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Approximate Controllability of Fractional Stochastic Perturbed Control Systems Driven by Mixed Fractional Brownian Motion

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Abstract In this paper, the approximate controllability of nonlinear Fractional order $0 < \alpha < 1$ Riemann-Liouville type stochastic perturbed control systems driven by mixed fractional Brownian motion in a real separable Hilbert spaces has been studied by using Krasnoselskii's fixed point theorem, stochastic analysis theory, fractional calculus and some sufficient conditions.

Keywords: *approximate controllability, mixed fractional brownian motion, fixed point theorem, perturbed control systems, mild solution, control function*

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1. Introduction

The aim main of this paper is to study the principle concepts of the approximate controllability for complicated classes of fractional order $0 < \alpha < 1$ Riemann-Liouville type stochastic perturbed control systems driven by mixed fractional Brownian motion The following form is the system under our consideration,

$$
\begin{aligned}
{}^{L}D_t^{\alpha} x(t) &= -(A + \Delta A)x(t) + Bu(t) \\
&+ F\left(t, x(t), \int_0^t h(t, s, x(s))ds\right) \\
&+ G\left(t, x(t), \int_0^t g(s, x(s))dW_{(s)}\right)\frac{d\,W_{(t)}}{dt} \\
&+ \sigma(t)\frac{dW_{(t)}^{H}}{dt}, \quad t \in [0, T]\ ,\ 0 < \alpha < 1 \\
{}^{L}D_t^{\alpha-1} x(t)|_{t=0} &= x_0
\end{aligned}
$$

where, ${}^{L}D_t^{\alpha}$ the Riemann-Liouville fractional derivative of order $0 < \alpha < 1$.$-A$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $S(t), t \geq 0$, $-\Delta A$ is a bounded linear operator in a real separable Hilbert space X. Assume that $0 \in \rho(A + \Delta A)$. The space X_β is a Hilbert such that $X_\beta = D(A + \Delta A)^{\beta})$, $0 \leq \beta < 1$, equipped with norm $\|x\|_\beta = \left\|(A + \Delta A)^{\beta} x\right\|_X$.

$$
\begin{aligned}
x(t) &\in C_{1-\alpha}\left([0, T]; L^2(\Omega, X_\beta)\right) \\
&= \left\{x(t); t^{1-\alpha}\, x(t) \in C\left([0, T]; L^2(\Omega, X_\beta)\right)\right\}
\end{aligned}
$$

is a continuous at $t \in [0, T]$ } with the norm $\|x\|_{C_{1-\alpha}} = (\sup_{t\epsilon[0,T]} (t^{1-\alpha}\, E\|x(t)\|_\beta)^2)^{\frac{1}{2}}$. x_0 is $\mathbf{F}_0$–measurable X- valued random variable independent of W and W^H which defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. The control function $u(.) \in L^2_{\mathcal{F}}([0,T]; U)$, U is a Hilbert space and the operator B from U into X is a bounded linear operator such that there exists constant $L_B > 0$, $\|Bu\| \leq L_B\|u\|$. $W = \{W_{(t)}, t \in [0, T]\}$ is a standard cylindrical Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with values in a Hilbert space K. Let Q be a positive, self –adjoint and trace class operator on K and let $L_2(K,X)$ be the space of all Q -Hilbert-Schmidt operators acting between K and X equipped with the Hilbert-Schmidt norm $\|.\|_{L_2}$. $W^H = \{W_{(t)}^H, t \in [0,T]\}$ is a Q-fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$ defined in a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with values in a Hilbert space Y, such that Q is a positive ,self –adjoint and trace class operator on Y and let $L_2^0(Y;X)$ be the space of all Q -Hilbert-Schmidt operators acting between Y and X equipped with the Hilbert-Schmidt norm $\|.\|_{L_2^0}$. The functions $F: [0,T] \times X_\beta \times X_\beta \to X$, $h: [0,T] \times [0,T] \times X_\beta \to X_\beta$, $G: [0,T] \times X_\beta \times X_\beta \to L_2(K;X)$ and $\sigma: [0,T] \to L_2^0(Y,X)$ are continuous functions.

Approximate controllability of stochastic control system driven by fractional Brownian motion has been interested by many authors; Sakthivel [19] study for the approximate controllability of impulsive stochastic systems with fractional Brownian motion. Guendouzi and Idrissi, [7] established and discussed the approximate controllability result of a class of dynamic control systems described by nonlinear fractional stochastic functional

differential equations in Hilbert space driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Ahmed [2] investigate the approximate controllability problem for the class of impulsive neutral stochastic functional differential equations with finite delay and fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ in a Hilbert space. Abid, Hasan and Quaez [1] studied the Approximate controllability of fractional stochastic integro-differential equations which is derived by mixed type of fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ and wiener process in real separable Hilbert space.

In this paper we will study the approximate controllability of nonlinear stochastic system. More precisely, we shall formulate and prove sufficient conditions for the Approximate controllability of Fractional order $0 < \alpha < 1$ Riemann-Liouville type stochastic perturbed control systems driven by mixed fractional Brownian motion in a real separable Hilbert spaces.

The rest of this paper is organized as follows, in section 2, we will introduced some concepts, definitions and some lemmas of semigroup theory and fractional stochastic calculus which are useful for us here. In section 3, we will prove our main result.

2. Preliminaries

In this section, we introduce some notations and preliminary results, which we needed to establish our results.

Definition (2.1), [5]:

Let H be a constant belonging to (0, 1). A one dimensional fractional Brownian motion $B^H = \{B^H_{(t)}, t \geq 0\}$ of Hurst index H is a continuous and centered Gaussian process with covariance function

$$E\left(B^H_{(t)} B^H_{(s)}\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \text{ for } t, s \geq 0. \quad (1)$$

- If $H = \frac{1}{2}$, then the increments of B^H are non-correlated, and consequently independent. So B^H is a Wiener Process which we denote further by B.
- If $H \in (\frac{1}{2}, 1)$ then the increments are positively correlated.
- If $H \in (0, \frac{1}{2})$ then the increments are negative correlated.

B^H has the integral representation

$$B^H_{(t)} = \int_0^t K_H(t,s)\, dB_{(s)} \quad (2)$$

where, B is a wiener process and the kernel $K_H(t,s)$ defined as

$$K_H(t,s) = cH\, s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\, du \quad (3)$$

$$\frac{\partial K}{\partial t}(t,s) = cH \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}} \quad (4)$$

$cH = \left[\frac{H(2H-1)}{\beta\left(2-2H, H-\frac{1}{2}\right)}\right]^{\frac{1}{2}}$, $t > s$ and β is a beta function.

In the case $H = \frac{1}{2}$, we shall use Ito Isometry theorem

***Lemma (2.1)*, "Ito isometry theorem", [11]:**

Let V [0,T] be the class of functions such that $f: [0,T] \times \Omega \to R$, f is measurable, $\mathcal{F}_t$- adapted and $E\left[\int_0^T \left(f(t,\omega)\right)^2 dt\right] \leq \infty$. Then for every $f \in V$ [0,T], we have

$$E\left[\int_0^T f(t,\omega) dB(t)\right]^2 = E\left[\int_0^T \left(f(t,\omega)\right)^2 dt\right] \quad (5)$$

where B is a wiener process.

Now, we denote by ξ the set of step functions on [0, T]. If $\Phi \in \xi$ then, we can write it the form as:

$$\Phi(t) = \sum_{k=1}^n a_k\, 1_{[t_k, t_{k+1}]}(t), \text{ where } t \in [0,T].$$

The integral of a step function $\Phi \in \xi$ with respect to one dimensional fractional Brownian motion is defined

$$\int_0^T \Phi(t) dB^H_t = \sum_{k=1}^n a_k (B^H_{t_{k+1}} - B^H_{t_k}),$$

where $a_k \in R$,

$$0 = t_1 < t_2 < \ldots\ldots < t_{n+1} = T.$$

Let $\mathcal{H}$ be the Hilbert space defined as the closure of ξ with respect to the scalar product $\langle 1_{[0,t]}, 1_{[0,s]} \rangle = R_H(t,s) = E(B^H_t B^H_s)$. The mapping $1_{[0,t]} \to \{B^H(t), t \in [0,T]\}$ can be extended to an isometry between $\mathcal{H}$ and $\text{span}^{L^2(\Omega)}\left\{B^H_{(t)}, t \in [0,T]\right\}$ .i.e. the mapping $\mathcal{H} \to L^2(\Omega, \mathcal{F}, P)$, $\Phi \to \int_0^T \Phi(t) dB^H_t$ is isometry.

