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Some Properties of Skew Uniform Distribution

Salah H Abid*

Mathematics Department, Education College, Al-Mustansirya University, Baghdad, Iraq
*Corresponding author: abidsalah@gmail.com

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Abstract There is one work that appears to give some details of the skew uniform distribution, this work due to Aryal and Nadarajah [Random Operators and stochastic equations, Vol.12, No.4, pp.319-330, 2004]. They defined a random variable X to have the skew uniform distribution such that $f_X(x) = 2\,g(x)G(\theta x)$, where $g(.)$ and $G(.)$ denote the probability density function (pdf) and the cumulative distribution function (cdf) of the uniform $(-a, a)$ distribution respectively. In this paper, we construct a new skewed distribution with pdf of the form $2\,f(x)G(\theta x)$, where θ is a real number, $f(.)$ is taken to be uniform $(-a, a)$ while $G(.)$ comes from uniform $(-b, b)$. We derive some properties of the new skewed distribution, the $r\ th$ moment, mean, variance, skewness, kurtosis, moment generating function, characteristic function, hazard rate function, median, Rènyi entropy and Shannon entropy. We also consider the generating issues.

Keywords: *Skew Uniform distribution, the $r\ th$ moment, characteristic function, hazard rate function, Entropy*

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1. Introduction

The skew-uniform distributions have been introduced by many authors, e.g. Gupta et al. [3], Aryal, G. and Nadarajah, S. [1], Nadarajah, S. and Kotz, S. [5]. This class of distributions includes the uniform distribution and possesses several properties which coincide or are close to the properties of the uniform family. Aryal, G. and Nadarajah, S. [1] defined a random variable X to have the skew uniform distribution such that $f_X(x) = 2\,g(x)G(\theta x)$, where $g(.)$ and $G(.)$ denote the probability density function (pdf) and the cumulative distribution function (cdf) of the uniform $(-a, a)$ distribution respectively. In this paper, we introduce a new skewed distribution with pdf of the form $2\,f(x)G(\theta x)$, where θ is a real number, $f(.)$ is taken to be uniform $(-a, a)$ while $G(.)$ comes from uniform $(-b, b)$.

The uniform $(-a, a)$ and the uniform $(-b, b)$ distributions [4] have the following $pdfs$ respectilely,

$$f(x) = 1/2a, \quad -a < x < a \tag{1}$$

$$g(x) = 1/2b\,, \quad -b < x < b \tag{2}$$

Where, $a\,, b > 0$ and $> b$.

A random variable X is said to have the skew-uniform distribution if its pdf is,

$$f_X(x) = 2\,f(x)G(\theta x) \tag{3}$$

Where,

$$G(\theta x) = (\theta x + b)/2b \tag{4}$$

The main feature of the skew-uniform distribution in (3) is that a new parameter θ is introduced to control skewness and kurtosis. Thus (3) allows for a greater degree of flexibility and we can expect this to be useful in many more practical situations.

It follows from (3) that the pdf and cdf of X are,

$$f_X(x) = \begin{cases} 0 & if \quad x < -b/\theta \\ (\theta x + b)/2ab & if \quad -b/\theta < x < b/\theta \\ 1/a & if \quad b/\theta < x < a \end{cases} \tag{5}$$

$$F_X(x) = \begin{cases} 0 & if \quad x < -b/\theta \\ (1/2ab)\{\theta\, x^2/2 + bx + b^2/2\theta\} & if \quad -b/\theta < x < b/\theta \\ x/a - b/\theta a & if \quad b/\theta < x < a \\ 1 & if \quad x \geq a \end{cases} \tag{6}$$

Respectively.

Throughout the rest of this paper (unless otherwise stated) we shall assume that $\theta > 0$, since the corresponding results for $\theta < 0$ can be obtained using the fact that $-X$ has the pdf $2\,f(x)G(-\theta x)$. When $\theta \to 0$ and $b/\theta \to a$, (5) reduces to the standard uniform pdf (1).

Figure 1.a illustrates the shape of the pdf (5) at different values of a and θ = 0.5, b=1.

