}(x_0,Y;t)\end{aligned} \qquad (19)$$

By applying equation (18) to equation (19) we get:

$$\begin{aligned}&M_{n+2}(x_0,Y;t) - 2M_{n+1}(x_0,Y;t)\\ &+M_n(x_0,Y;t) = -\frac{n}{ax}M_{n-1}(x_0,Y;t)\\ &+\left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right)M_n(x_0,Y;t)\\ &-\left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right)M_{n+1}(x_0,Y;t)\end{aligned} \qquad (20)$$

Now rewriting equation (20) we get:

$$\begin{aligned}&M_{n+2}(x_0,Y;t) =\\ &\left(2-\left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right)\right)M_{n+1}(x_0,Y;t)\\ &+\left(\left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right)-1\right)M_n(x_0,Y;t)\\ &-\frac{n}{ax}M_{n-1}(x_0,Y;t)\end{aligned} \qquad (21)$$

Through equation (21), the first moment $M_1(x_0,Y;t)$ and the second moment $M_2(x_0,Y;t)$ of the first –passage time can be approximated by

$$M_1(x_0,Y;t) \cong \left(2-\left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right)\right) \tag{22}$$

and

$$\begin{aligned} &M_2(x_0,Y;t) \\ &\cong \left(2-\left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right)\right)M_1(x_0,Y;t)+\left(\left(\frac{b}{a}+\frac{\varepsilon+c}{ax}\right)-1\right) \end{aligned} \tag{23}$$

Following the above example of such a process shown in Figure 1 assuming different values for the parameter c where $c = 0,\ 0.02,\ 0.1,\ 0.5,\ 1 \text{ and } 5$, we set the following Figures for the first moment $M_1(x_0,Y;t)$ and the second moment $M_2(x_0,Y;t)$ of the first –passage time.

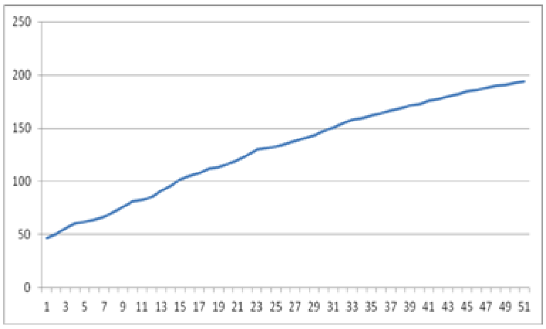

Figure 2. The First Moment of the First-Passage Time $M_1(t)$ of the Process X(t) when a=0.5, b=0.02, ε=0.02 and c=0

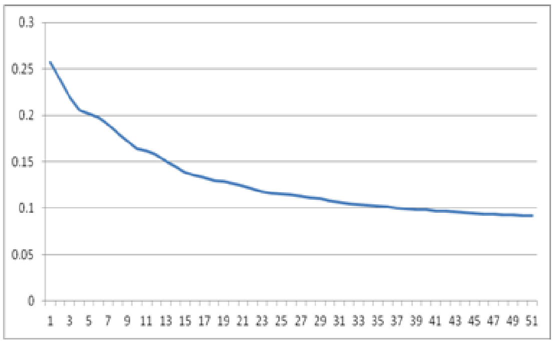

Figure 3. The Second Moment of the First-Passage Time $M_2(t)$ of the Process X(t) when a=0.5, b=0.02, ε=0.02 and c=0

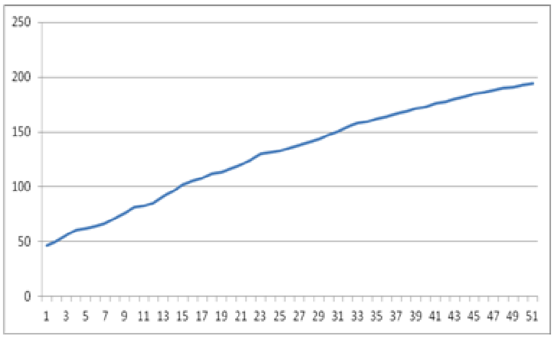

Figure 4. The First Moment of the First-Passage Time $M_1(t)$ of the Process X(t) when a=0.5, b=0.02, ε=0.02 and c=0.02