Remark (2.1):

- If $H = \frac{1}{2}$ and $\mathcal{H} = L^2([0,T])$ then by use Ito isometry, we have

$$E\left(\int_0^T \Phi(t) dB\right)^2 = \int_0^T (\Phi(t))^2 dt \quad (6)$$

- If $H > \frac{1}{2}$, we have

$$R_H(s,t) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), t, s \geq 0 \quad (7)$$

$$\frac{\partial R_H}{dt} = H\left(|t|^{2H-1} - |t-s|^{2H-1}\right) \quad (8)$$

$$\partial R^2_H = H(2H-1)|t-s|^{2H-2} dsdt$$

Lemma (2.2), [6]:

For any functions $\Phi, \varphi \in L^2[0,T] \cap L^1[0,T]$, we have

i) $E\left(\int_0^T \Phi(t) dB^H_{(t)} \int_0^T \varphi(s) dB^H_{(s)}\right) = H(2H-1) \times \int_0^T . \int_0^T \Phi(t)\, \varphi(s) |t-s|^{2H-2} dsdt$

ii) $E\left(dB^H_{(t)} dB^H_{(s)}\right) = \frac{\partial R^2_H}{dsdt} = H(2H-1)|t-s|^{2H-2} dsdt$

From this Lemma above, we obtain

$$E\left(\int_0^T \Phi(t) dB^H_{(t)}\right)^2 = H(2H-1) \times \int_0^T \int_0^T \Phi(s)\Phi(t)|t-s|^{2H-2} dsdt \quad (9)$$

Remark (2.2), [6]:

The space $\mathcal{H}$ contains the set of functions $\Phi \in L^2[0,T]$, such that, $\int_0^T . \int_0^T \Phi(s)\Phi(t) \left|t-s\right|^{2H-2} dsdt < \infty$, which includes $L^{\frac{1}{H}}([0,T])$.

Now,

Let $\hat{\mathcal{H}}$ be the Banach space of measurable functions on [0, T], such that

$$\|\Phi\|^2_{\hat{\mathcal{H}}} = H(2H-1) \int_0^T \int_0^T \Phi(s)\Phi(t) \left|t-s\right|^{2H-2} dsdt < \infty \quad (10)$$

Lemma (2.3), [10]:

$$L([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset \mathcal{H} \subset \mathcal{K}.$$

Suppose that there exists a complete orthonormal system $\{e_n\}_{n=1}^{\infty}$ in Y. Let $Q \in L(Y,Y)$ be the operator defined by $Qe_n = \lambda_n e_n$, where $\lambda_n \geq 0$ (n=1,2,….) are non-negative real numbers with finite trace $Tr\ Q = \sum_{n=1}^{\infty} \lambda_n < \infty$. The infinite dimensional fractional Brownian motion on Y can be defined by using covariance operator Q as

$$W_{(t)}^{H} = W_{Q\ (t)}^{H} = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\ e_n\ B_{n\ (t)}^{H},$$

where $B_{n\ (t)}^{H}$ are one dimensional fractional Brownian motions mutually independent on $(\Omega, \mathcal{F}, P)$.

In order to defined stochastic integral with respect to the Q-fractional Brownian motion. We introduce the space $L_2^0(Y,X)$ of all Q-Hilbert- Schmidt operators that is with the inner product $\langle \Phi, \varphi \rangle_{L_2^0} = \sum_{n=1}^{\infty} \langle \Phi e_n, \varphi e_n \rangle$ is a separable Hilbert space.

***Lemma (2.4),* [10]**:

Let $\{\Phi(t)\}_{t\in[0,T]}$ be a deterministic function with values in $L_2^0(Y,X)$ The stochastic integral of Φ with respect to W^H is defined by

$$\begin{aligned}\int_0^t \Phi(s) dW_{(s)}^{H} &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\, \Phi(s) e_n dB_{n(s)}^{H} \\ &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\, (\sqrt{K_H^*}(\Phi e_n))(s) dB_{n(s)}\end{aligned} \quad (11)$$

***Lemma (2.5),* [10]**:

If φ: $[0, b] \to L_2^0(Y,X)$ satisfies $\int_0^T \|\varphi(s)\|_{L_2^0}^2 ds < \infty$ then the above sum in (11) is well defined as an X-valued random variable and we have

$$E\left\| \int_0^t \varphi(s) dW_{(S)}^{H} \right\|^2 \leq 2H\, t^{2H-1} \int_0^t \|\varphi(s)\|_{L_2^0}^2 ds \quad (12)$$

***Definition (2.2),* [18]**:

The Riemann - Liouvill derivative of order $\alpha > 0$ with lower limit zero for a function f can be written as:

$${}^{L}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} \quad (13)$$

where, $t > 0$, $n-1 < \alpha < n$.

***Definition (2.4),* [18]**:

The Laplace transform of the Riemann-Liouville fractional derivation of order $\alpha > 0$ gives as:

$$\mathcal{L}\{LD_t^{\alpha} f(t)\} = \lambda^{\alpha}\mathcal{L}(f(t))(\lambda) - \sum_{k=0}^{n-1} \lambda^k [\, D_t^{\alpha-k-1} f(t)]_{t=0}(\lambda) \quad (14)$$

where, $n-1 < \alpha < n$.

***Lemma (2.6),* [17]**:

Let $-A$ be the infinitesimal generator of an analytic semigroup S(t), $t \geq 0$ on a Hilbert space X. If $-\Delta A$ is a bounded linear operator on X then $-(A+\Delta A)$ is the infinitesimal generator of an analytic semigroup $\check{T}(t)$, $t \geq 0$ on X.

Remark (2.3):

Assume that $\check{T}(t)$, $t \geq 0$ is a compact analytic semigroup of uniformly bounded operators in X , that is , there exists $M > 1$ such that $\|\check{T}(t)x\| \leq M$.

Definition (2.5):

An X_β-valued process x(t) is called a mild solution of the system (1) if $x(t) \in C_{1-\alpha}\left([0,T]; L^2(\Omega, X_\beta)\right)$ and, for $t \in [0,T]$ satisfies the integral equation

$$\begin{aligned} x(t) &= \check{T}_{\alpha}(t)(t)^{\alpha-1} x_0 \\ &+ \int_0^t \check{T}_{\alpha}(t-s)(t-s)^{\alpha-1} Bu(s) ds \\ &+ \int_0^t \check{T}_{\alpha}(t-s)(t-s)^{\alpha-1} F\left(s, x(s), \int_0^s h(s,r,x(r)) dr\right) ds \quad (15) \\ &+ \int_0^t \check{T}_{\alpha}(t-s)(t-s)^{\alpha-1} G\begin{pmatrix} s, x(s), \\ \int_0^s g(s, x(r)) dW_{(r)} \end{pmatrix} dW_{(s)} \\ &+ \int_0^t \check{T}_{\alpha}(t-s)(t-s)^{\alpha-1} \sigma(s) dW_{(s)}^{H} \end{aligned}$$

where, $\check{T}_{\alpha}(t) = \int_0^{\infty} \alpha r M_{\alpha}(r) T(t^{\alpha} r)\, dr$

$M_{\alpha}(r)$ is a Mainardi's function.