2. Moments

Using direct integration, it is easy to show that the rth moment of X is given by,

$$EX^r = \int_{-b/\theta}^{b/\theta} \frac{X^r(\theta x + b)}{2ab} dx + \int_{b/\theta}^{a} \frac{x^r}{a} dX = \mu_r'$$

$$= \frac{\theta X^{r+2}}{2ab(r+2)} + \frac{X^{r+1}}{2a(r+1)}\Bigg|_{-b/\theta}^{b/\theta} + \frac{X^{r+1}}{a(r+1)}\Bigg|_{b/\theta}^{a}$$

$$= \frac{(b/\theta)^{r+1}}{2a}\left\{\frac{1}{r+2}+\frac{1}{r+1}-\frac{(-1)^{r+2}}{r+2}-\frac{(-1)^{r+1}}{r+1}\right\}$$

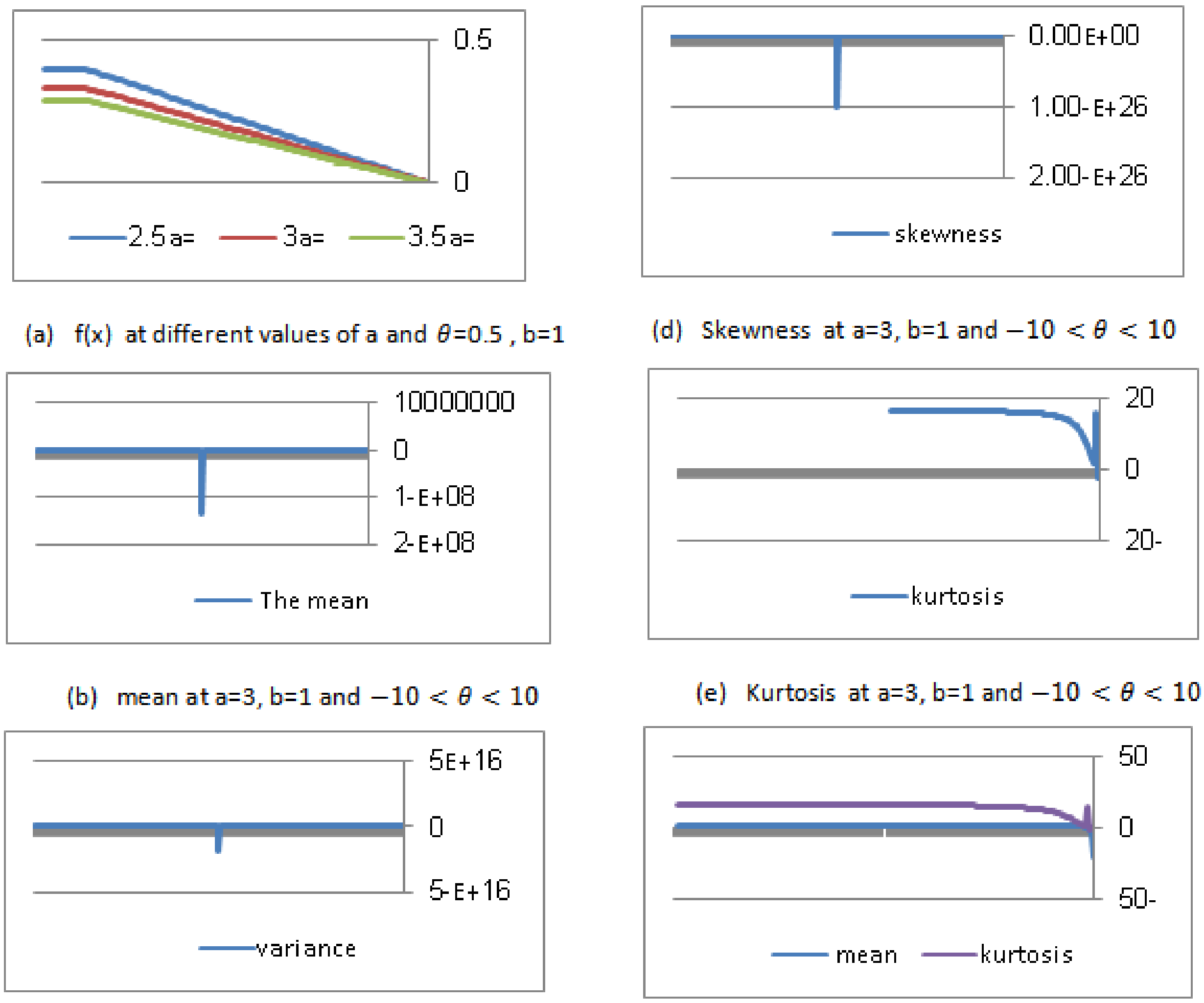
$$+\frac{1}{a}\left\{\frac{a^{r+1}}{r+1}-\frac{(b/\theta)^{r+1}}{r+1}\right\}$$