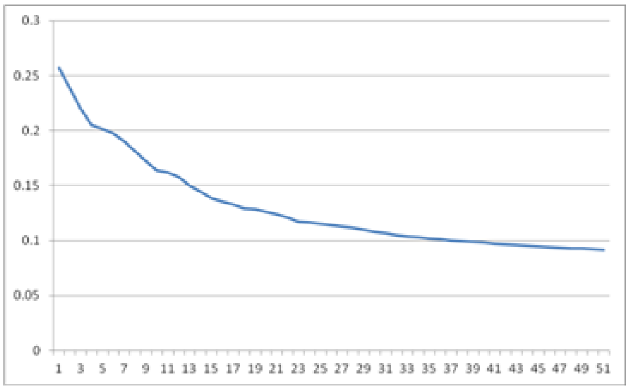

Figure 5. The Second Moment of the First-Passage Time $M_2(t)$ of the Process X(t) when a=0.5, b=0.02, ε=0.02 and c=0.02

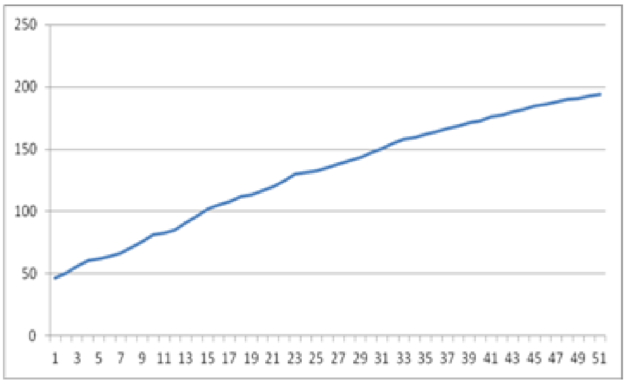

Figure 6. The First Moment of the First-Passage Time $M_1(t)$ of the Process X(t) when a=0.5, b=0.02, ε=0.02 and c=0.1

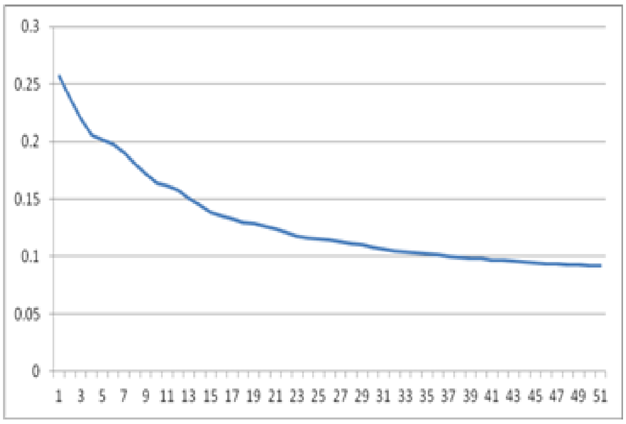

Figure 7. The Second Moment of the First-Passage Time $M_2(t)$ of the Process X(t) when a=0.5, b=0.02, ε=0.02 and c=0.1

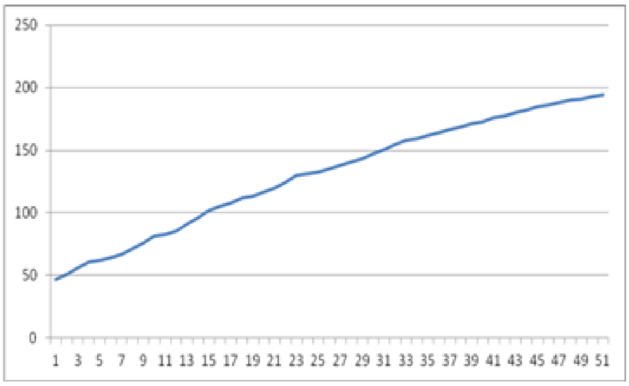

Figure 8. The First Moment of the First-Passage Time $M_1(t)$ of the Process X(t) when a=0.5, b=0.02, ε=0.02 and c=0.5

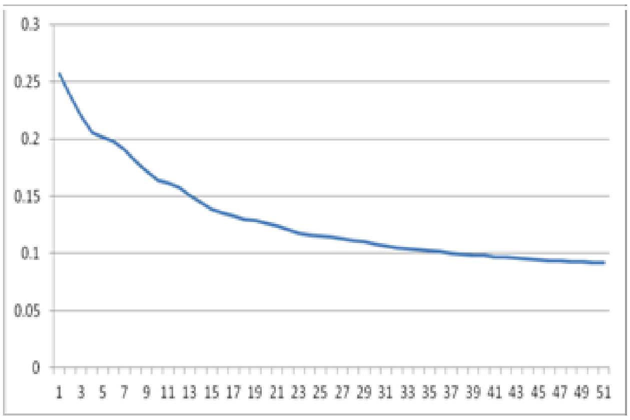

Figure 9. The Second Moment of the First-Passage Time $M_2(t)$ of the Process X(t) when a=0.5, b=0.02, ε=0.02 and c=0.5