Lemma (2.7):

If $\check{T}(t), t \geq 0$ is a compact analytic semigroup then the family of operators $\check{T}_{\alpha}(t)$, $t \geq 0$ have the following properties:

i. For any fixed $t \geq 0$, the operator $\check{T}_{\alpha}(t)$ is a linear and bounded, i.e. for any $x \in X$, there exists $M \geq 1$ such that $\|\check{T}_{\alpha}(t)x\|_X \leq \frac{M}{\Gamma(\alpha)} \|x\|_X$.

ii. For any $x \in X_\beta$, there exists $M \geq 1$ such that $\|\check{T}_{\alpha}(t)x\|_\beta \leq \frac{M}{\Gamma(\alpha)} \|x\|_\beta$.

iii. For any $x \in X$, $\|\check{T}_{\alpha}(t)x\|_\beta \leq \tilde{N}_{\alpha,\beta}\, t^{-\alpha\beta} \|x\|_X$ where, $\tilde{N}_{\alpha,\beta} = \alpha \acute{K}_{\beta} \frac{\Gamma(2-\beta)}{\Gamma(\alpha(1-\beta)+1)}$

iv. $\{\check{T}_{\alpha}(t), t \geq 0\}$ is a strongly continuous, which mean that for every $x \in X$ and $0 \leq t_1 < t_2 \leq T$ then $\|\check{T}_{\alpha}(t_2)x - \check{T}_{\alpha}(t_1)x\|_X \to 0$ if $t_2 \to t_1$.

v. The operator $\check{T}_{\alpha}(t)$ is a compact operator in X for $t > 0$.

3. Main Result of the Approximately Controllable

In this section, we formulate and prove the result on approximate controllability of nonlinear fractional stochastic perturbed control system driven by mixed fractional Brownian motion in (1). To establish our results, we introduce the following assumptions:

a) The operator $\check{T}(t)$ is a compact for any $t > 0$.

b) The linear fractional order system of corresponding the system (3.41) which has following form:

$$\begin{aligned} {}^{L}D_t^{\alpha} x(t) &= Ax(t) + Bu(t), t \in [0,T] \\ {}^{L}D_t^{\alpha-1} x(t)|_{t=0} &= x_0, \ \frac{1}{2} < \alpha < 1 \end{aligned} \quad (16)$$

is an approximately controllable on [0,T].

c) The functions $F:[0,T] \times X_\beta \times X_\beta \to X$, $h:[0,T] \times [0,T] \times X_\beta \to X_\beta$ are satisfying linear growth and Lipschitz conditions. This mean that, for any $x , y \in X_\beta$, there exists positive constants $K_1, K_2 > 0\ and\ K_3, K_4 > 0$ such that

$$\left\| F\left(t, x(t), \int_0^t h(t,s,x(s)) ds\right) - \right.$$
$$F t, y, s, 0 t h t, s, y(s) ds\ X2 \leq K1x - y\beta 2$$

$$\left\| F\left(t, x(t), \int_0^t h(t,s,x(s)) ds\right) \right\|_X^2 \leq K_2(1 + \|x\|_\beta^2)$$
$$\|h(t,s,x) - h(t,s,y)\|_\beta^2 \leq K_3 \|x - y\|_\beta^2 \quad ,$$
$$\|h(t,s,x)\|_\beta^2 \leq K_4(1 + \|x\|_\beta^2)$$

Also, F is a uniformly bounded. In other word, there exists $D_1 > 0$ such that

$$\left\| F\left(t, x(t), \int_0^t h(t,s,x(s))ds\right)\right\|_X^2 < D_1, \text{ for } t \in [0, T]$$

d) The function σ: $[0,T] \to L_2^0(Y;X)$ satisfies, for every $t \in [0, T]$

$\int_0^t \|\sigma(s)\|_{L_2^0}^2 ds < \infty$ and there exists $C_1 > 0$ such that $\sup_{t\in[0,T]} \|\sigma(s)(t)\|_{L_2^0}^2 \le C_1$.

e) $\tilde{G}$ is F_t – adapted with respect to t, such that, for every $t \in [0,T]$, satisfy the following:

i. $E\left\|(\tilde{G}x)(s)\right\|_{L_2}^2$ Exists.

ii. $\int_0^t E\left\|(\tilde{G}x)(t)\right\|_{L_2}^2 ds < \infty$.

iii. There exists $D_2 > 0$, such that

$$\sup_{t\in[0,T]} E\left\|(\tilde{G}x)(t)\right\|_{L_2}^2 < D_2.$$

where, $(\tilde{G}x)(t) = G\left(t, x(t), \int_0^t g(s, x(s))dW_{(s)}\right)$

Definition (3.1):

The system (1) is said to be approximately controllable on [0,T] if the reachable set $\mathcal{R}(T)$ is dense in the space $L^2(\Omega, X)$. This mean that $\overline{\mathcal{R}(T)} = L^2(\Omega, X)$. where, $\mathcal{R}(T) = \{ x(T,u) : u \in L^2_{\mathcal{F}}([0,T]; U)\}$

Now,

The controllability operator Γ_s^T associated with control system (16) is defined by

$$\Gamma_s^T = \int_s^T (T-t)^{\alpha-1} \check{T}_\alpha(T-t)\, BB^* \check{T}_\alpha^*(T-t)dt \quad (17)$$

Also, for any $\theta > 0$ and $0 \le s < T$, the operator $R(\theta, \Gamma_s^T)$ is defined by

$$R(\theta, \Gamma_s^T) = (\theta I + \Gamma_s^T)^{-1} \quad (18)$$

where, B^* and $\check{T}_\alpha^*$ are the adjoint operators for B and $\check{T}_\alpha$ respectively.

Lemma (3.1), [12]:

The linear fractional order deterministic system in (1) is an approximately controllable on [0, T] if and only if the operator $\theta R(\theta, \Gamma_0^T) \to 0$ as $\theta \to 0^+$, $\|\theta R(\theta, \Gamma_s^T)\| \le 1$.

Lemma (3.2), [13]: For any $x_T \in L^2(\Omega, \mathcal{F}_T, X_\beta)$, there exists $\hat{H} \in L^2\left(\Omega; L^2([0,T]; L_2^0(Y,X))\right)$ and $\phi \in L^2\left(\Omega; L^2([0,T]; L_2(K;X))\right)$, such that

$$x_T = Ex_T + \int_0^T \phi(s)dW_{(s)} + \int_0^T \hat{H}(s)dw_{(s)}^H \quad (19)$$

where, $\sup_{t\in[0,T]} E\|\phi(t)\|_{L_2}^2 \le \check{C}_1$,

$$\sup_{t\in[0,T]} E\left\|\hat{H}(t)\right\|_{L_2^0}^2 \le \check{C}_2$$

Now, For any $\theta > 0$ and any $x_T \in L^2(\Omega, \mathcal{F}_T, X_\beta)$, we defined the control function of the system (1) in the following form:

$$\begin{aligned} u^\theta(t,x) &= B^*\check{T}_\alpha^*(T-t)R(\theta,\Gamma_0^T)\left(Ex_T - \check{T}_\alpha(T)(T)^{\alpha-1}x_0\right) \\ &+B^*\check{T}_\alpha^*(T-t)\int_0^t R\left(\theta,\Gamma_s^T\right)\phi(s)dW_{(s)} \\ &+B^*\check{T}_\alpha^*(T-t)\int_0^t R\left(\theta,\Gamma_s^T\right)\hat{H}(s)dW_{(s)}^H \\ &-B^*\check{T}_\alpha^*(T-t)\int_0^t \begin{bmatrix} R(\theta,\Gamma_s^T)\check{T}_\alpha(T-s)(T-s)^{\alpha-1} \\ F\left(s, x(s)\int_0^s h(s,r,x(r))dr\right)\end{bmatrix} ds \\ &-B^*\check{T}_\alpha^*(T-t)\int_0^t \begin{bmatrix} R(\theta,\Gamma_s^T)\check{T}_\alpha(T-s)(T-s)^{\alpha-1} \\ G\left(s, x(s), \int_0^s g(v,x(v))dW_{(v)}\right)\end{bmatrix} dW_{(s)} \quad (20) \\ &-B^*\check{T}_\alpha^*(T-t)\int_0^t \begin{bmatrix} R(\theta,\Gamma_s^T)\check{T}_\alpha(T-s) \\ (T-s)^{\alpha-1}\sigma(s)\end{bmatrix} dW_{(s)}^H \end{aligned}$$