Since,

$$\frac{1}{r+2}+\frac{1}{r+1}-\frac{(-1)^{r+2}}{r+2}-\frac{(-1)^{r+1}}{r+1}=\begin{cases}2/(r+2)\; if\;\; r\; is\; odd\\ 2/(r+1) if\;\; r\; is\; even\end{cases}$$

Then,

$$EX^r = \begin{cases}\frac{1}{r+1}\left(a^r - \frac{(b/\theta)^{r+1}}{a(r+2)}\right)\; if\;\; r\;\; is\; odd\\ \frac{a^r}{r+1} \qquad\qquad\qquad if\;\; r\;\; is\; even\end{cases} \tag{7}$$



(a) f(x) at different values of a and θ=0.5 , b=1

(d) Skewness at a=3, b=1 and $-10 < \theta < 10$

(b) mean at a=3, b=1 and $-10 < \theta < 10$

(e) Kurtosis at a=3, b=1 and $-10 < \theta < 10$

(c) Var(x) at a=3, b=1 and $-10 < \theta < 10$

(f) Mean and Kurtosis at a=3, b=1 and $-10 < \theta < 10$

Figure 1. (a) the shape of f(x) at different values of a and θ=0.5 , b=1; (b) mean at a=3, b=1 and $-10 < \theta < 10$; (c) Var(x) at a=3, b=1 and $-10 < \theta < 10$; (d) Skewness at a=3, b=1 and $-10 < \theta < 10$; (e) Kurtosis at a=3, b=1 and $-10 < \theta < 10$; (f) Mean and Kurtosis at a=3, b=1 and $-10 < \theta < 10$

It follows from (7) that the mean, variance, skewness and the kurtosis of X are

$$EX = \tfrac{1}{2}\left(a - \tfrac{(b/\theta)^2}{3a}\right) = \mu_1' \tag{8}$$

$$Var(X) = \mu_2 = EX^2 - (EX)^2$$

$$= \frac{a^2}{3} - \frac{1}{4}\left(a - \frac{(b/\theta)^2}{3a}\right)^2 = \frac{a^2}{12} - \frac{(b/\theta)^4}{36a^2} + \frac{(b/\theta)^2}{6} \tag{9}$$

$$Skewness = \gamma_1 = \sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}}$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2\left(\mu_1'\right)^3$$

$$= \frac{1}{4}\left(a^3 - \frac{(b/\theta)^4}{5a}\right) - 3\frac{1}{2}\left(a - \frac{(b/\theta)^2}{3a}\right)\frac{a^2}{3}$$

$$+2\left[\frac{1}{2}\left(a - \frac{(b/\theta)^2}{3a}\right)\right]^3$$

$$= \frac{-1}{12}a(b/\theta)^2 + \frac{(b/\theta)^4}{30a} - \frac{(b/\theta)^6}{108a^3}$$

Which implies to,

$$\gamma_1 = \frac{(b/\theta)^2 \frac{1}{6}\left[\frac{(b/\theta)^2}{5a} - \frac{a}{2} - \frac{(b/\theta)^4}{18a^3}\right]}{\left[\frac{a^2}{12} - \frac{(b/\theta)^4}{36a^2} + \frac{(b/\theta)^2}{6}\right]^{3/2}}$$

$$= \frac{(b/\theta)^2\left[\frac{(b/\theta)^2}{5a} - \frac{a}{2} - \frac{(b/\theta)^4}{18a^3}\right]}{6\left[\frac{1}{6}\left(\frac{a^2}{2} - \frac{(b/\theta)^4}{6a^2} + (b/\theta)^2\right)\right]^{3/2}} \tag{10}$$