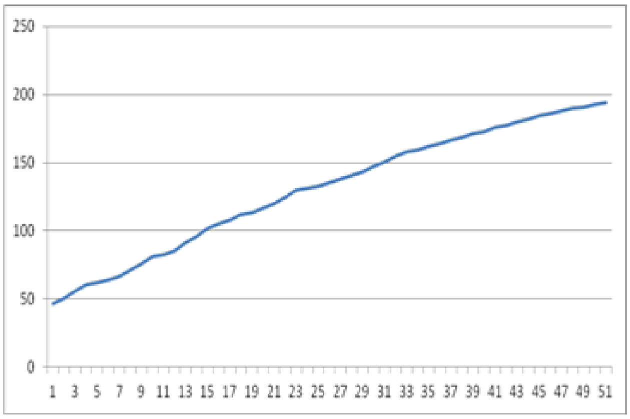

Figure 10. The First Moment of the First-Passage Time $M_1(t)$ of the Process X(t) when a=0.5, b=0.02, ε=0.02 and c=1

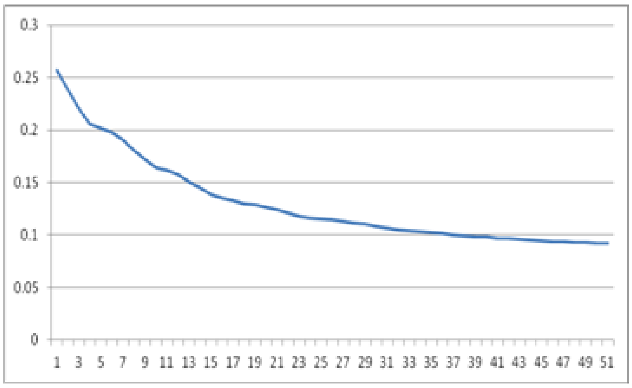

Figure 11. The Second Moment of the First-Passage Time $M_2(t)$ of the Process X(t) when a=0.5, b=0.02, ε=0.02 and c=1

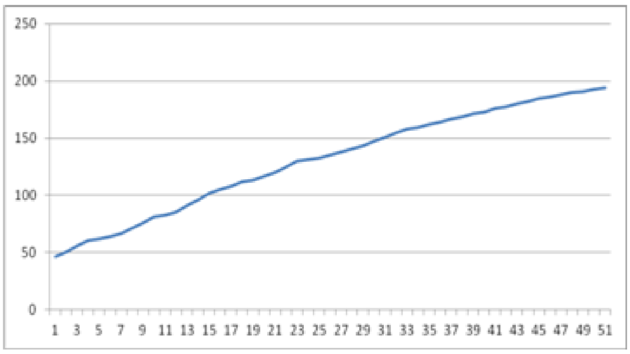

Figure 12. The First Moment of the First-Passage Time $M_1(t)$ of the Process X(t) when a=0.5, b=0.02, ε=0.02 and c=5

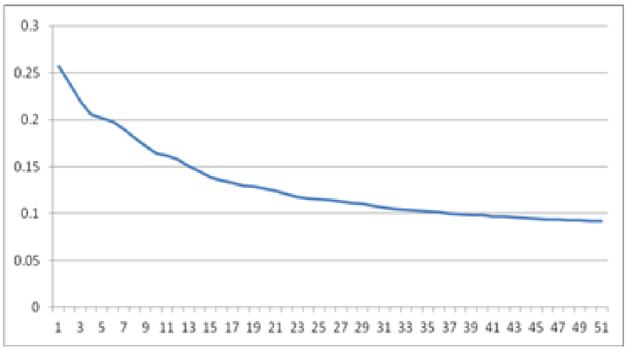

Figure 13. The Second Moment of the First-Passage Time $M_2(t)$ of the Process X(t) when a=0.5, b=0.02, ε=0.02 and c=5

The above figures of the first monent $M_1(x_0, Y; t)$ of the first-passage time of the process show the growth exponentially for the suggested values of the parameter c even when $c = 0$ and converges as the time increased. But on the contrary the second moment $M_2(x_0, Y; t)$ of the first-passage time of the process show decline for the same suggested values of c even when $c = 0$ and converges as time increased too.

4. Conclusion

In conclusion the advantage of this technique is to use the difference equation to approximate the ordinary differential equation since it is the discretization of the ODE. Also, the system of the solutions in equation (14) gives an explicit solution to the first – passage time moments for the birth and death diffusion process with immigration to a moving linear barriers. This increases the applicability of the diffusion process in stochastic modeling or in all area of applied probability theory. For further study I think it is possible to set up an exact or an approximated technique to predict the parameters of the suggested process and the open a possibility to be applied to a real life problem.

Support

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