Lemma (3.3): There exists positive real constant N_C, such that for all $x \in C\left([0,T]; L^2(\Omega, X_\beta)\right)$,

$$E\left\|u^\theta(t,x)\right\|^2 \le N_C \quad (21)$$

where,

$$\begin{aligned} N_C &= \frac{12L_B^2M^4}{\theta^2(\Gamma(\alpha))^4}(E\|x_T\|_\beta^2 + (T)^{2\alpha-2}E\|x_0\|_\beta^2) \\ &+6\frac{L_B^2}{\theta^2(\Gamma(\alpha))^2}M^2T\check{C}_1 + \frac{12HT^{2H}L_B^2M^2}{\theta^2(\Gamma(\alpha))^2}\check{C}_2 \\ &+\frac{6L_B^2M^2T^{2\alpha-2\alpha\beta}}{\theta^2(\Gamma(\alpha))^2(1-2\alpha\beta)(2\alpha-1)}\check{N}_{\alpha,\beta}{}^2D_1 \\ &+\frac{6L_B^2M^2T^{2\alpha-2\alpha\beta}}{\theta^2(\Gamma(\alpha))^2(1-2\alpha\beta)(2\alpha-1)}\check{N}_{\alpha,\beta}{}^2D_2 \\ &+\frac{12HL_B^2M^2T^{2\alpha-2\alpha\beta+2H-1}}{\theta^2(\Gamma(\alpha))^2(2\alpha-1)(1-2\alpha\beta)}\check{N}_{\alpha,\beta}{}^2C_1 \end{aligned}$$

Proof

Let $x \in C\left([0,T]; L^2(\Omega, X_\beta)\right)$ and $T > 0$ be a fixed. From the equation (21), we have:

$$\begin{aligned} &E\left\|u^\theta(t,x)\right\|^2 \\ &\le 6E\left\|B^*\check{T}_\alpha^*(T-t)R(\theta,\Gamma_0^T)\begin{pmatrix} Ex_T - \\ \check{T}_\alpha(T)(T)^{\alpha-1}x_0\end{pmatrix}\right\|^2 \\ &+6E\left\|B^*\check{T}_\alpha^*(T-t)\int_0^t R\left(\theta,\Gamma_s^T\right)\phi(s)dW_{(s)}\right\|^2 \\ &+6E\left\|B^*\check{T}_\alpha^*(T-t)\int_0^t R\left(\theta,\Gamma_s^T\right)\hat{H}(s)dW_{(s)}^H\right\|^2 \\ &+6E\left\|B^*\check{T}_\alpha^*(T-t)\int_0^t \begin{bmatrix} R(\theta,\Gamma_s^T)\check{T}_\alpha(T-s)(T-s)^{\alpha-1} \\ F\left(s, x(s), \int_0^t h(s,r,x(r))dr\right)\end{bmatrix} ds\right\|^2 \\ &+6E\left\|B^*\check{T}_\alpha^*(T-t)\int_0^t \begin{bmatrix} R(\theta,\Gamma_s^T)\check{T}_\alpha(T-s)(T-s)^{\alpha-1} \\ G\left(s, x(s), \int_0^s g(v,x(v))dW_{(v)}\right)\end{bmatrix} dW_{(s)}\right\|^2 \end{aligned}$$

$$+6E\left\|B^{*}\breve{T}_{\alpha}^{*}(T-t)\int_{0}^{t}\begin{bmatrix}R(\theta,\Gamma_{s}^{T})\breve{T}_{\alpha}(T-s)\\(T-s)^{\alpha-1}\sigma(s)\end{bmatrix}dW_{(s)}^{H}\right\|^{2}$$

Applying Holder's inequality and by using Ito isometry , Lemmas (2.5), (2.7) and the assumptions (a)-(e), we obtain

$$E\left\|u^{\theta}(t,x)\right\|^{2}\leq N_{C}$$

where,

$$N_{C}=\frac{12L_{B}^{2}M^{4}}{\theta^{2}(\Gamma(\alpha))^{4}}(E\|x_{T}\|_{\beta}^{2}+(T)^{2\alpha-2}E\|x_{0}\|_{\beta}^{2})$$
$$+6\frac{L_{B}^{2}}{\theta^{2}(\Gamma(\alpha))^{2}}M^{2}T\breve{C}_{1}+\frac{12HT^{2H}L_{B}^{2}M^{2}}{\theta^{2}(\Gamma(\alpha))^{2}}\breve{C}_{2}$$
$$+\frac{6L_{B}^{2}M^{2}T^{2\alpha-2\alpha\beta}}{\theta^{2}(\Gamma(\alpha))^{2}(1-2\alpha\beta)(2\alpha-1)}\breve{N}_{\alpha,\beta}{}^{2}D_{1}$$
$$+\frac{6L_{B}^{2}M^{2}T^{2\alpha-2\alpha\beta}}{\theta^{2}(\Gamma(\alpha))^{2}(1-2\alpha\beta)(2\alpha-1)}\breve{N}_{\alpha,\beta}{}^{2}D_{2}$$
$$+\frac{12HL_{B}^{2}M^{2}T^{2\alpha-2\alpha\beta+2H-1}}{\theta^{2}(\Gamma(\alpha))^{2}(2\alpha-1)(1-2\alpha\beta)}\breve{N}_{\alpha,\beta}{}^{2}C_{1}$$

Now,

For any $\theta > 0$, consider the operator Ψ_{θ} on $C_{1-\alpha}\left([0,T];\,L^{2}(\Omega,X_{\beta})\right)$ defined as follows:

$$(\Psi_{\theta}x)(t)=\breve{T}_{\alpha}(t)(t)^{\alpha-1}x_{0}$$
$$+\int_{0}^{t}\breve{T}_{\alpha}(t-s)(t-s)^{\alpha-1}Bu^{\theta}(t,x)ds$$
$$+\int_{0}^{t}\breve{T}_{\alpha}(t-s)(t-s)^{\alpha-1}F\left(s,x(s),\int_{0}^{t}h(s,r,x(r))dr\right)ds$$
$$+\int_{0}^{t}\begin{bmatrix}\breve{T}_{\alpha}(t-s)(t-s)^{\alpha-1}\\G\left(s,x(s),\int_{0}^{s}g(v,x(v))dW_{(v)}\right)\end{bmatrix}dW_{(s)}$$
$$+\int_{0}^{t}\breve{T}_{\alpha}(t-s)(t-s)^{\alpha-1}\sigma(s)dW_{(s)}^{H}$$
$$(22)$$

Also, for any $\delta > 0$, the subset B_{δ} of $C_{1-\alpha}\left([0,T];\,L^{2}(\Omega,X_{\beta})\right)$ is define as $B_{\delta}=\left\{x(t)\in C_{1-\alpha}\left([0,T];\,L^{2}(\Omega,X_{\beta})\right):\|x(t)\|_{C_{1-\alpha}}^{2}\leq\delta\right\}$.

Lemma (3.4):

for any $\theta > 0$, There exists $\breve{\delta} > 0$ *such that* $\Psi_{\theta}(B_{\breve{\delta}})\subset B_{\delta}$.

Proof:

To prove that there exists $\delta > 0$ *such that* $\Psi_{\theta}(B_{\delta})\subset B_{\delta}$, in other word, $\|\Psi_{\theta}x(t)\|_{C_{1-\alpha}}^{2}\leq\delta$, for each $x(t)\in B_{\delta}$.