$$Kurtosis = \gamma_2 = \beta_2 - 3 = \frac{\mu_4}{\mu_2^2} - 3$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6\left(\mu_1'\right)^2 \mu_2' - 3\left(\mu_1'\right)^4$$

$$= \frac{a^4}{5} - 4\left[\frac{1}{2}\left(a - \frac{(b/\theta)^2}{3a}\right)\right]\frac{1}{4}\left(a^3 - \frac{(b/\theta)^4}{5a}\right)$$

$$+6\left[\frac{1}{2}\left(a - \frac{(b/\theta)^2}{3a}\right)\right]^2 \frac{a^2}{3} - 3\left[\frac{1}{2}\left(a - \frac{(b/\theta)^2}{3a}\right)\right]^4$$

$$= \frac{11}{80}a^4 + \frac{11}{360}(b/\theta)^4 + \frac{a^2}{12}(b/\theta)^2 - \frac{(b/\theta)^6}{180a^2} - \frac{(b/\theta)^8}{432a^4}$$

Which implies to,

$$\gamma_2 = \frac{\left[\frac{11}{80}a^4 + \frac{11}{360}(b/\theta)^4 + \frac{a^2}{12}(b/\theta)^2 - \frac{(b/\theta)^6}{180a^2} - \frac{(b/\theta)^8}{432a^4}\right]}{\left[\frac{a^4}{144} + \frac{5(b/\theta)^4}{216} + \frac{a^2(b/\theta)^2}{36} - \frac{(b/\theta)^6}{108a^2} - \frac{(b/\theta)^8}{1296a^4}\right]} - 3 \tag{11}$$

Figure 1.b,c,d,e,f illustrates the behavior of the above four measures at $a = 3,\ b = 1$ and $-10 < \theta < 10$.

3. The Characteristic Function

The characteristic function (cf) [7] of a random variable X is defined by $(t) = Ee^{i\,t\,X}$, where $i = \sqrt{-1}$. When X has the pdf (5) direct integration yields that,

$$\psi(t) = Ee^{i\,t\,X} = \int_{-b/\theta}^{b/\theta} e^{i\,t\,X}\frac{(\theta X + b)}{2ab}dX + \int_{b/\theta}^{a} e^{i\,t\,X}\frac{1}{a}dX$$

$$= \frac{1}{2ab}\left\{\theta\left(\frac{X}{it} + \frac{1}{t^2}\right) + \frac{b}{it}\right\}e^{i\,t\,X}\Bigg|_{-b/\theta}^{b/\theta} + \frac{1}{ait}e^{i\,t\,X}\Bigg|_{b/\theta}^{a}$$

$$= \frac{1}{2ab}\left\{\begin{array}{l}\left(\theta\left(\frac{b/\theta}{it} + \frac{1}{t^2}\right) + \frac{b}{it}\right)e^{itb/\theta} \\ -\left(\theta\left(\frac{-b/\theta}{it} + \frac{1}{t^2}\right) + \frac{b}{it}\right)e^{-itb/\theta}\end{array}\right\} + \frac{1}{ait}\left(\begin{array}{l}e^{i\,t\,a} \\ -e^{itb/\theta}\end{array}\right)$$

$$= \frac{\theta}{2abt^2}\left(e^{itb/\theta} - e^{-itb/\theta}\right) + \frac{1}{ait}e^{i\,t\,a} \tag{12}$$

$$= \frac{\theta i}{abt^2}Sin(tb/\theta) + \frac{1}{ait}e^{i\,t\,a} \tag{13}$$

Since the moment generating function (mgf) is $M(t) = \psi(t/i)$ then,

$$M(t) = \frac{\theta}{2ab\,t^2}\left(e^{-itb/\theta} - e^{itb/\theta}\right) + \frac{1}{at}e^{\,t\,a} \tag{14}$$

4. Hazard Rate Function

Since the reliability function, $R(x) = 1 - F_X(x)$, is,

$$R(x) = \begin{cases} 1 & if\ x < -b/\theta \\ 1 - (1/2ab)\left\{\theta x^2/2 + +bx + b^2/2\theta\right\} & \\ \quad if\ b/\theta < x < b/\theta & \\ 1 - x/a + b/\theta a & if\ b/\theta < x < a \\ 0 & f\ x \geq a \end{cases} \tag{15}$$