Suppose that this is not true, then for each $\delta > 0$, there exists $x(t)\in B_{\delta}$ such that $\|\Psi_{\theta}x(t)\|_{C_{1-\alpha}}^{2}>\delta$, for $t\in[0,T]$, t may depending upon δ. However, on the other hand, we have

$$t^{2-2\alpha}E\|\Psi_{\theta}x(t)\|_{\beta}^{2}\leq 5t^{2-2\alpha}E\left\|\breve{T}_{\alpha}(t)(t)^{\alpha-1}x_{0}\right\|_{\beta}^{2}$$
$$+5t^{2-2\alpha}E\left\|\int_{0}^{t}\breve{T}_{\alpha}(t-s)(t-s)^{\alpha-1}Bu^{\theta}(s,x)ds\right\|_{\beta}^{2}$$
$$+5t^{2-2\alpha}\left\|E\int_{0}^{t}\begin{bmatrix}\breve{T}_{\alpha}(t-s)(t-s)^{\alpha-1}\\F\left(s,x(s),\int_{0}^{t}h(s,r,x(r))dr\right)\end{bmatrix}ds\right\|_{\beta}^{2}$$
$$+5t^{2-2\alpha}E\left\|\int_{0}^{t}\begin{bmatrix}\breve{T}_{\alpha}(t-s)(t-s)^{\alpha-1}\\G\left(s,x(s),\int_{0}^{s}g(v,x(v))dW_{(v)}\right)\end{bmatrix}dW_{(s)}\right\|_{\beta}^{2}$$
$$+5t^{2-2\alpha}E\left\|\int_{0}^{t}\breve{T}_{\alpha}(t-s)(t-s)^{\alpha-1}\sigma(s)dW_{(s)}^{H}\right\|_{\beta}^{2}$$

Applying Holder's inequality and by using Ito isometry , Lemmas (2.5), (2.7), (3.3) and the assumptions (a)-(e), we obtain

$$\sup_{t\in[0,T]}t^{2-2\alpha}E\|\Psi_{\theta}x(t)\|_{\beta}^{2}$$
$$\leq 5\frac{M^{2}}{(\Gamma(\alpha))^{2}}E\|x_{0}\|_{\beta}^{2}+\frac{5L_{B}^{2}T^{2-2\alpha\beta}N_{C}\breve{N}_{\alpha,\beta}^{2}}{(2\alpha-1)(1-2\alpha\beta)}$$
$$+\frac{5T^{2-2\alpha\beta}D_{1}\breve{N}_{\alpha,\beta}^{2}}{\left|(2\alpha-1)(1-2\alpha\beta)\right|}+\frac{5T^{2-2\alpha\beta}\breve{N}_{\alpha,\beta}^{2}D_{2}}{\left|(2\alpha-1)(1-2\alpha\beta)\right|}$$
$$+\frac{10HT^{1+2H-2\alpha\beta}}{\left|(2\alpha-1)(1-2\alpha\beta)\right|}\breve{N}_{\alpha,\beta}^{2}C_{1}$$

Hence,

$$\|\Psi_{\theta}x(t)\|_{C_{1-\alpha}}^{2}$$
$$\leq 5\frac{M^{2}}{(\Gamma(\alpha))^{2}}E\|x_{0}\|_{\beta}^{2}+\frac{5L_{B}^{2}T^{2-2\alpha\beta}N_{C}\breve{N}_{\alpha,\beta}^{2}}{(2\alpha-1)(1-2\alpha\beta)}$$
$$+\frac{5T^{2-2\alpha\beta}D_{1}\breve{N}_{\alpha,\beta}^{2}}{\left|(2\alpha-1)(1-2\alpha\beta)\right|}+\frac{5T^{2-2\alpha\beta}\breve{N}_{\alpha,\beta}^{2}D_{2}}{\left|(2\alpha-1)(1-2\alpha\beta)\right|}$$
$$+\frac{10HT^{1+2H-2\alpha\beta}}{\left|(2\alpha-1)(1-2\alpha\beta)\right|}\breve{N}_{\alpha,\beta}^{2}C_{1}$$

Therefore,

$$\delta<5\frac{M^{2}}{(\Gamma(\alpha))^{2}}E\|x_{0}\|_{\beta}^{2}+\frac{5L_{B}^{2}T^{2-2\alpha\beta}N_{C}\breve{N}_{\alpha,\beta}^{2}}{(2\alpha-1)(1-2\alpha\beta)}$$
$$+\frac{5T^{2-2\alpha\beta}D_{1}\breve{N}_{\alpha,\beta}^{2}}{\left|(2\alpha-1)(1-2\alpha\beta)\right|}+\frac{5T^{2-2\alpha\beta}\breve{N}_{\alpha,\beta}^{2}D_{2}}{\left|(2\alpha-1)(1-2\alpha\beta)\right|}$$
$$+\frac{10HT^{1+2H-2\alpha\beta}}{\left|(2\alpha-1)(1-2\alpha\beta)\right|}\breve{N}_{\alpha,\beta}^{2}C_{1}$$

By dividing both sides of above inequality by δ and taking the limit as $\delta\to\infty$, which is a contradiction. Thus, for each $\theta > 0$, there exists positive number $\breve{\delta}$ such that $\Psi_{\theta}(B_{\breve{\delta}})\subset B_{\breve{\delta}}$.

Now,

Let $\Psi_{\theta} = \Psi_{1} + \Psi_{2}$ where,

$$(\Psi_1 x)(t)=\int_0^t\begin{bmatrix}\breve{T}_\alpha(t-s)(t-s)^{\alpha-1}\\ F\left(s,x(s),\int_0^t h(s,r,x(r))dr\right)\end{bmatrix}ds \quad (23)$$

$$\begin{aligned}(\Psi_2 x)(t)&=\breve{T}_\alpha(t)(t)^{\alpha-1}x_0\\&+\int_0^t\breve{T}_\alpha(t-s)(t-s)^{\alpha-1}Bu^\theta(s,x)ds\\&+\int_0^t\begin{bmatrix}\breve{T}_\alpha(t-s)(t-s)^{\alpha-1}\\ G\left(s,x(s),\int_0^s g(v,x(v))dW_{(v)}\right)\end{bmatrix}dW_{(s)}\\&+\int_0^t\breve{T}_\alpha(t-s)(t-s)^{\alpha-1}\sigma(s)dW^H_{(s)}\end{aligned} \quad (24)$$

Lemma (3.5):

Assume that the assumptions (a) – (e) hold, then for any $\theta > 0$, and for any x, $y \in B_{\breve{\delta}}$, $(\Psi_1 y)(t) + (\Psi_2 x)(t) \in B_{\breve{\delta}}$, for $t \in [0,T]$.

Proof

Let x, $y \in B_{\breve{\delta}}$ and $0 \le t \le T$, we have

$$\begin{aligned}&t^{2-2\alpha}E\|(\Psi_1 y)(t)+(\Psi_2 x)(t)\|^2_\beta\\&\le 5t^{2-2\alpha}E\left\|\breve{T}_\alpha(t)(t)^{\alpha-1}x_0\right\|^2_\beta\\&+5t^{2-2\alpha}E\left\|\int_0^t\breve{T}_\alpha(t-s)(t-s)^{\alpha-1}Bu^\theta(s,x))ds\right\|^2_\beta\\&+5t^{2-2\alpha}\left\|E\int_0^t\begin{bmatrix}\breve{T}_\alpha(t-s)(t-s)^{\alpha-1}\\ F\left(s,y(s),\int_0^t h(s,r,y(r))dr\right)\end{bmatrix}ds\right\|^2_\beta\\&+5t^{2-2\alpha}E\left\|\int_0^t\begin{bmatrix}\breve{T}_\alpha(t-s)(t-s)^{\alpha-1}\\ G\left(s,x(s),\int_0^s g(v,x(v))dW_{(v)}\right)\end{bmatrix}dW_{(s)}\right\|^2_\beta\\&+5t^{2-2\alpha}E\left\|\int_0^t\breve{T}_\alpha(t-s)(t-s)^{\alpha-1}\sigma(s)dW^H_{(s)}\right\|^2_\beta\end{aligned}$$

Applying Holder's inequality and by using Ito isometry , Lemmas (2.5), (2.7), (3.3) and the assumptions (a)-(e), we obtain

$$\begin{aligned}&t^{2-2\alpha}E\|(\Psi_1 y)(t)+(\Psi_2 x)(t)\|^2_\beta\\&\le 5\frac{M^2}{(\Gamma(\alpha))^2}E\|x_0\|^2_\beta+\frac{5L_B^2T^{2-2\alpha\beta}N_C\breve{N}^2_{\alpha,\beta}}{(2\alpha-1)(1-2\alpha\beta)}\\&+\frac{5T^{2-2\alpha\beta}D_1\breve{N}^2_{\alpha,\beta}}{|(2\alpha-1)(1-2\alpha\beta)|}+\frac{5T^{2-2\alpha\beta}\breve{N}^2_{\alpha,\beta}D_2}{|(2\alpha-1)(1-2\alpha\beta)|}\\&+\frac{10HT^{1+2H-2\alpha\beta}}{|(2\alpha-1)(1-2\alpha\beta)|}\breve{N}^2_{\alpha,\beta}C_1\end{aligned}$$

which mean $\|(\Psi_1 y)(t) + (\Psi_2 x)(t)\|^2_{C_{1-\alpha}} \le \breve{\delta}$,then $(\Psi_1 y)(t) + (\Psi_2 x)(t) \in B_{\breve{\delta}}$

***Lemma (3.6)*:**

For any $t \in [0,T]$, the operator Ψ_1 is a contraction on $B_{\breve{\delta}}$, provided that $\gamma = \frac{\breve{N}^2_{\alpha,\beta}T^{2-2\alpha\beta}K_1}{|(2\alpha-1)(1-2\alpha\beta)|} < 1$

Proof:

Let $t \in [0,T]$, and $y_1, y_2 \in B_{\breve{\delta}}$, we have

$$\begin{aligned}&t^{2-2\alpha}E\|\Psi_1 y_1(t)-\Psi_1 y_2(t)\|^2_\beta\\&=t^{2-2\alpha}E\int_0^t\left\{\begin{bmatrix}\breve{T}_\alpha(t-s)(t-s)^{\alpha-1}\\ \times\begin{bmatrix}F\left(s,y_1(s),\int_0^t h(s,r,y_1(r))dr\right)\\ -F\left(s,y_2(s),\int_0^t h(s,r,y_2(r))dr\right)\end{bmatrix}_\beta\end{bmatrix}^2\right\}ds\end{aligned}$$

By using lemma (2.7) and assumption (c), we get

$$\begin{aligned}&t^{2-2\alpha}E\|\Psi_1 y_1(t)-\Psi_1 y_2(t)\|^2_\beta\\&\le\frac{t}{|2\alpha-1|}\int_0^t\breve{N}^2_{\alpha,\beta}(t-s)^{-2\alpha\beta}K_1E\|y_1(s)-y_2(s)\|^2_\beta ds\end{aligned}$$

By taking the supremum over $t \in [0,T]$ for both sides, we have

$$\begin{aligned}&\sup_{t\in[0,T]}\{t^{2-2\alpha}E\|\Psi_1 y_1(t)-\Psi_1 y_2(t)\|^2_\beta\}\\&\le\frac{\breve{N}^2_{\alpha,\beta}T^{2-2\alpha\beta}K_1}{|(2\alpha-1)(1-2\alpha\beta)|}\sup_{t\in[0,T]}\left\{E\|y_1(t)-y_2(t)\|^2_\beta\right\}\end{aligned}$$

Therefore,

$$\begin{aligned}&\|\Psi_1 y_1(t)-\Psi_1 y_2(t)\|^2_{C_{1-\alpha}}\\&\le\frac{\breve{N}^2_{\alpha,\beta}T^{2-2\alpha\beta}K_1}{|(2\alpha-1)(1-2\alpha\beta)|}\|y_1(t)-y_2(t)\|^2_{C_{1-\alpha}}\\&\le\gamma\|y_1(t)-y_2(t)\|^2_{C_{1-\alpha}}\end{aligned}$$

Where, $\gamma = \frac{\breve{N}^2_{\alpha,\beta}T^{2-2\alpha\beta}K_1}{|(2\alpha-1)(1-2\alpha\beta)|} < 1$, hence Ψ_1 is a contraction.

Lemma (3.7):

Assume that the assumptions (a) – (e) hold, then the operator Ψ_2 maps bounded sets into bounded sets in $B_{\breve{\delta}}$.

Proof:

Let $x(t) \in B_{\breve{\delta}}$, for $t \in [0, T]$, we have

$$\begin{aligned}&t^{2-2\alpha}E\|\Psi_2 x(t)\|^2_\beta\\&\le 4t^{2-2\alpha}E\left\|\breve{T}_\alpha(t)(t)^{\alpha-1}x_0\right\|^2_\beta\\&+4t^{2-2\alpha}E\left\|\int_0^t\breve{T}_\alpha(t-s)(t-s)^{\alpha-1}Bu^\theta(s,x)ds\right\|^2_\beta\\&+4t^{2-2\alpha}E\left\|\int_0^t\begin{bmatrix}\breve{T}_\alpha(t-s)(t-s)^{\alpha-1}\\ G\left(s,x(s),\int_0^s g(v,x(v))dW_{(v)}\right)\end{bmatrix}dW_{(s)}\right\|^2_\beta\\&+4t^{2-2\alpha}E\left\|\int_0^t\breve{T}_\alpha(t-s)(t-s)^{\alpha-1}\sigma(s)dW^H_{(s)}\right\|^2_\beta\end{aligned}$$

Applying Holder's inequality and by using Ito isometry , Lemmas (2.5), (2.7), (3.3) and the assumptions (a)-(e), we obtain

$$t^{2-2\alpha}E\,\|\Psi_2x(t)\|_\beta^2 \le \text{Đ}$$

where, $\text{Đ} = 4\frac{M^2}{(\Gamma(\alpha))^2}\quad E\,\|x_0\|_\beta^2 + 4\frac{L_B{}^2T^{2-2\alpha\beta}N_C\breve{N}_{\alpha,\beta}^2}{(2\alpha-1)(1-2\alpha\beta)} + 4\frac{T^{2-2\alpha\beta}\breve{N}_{\alpha,\beta}^2D_2}{|(2\alpha-1)(1-2\alpha\beta)|} + 8\frac{H\,T^{1+2H-2\alpha\beta}}{|(2\alpha-1)(1-2\alpha\beta)|}\breve{N}_{\alpha,\beta}^2C_1$

By taking the supremum over $t\in[0,T]$ for both sides, we get $\sup_{t\in[0,T]} t^{2-2\alpha}E\|\Psi_2x(t)\|_\beta^2 \le \text{Đ}$

Therefore, for each $x \in B_{\breve{\delta}}$, we get $\|\Psi_2x(t)\|_{C_{1-\alpha}}^2 \le$ Đ. Then Ψ_2 maps bounded sets into bounded sets in $B_{\breve{\delta}}$.

***Lemma (3.8)*:**

Assume that the assumptions (a) – (e) hold, then Ψ_2 is a continuous on $B_{\breve{\delta}}$.

Proof:

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $B_{\breve{\delta}}$ such that $x_n \to x$ as $n\to\infty$ in $C_{1-\alpha}\left([0,T];L^2(\Omega,X_\beta)\right)$. For each $t\in[0,T]$, we have

$$t^{2-2\alpha}E\|(\Psi_2x_n)(t)-(\Psi_2x)(t)\|_\beta^2$$
$$\le 2t^{2-2\alpha}E\left\|\int_0^t\begin{Bmatrix}\breve{T}_\alpha(t-s)(t-s)^{\alpha-1}\\ B[u^\theta(s,x_n)-u^\theta(s,x)]\end{Bmatrix}ds\right\|_\beta^2$$
$$+2t^{2-2\alpha}E\left\|\int_0^t\begin{Bmatrix}\breve{T}_\alpha(t-s)(t-s)^{\alpha-1}\\ [(\tilde{G}x_n)(s)-(\tilde{G}x)(s)]\end{Bmatrix}dW_{(s)}\right\|_\beta^2$$

From Ito isometry and lemma (2.7), we obtain

$$t^{2-2\alpha}E\|(\Psi_2x_n)(t)-(\Psi_2x)(t)\|_\beta^2$$
$$\le 2L_B^2\breve{N}_{\alpha,\beta}^2t^{2-2\alpha}\int_0^t\begin{bmatrix}(t-s)^{-2\alpha\beta+2\alpha-2}\\ E\left\|u^\theta(s,x_n)-u^\theta(s,x)\right\|^2\end{bmatrix}ds$$
$$+2\breve{N}_{\alpha,\beta}^2t^{2-2\alpha}\int_0^t\begin{bmatrix}(t-s)^{-2\alpha\beta+2\alpha-2}\\ E\left\|(\tilde{G}x_n)(s)-(\tilde{G}x)(s)\right\|^2\end{bmatrix}ds$$

Therefore, it follows from the continuity of G and u^θ that for each $t\in[0,T]$, $G\left(s,x_n(s),\int_0^s g(v,x_n(v))dW_{(v)}\right) \to G\left(s,x(s),\int_0^s g(v,x(v))dW_{(v)}\right)$ and $u^\theta(s,x_n)\to u^\theta(s,x)$, using the Lebesgue dominated convergence theorem that for all $t\in[0,T]$, we conclude $\|(\Psi_2x_n)(t)-(\Psi_2x)(t)\|_{C_{1-\alpha}}\to 0$, as $n\to\infty$, Implying that $\|\Psi_2x_n-\Psi_2x\|_{C_{1-\alpha}}\to 0$, as $n\to\infty$. Hence, Ψ_2 is continuous on $B_{\breve{\delta}}$.

***Lemma (3.9)*:**

If the assumptions (a) – (e) are hold, then for $x\in B_{\breve{\delta}}$, the set $\{(\Psi_2x)(t), t\in[0,T]\}$ is equicontinuous.

Proof:

Let $t_1,t_2\in[0,T]$ such that $0<t_1<t_2\le T$. Then, from the equation (24), we have

$$E\|\Psi_2x(t_2)-\Psi_2x(t_1)\|_\beta^2 \le 7E\|[(\breve{T}_\alpha(t_2)-\breve{T}_\alpha(t_1)]x_0\|_\beta^2$$
$$+14E\left\|\int_0^{t_1}(t_2-s)^{\alpha-1}\begin{bmatrix}\breve{T}_\alpha(t_2-s)\\ -\breve{T}_\alpha(t_1-s)\end{bmatrix}Bu^\theta(s,x)ds\right\|_\beta^2$$
$$+14E\left\|\int_0^{t_1}\breve{T}_\alpha(t_1-s)\begin{bmatrix}(t_2-s)^{\alpha-1}\\ -(t_1-s)^{\alpha-1}\end{bmatrix}Bu^\theta(s,x)ds\right\|_\beta^2$$
$$+7E\left\|\int_0^{t_2}\breve{T}_\alpha(t_2-s)(t_2-s)^{\alpha-1}Bu^\theta(s,x)ds\right\|_\beta^2$$
$$+14E\left\|\int_0^{t_1}(t_2-s)^{\alpha-1}\begin{bmatrix}\breve{T}_\alpha(t_2-s)\\ -\breve{T}_\alpha(t_1-s)\end{bmatrix}(\tilde{G}x)(s)dW_{(s)}\right\|_\beta^2$$
$$+14E\left\|\int_0^{t_1}\breve{T}_\alpha(t_1-s)\begin{bmatrix}(t_2-s)^{\alpha-1}\\ -(t_1-s)^{\alpha-1}\end{bmatrix}(\tilde{G}x)(s)dW_{(s)}\right\|_\beta^2$$
$$+7E\left\|\int_{t_1}^{t_2}\breve{T}_\alpha(t_2-s)(t_2-s)^{\alpha-1}(\tilde{G}x)(s)dW_{(s)}\right\|_\beta^2$$
$$+14E\left\|\int_0^{t_1}(t_2-s)^{\alpha-1}\begin{bmatrix}\breve{T}_\alpha(t_2-s)\\ -\breve{T}_\alpha(t_1-s)\end{bmatrix}\sigma(s)dW_{(s)}^H\right\|_\beta^2$$
$$+14E\left\|\int_0^{t_1}\breve{T}_\alpha(t_1-s)\begin{bmatrix}(t_2-s)^{\alpha-1}\\ -(t_1-s)^{\alpha-1}\end{bmatrix}\sigma(s)dW_{(s)}^H\right\|_\beta^2$$
$$+7E\left\|\int_{t_1}^{t_2}\breve{T}_\alpha(t_2-s)(t_2-s)^{\alpha-1}\sigma(s)dW_{(s)}^H\right\|_\beta^2$$

Now, from Lemma (2.7), noting the fact that for every $\epsilon>0$, there exists $\tau>0$ such that, whenever $t_2-t_1<\tau$, for every $t_1,t_2\in[0,T]$, $\|\breve{T}_\alpha(t_2)-\breve{T}_\alpha(t_1)\|_\beta^2<\epsilon$. Therefore, when $t_2-t_1<\tau$, we have

$$E\|\Psi_2x(t_2)-\Psi_2x(t_1)\|_\beta^2$$
$$\le 7\epsilon^2E\|x_0\|_\beta^2 + 14\frac{L_B^2\epsilon^2N_CT^{2\alpha}}{|2\alpha-1|}$$
$$+14\frac{L_B^2N_C\breve{N}_{\alpha,\beta}^2T^{1-2\alpha\beta}}{|1-2\alpha\beta|}\int_0^{t_1}\begin{bmatrix}(t_2-s)^{\alpha-1}\\ -(t_1-s)^{\alpha-1}\end{bmatrix}^2ds$$
$$+7N_C\breve{N}_{\alpha,\beta}^2L_B^2\int_{t_1}^{t_2}(t_2-s)^{2\alpha-2\alpha\beta-2}ds + 14\frac{\epsilon^2D_2T^{2\alpha}}{|2\alpha-1|}$$
$$+14\frac{\breve{N}_{\alpha,\beta}^2D_2T^{1-2\alpha\beta}}{|1-2\alpha\beta|}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]^2ds$$
$$+7\breve{N}_{\alpha,\beta}^2D_2\int_{t_1}^{t_2}(t_2-s)^{2\alpha-2\alpha\beta-2}ds$$
$$+28H\epsilon^2C_1\frac{T^{2\alpha+2H-2}}{2\alpha-1}$$
$$+28H\,T^{2H-1}C_1\breve{N}_{\alpha,\beta}^2\int_0^{t_1}(t_1-s)^{-2\alpha\beta}\begin{bmatrix}(t_2-s)^{\alpha-1}\\ -(t_1-s)^{\alpha-1}\end{bmatrix}^2ds$$
$$+28HT^{2H-1}C_1\breve{N}_{\alpha,\beta}^2\int_{t_1}^{t_2}(t_2-s)^{2\alpha-2\alpha\beta-2}ds$$

The right hand of the inequality above tends to 0 as $t_2 \to t_1$ and $\epsilon \to 0$. Hence for $x \in B_{\delta}$, the set $\{\Psi_2 x, t \in [0,T]\}$ is equicontinuous.

***Lemma (3.10)*:**

If the assumption (a) is hold. Then for each $t \in [0, T]$, the set $\ddot{U}(t) = \{\Psi_2 x, x \in B_{\delta}\}$ is relatively compact in B_{δ}.

Proof:

Let $t \in (0,T]$ be a fixed and $0 < \hat{a} < t$, for every $\gamma > 0$, $x \in B_{\delta}$, we define

$$\begin{aligned}\Psi_2^{\hat{a},\gamma} x(t) &= \int_{\gamma}^{\infty} \alpha r M_{\alpha}(r) \check{T}(t^{\alpha} r)(t)^{\alpha-1} x_0 dr \\ &+ \int_0^{t-\hat{a}} \int_{\gamma}^{\infty} \alpha r M_{\alpha}(r) \check{T}((t-s)^{\alpha} r)(t-s)^{\alpha-1} Bu^{\theta}(t,x) dr ds \\ &+ \int_0^{t-\hat{a}} \int_{\gamma}^{\infty} \alpha r M_{\alpha}(r) \check{T}((t-s)^{\alpha} r)(t-s)^{\alpha-1} (\tilde{G}x)(s) dr dW_{(s)} \\ &+ \int_0^{t-\hat{a}} \int_{\gamma}^{\infty} \alpha r M_{\alpha}(r) \check{T}((t-s)^{\alpha} r)(t-s)^{\alpha-1} \sigma(s) dr dW_{(s)}^{H}\end{aligned} \tag{25}$$

Then, from the definition of semigroup $\check{T}(t^{\alpha} r)$, $t^{\alpha} r > 0$, we can easily be written in the form, $\check{T}(t^{\alpha} r) = \check{T}(\hat{a}^{\alpha}\gamma)\check{T}(t^{\alpha} r - \hat{a}^{\alpha}\gamma)$, from the equality (25), we have

$$\begin{aligned}\Psi_2^{\hat{a},\gamma} x(t) &= \check{T}(\hat{a}^{\alpha}\gamma) \int_{\gamma}^{\infty} \alpha r M_{\alpha}(r) \check{T}(t^{\alpha} r - \hat{a}^{\alpha}\gamma)(t)^{\alpha-1} x_0 dr \\ &+ \check{T}(\hat{a}^{\alpha}\gamma) \int_0^{t-\hat{a}} \int_{\gamma}^{\infty} \begin{bmatrix} \alpha r M_{\alpha}(r) \check{T}((t-s)^{\alpha} r - \hat{a}^{\alpha}\gamma) \\ (t-s)^{\alpha-1} Bu^{\theta}(t,x) \end{bmatrix} dr ds \\ &+ \check{T}(\hat{a}^{\alpha}\gamma) \int_0^{t-\hat{a}} \int_{\gamma}^{\infty} \begin{bmatrix} \alpha r M_{\alpha}(r) \check{T}((t-s)^{\alpha} r - \hat{a}^{\alpha}\gamma) \\ (t-s)^{\alpha-1} (\tilde{G}x)(s) \end{bmatrix} dr dW_{(s)} \\ &+ \check{T}(\hat{a}^{\alpha}\gamma) \int_0^{t-\hat{a}} \int_{\gamma}^{\infty} \begin{bmatrix} \alpha r M_{\alpha}(r) \check{T}((t-s)^{\alpha} r \\ -\hat{a}^{\alpha}\gamma)(t-s)^{\alpha-1} \sigma(s) \end{bmatrix} dr dW_{(s)}^{H}\end{aligned}$$

Then, from the compactness of $\check{T}(\hat{a}^{\alpha}\gamma)$, we obtain that the set $\{\Psi_2^{\hat{a},\gamma} x(t), x \in B_{\delta}\}$ is relatively compact in X for every $\hat{a}$ such that $0 < \hat{a} < t$. Moreover, for $x \in B_{\delta}$, we can easily prove that $\Psi_2^{\hat{a},\gamma} x(t)$ is convergent to $\Psi_2 x(t)$ in B_{δ}, as $\hat{a} \to 0^+$ and $\gamma \to 0$, hence the set $\ddot{U}(t) = \{\Psi_2 x, x \in B_{\delta}\}$ is relatively compact in B_{δ}.

***Lemma (3.11)*:**

If the assumptions (a) – (e) are hold. Then Ψ_2 is a completely continuous.

Proof:

From lemma (3.7) and (3.9), for each $x \in B_{\delta}$ the operator Ψ_2 is uniformly bounded and the set $\{\Psi_2 x, t \in [0,T]\}$ is equicontinuous by Appling the Arzela –Ascoli theorem, it results that for $x \in B_{\delta}$, the set $\{\Psi_2 x, t \in [0,T]\}$ is relatively compact.

we obtain that Ψ_2 is a completely continuous.

***Theorem (3.1)*:**

If the assumptions (a)-(e) are satisfied. Then for each $\theta > 0$, the control system (1) has a mild solution on [0, T], provided that $\gamma = \frac{\tilde{N}_{\alpha,\beta}^2 T^{2-2\alpha\beta} K_1}{|(2\alpha-1)(1-2\alpha\beta)|} < 1$.

Proof:

For any $\theta > 0$ and for any $x, y \in B_{\delta}$, $(\Psi_1 y)(t) + (\Psi_2 x)(t) \in B_{\delta}$, for $t \in [0,T]$. By using lemma (3.6) and (3.11) with applying Krasnoselskii's fixed point theorem, we conclude that the operator Ψ_{θ} has a fixed point, which gives rise to mild solution of system (1) with stochastic control function given in (20). This completes the proof.

***Theorem (3.2)*:**

If the assumptions (a) – (e) are satisfied. Then the stochastic control system (1) is approximately controllable on [0,T].

Proof:

For every $\theta > 0$, let x_{θ} be a fixed point of the operator Ψ_{θ} in the space $C_{1-\alpha}([0,T]; L^2(\Omega, X_{\beta}))$, which is a mild solution under the stochastic control function in (20) of the stochastic control system (1). Then, we have

$$\begin{aligned}x_{\theta}(T) &= x_T - \epsilon R(\epsilon, \Gamma_0^T)\left(Ex_T - \check{T}_{\alpha}(T)(T)^{\alpha-1} x_0\right) \\ &- \int_0^T \theta R(\theta, \Gamma_s^T) \hat{a}(s) dW_{(s)} \\ &+ \int_0^T \begin{bmatrix} \theta R(\theta, \Gamma_s^T) \check{T}_{\alpha}(T-s)(T-s)^{\alpha-1} \\ F\left(s, x_{\theta}(s), \int_0^s h(s,r,x_{\theta}(r)) dr\right) \end{bmatrix} ds \\ &+ \int_0^T \begin{bmatrix} \theta R(\theta, \Gamma_s^T) \check{T}_{\alpha}(T-s)(T-s)^{\alpha-1} \\ G\left(s, x_{\theta}(s), \int_0^s g(v, x_{\theta}(v)) dW_{(v)}\right) \end{bmatrix} dW_{(s)} \\ &+ \int_0^T \theta R(\theta, \Gamma_s^T) \check{T}_{\alpha}(T-s)(T-s)^{\alpha-1} \sigma(s) dW_{(s)}^{H}\end{aligned} \tag{25}$$

By using the assumptions (c) the function F is uniformly bounded, such that $E\left\|F(t,x,\int_0^t h(t,s,x) ds)\right\|^2 \le D_1$, for any $0 \le s \le T$. Then, for all $(s,\omega) \in [0,T] \times \Omega$, there is a subsequence of sequence $\{F(s, x_{\theta}(s), \int_0^s h(s,r,x_{\theta}(r)) dr)\}$ denoted by $\{F(s, x_{\theta}(s), \int_0^s h(s,r,x_{\theta}(r)) dr)\}$ which is weakly converging to say F(s) in X. Also, there is a subsequence of sequence $\{G(s, x_{\theta}(s), \int_0^s g(v, x_{\theta}(v)) dW_{(v)})\}$ denoted by $\{G(s, x_{\theta}(s), \int_0^s g(v, x_{\theta}(v)) dW_{(v)})\}$ which is weakly converging to say G(s) in $L_2(K,X)$. On the other hand, from the assumption (b), for all $0 \le s < T$ the operator $\theta R(\theta, \Gamma_s^T) \to 0$ Strongly as $\theta \to 0^+$ and $\|\theta R(\theta, \Gamma_s^T)\| \le 1$.

$$\begin{aligned}&E\|x_{\theta}(T) - x_T\|^2 \\ &\le 8E\left\|\theta R(\theta, \Gamma_s^T)\left(Ex_T - \check{T}_{\alpha}(T)(T)^{\alpha-1} x_0\right)\right\|^2 \\ &+ 8E\left\|\int_0^T \theta R(\theta, \Gamma_s^T) \phi(s) dW_{(s)}\right\|^2 \\ &+ 8E\left\|\int_0^T R(\theta, \Gamma_s^T) \hat{H}(s) dW_{(s)}^{H}\right\|^2 \\ &+ 8E\left\|\int_0^T \begin{Bmatrix} R(\theta, \Gamma_s^T) \check{T}_{\alpha}(T-s)(T-s)^{\alpha-1} \\ [F\left(s, x_{\theta}(s), \int_0^s h(s,r,x_{\theta}(r)) dr\right) - F(s)] \end{Bmatrix} ds\right\|^2 \\ &+ 8E\left\|\int_0^T \theta R(\theta, \Gamma_s^T) \check{T}_{\alpha}(T-s)(T-s)^{\alpha-1} F(s) ds\right\|^2\end{aligned}$$

$$+8E\left\|\int_0^T\left\{\begin{array}{l}\theta\, R\left(\theta,\Gamma_s^T\right)\check{T}_\alpha\left(T-s\right)\left(T-s\right)^{\alpha-1}\\ \left[\begin{array}{l} G\left(s,x_\theta\left(s\right),\int_0^s g(v,x_\theta(v))dW_{(v)}\right)\\ -G(s)\end{array}\right]\end{array}\right\}dW_{(s)}\right\|^2$$

By using the Lebesgue dominated convergence theorem, we obtain $E\|x_\theta(T)-x_T\|^2\to 0$ as $\theta\to 0^+$. This gives the approximate controllability.

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