Then, after some simple steps, one can get the hazard function $\lambda(x) = f(x)/R(x)$ as follows,

$$\lambda(x) = \begin{cases} 0 & if\ x < -b/\theta \\ 2\theta(\theta x + b)/\left(4ab\theta - (\theta x + b)^2\right) & if\ -b/\theta \leq x < b/\theta \\ \theta/\left(\theta(a - x) + b\right) & if\ b/\theta < x < a \end{cases} \tag{16}$$

The hazard rate function is an important quantity characterizing life phenomena.

5. Entropy

An entropy of a random variable X is a measure of variation of the uncertainty. Rènyi entropy is defined by [6],

$$\mathfrak{Z}_\omega = \frac{1}{1-\omega}Ln\{\int f^\omega(x)\,dx\}$$

Now, since

$$\int f^\omega(x)dx = \int_{-b/\theta}^{b/\theta}\left(\frac{\theta X + b}{2ab}\right)^\omega dx + \int_{b/\theta}^{a}\left(\frac{1}{a}\right)^\omega$$

$$= \left(\frac{1}{2ab}\right)^\omega \frac{(\theta X + b)^{\omega+1}}{\theta(\omega+1)}\Bigg|_{-b/\theta}^{b/\theta} + \frac{X}{a^\omega}\Bigg|_{b/\theta}^{a}$$

$$= \frac{1}{a^\omega}\left\{\frac{2b}{\theta(\omega+1)} + a - b/\theta\right\}$$

So,

$$З_{\omega} = \frac{1}{1-\omega}\left\{-\omega Ln(a) + Ln\left(\frac{2b}{\theta(\omega+1)} + a - b/\theta\right)\right\} \quad (17)$$

The Shannon entropy [2] can be found as follows,

$$H = -\int f(x) Ln\big(f(x)\big) dx$$

$$= -\int_{-b/\theta}^{b/\theta} \frac{(\theta X+b)}{2ab}\ Ln\left(\frac{(\theta X+b)}{2ab}\right) dx + \int_{b/\theta}^{a} \frac{1}{a}\ Ln(a)\ dx$$

Since,

$$\int_{b/\theta}^{a} \frac{1}{a} Ln(a) dx = Ln(a) - \frac{b\ Ln(a)}{\theta\ a}\ and$$

$$= -\int_{-b/\theta}^{b/\theta} \frac{(\theta X+b)}{2ab} Ln\left(\frac{(\theta X+b)}{2ab}\right) dx$$

$$= -\int_{-b/\theta}^{b/\theta} \frac{\theta X}{2ab} Ln\left(\frac{\theta X}{2ab}+\frac{1}{2a}\right) dx - \frac{1}{2a}\int_{-b/\theta}^{b/\theta} Ln\left(\frac{\theta X}{2ab}+\frac{1}{2a}\right)$$

$$= \frac{-\theta}{2ab}\left\{\frac{1}{2}\left(X^2 - \frac{b^2}{\theta^2}\right) Ln\left(\frac{\theta X}{2ab}+\frac{1}{2a}\right) - \frac{1}{2}\left(\frac{X^2}{2} - \frac{bX}{\theta}\right)\right\}$$

$$- \frac{1}{2a}\left\{\frac{2ab}{\theta}\left(\frac{(\theta X+b)}{2ab}\right) Ln\left(\frac{(\theta X+b)}{2ab}\right) - X\right\}\Bigg|_{-b/\theta}^{b/\theta}$$

$$= \frac{b}{2a\theta} + \frac{b\ Ln(a)}{\theta\ a}$$

Then,

$$H = \frac{b}{2a\theta} + Ln(a) \quad (18)$$

6. Generation procedure

By using the inverse transform method and some logical aspects, one can generate the random variable X as follows,

$$X = \begin{cases} 2\sqrt{abU/\theta} - b/\theta & if\ \ 0 < U < b/a\theta \\ aU + b/\theta & if\ \ b/a\theta \le U < 1 \end{cases} \quad (19)$$

Where U is a uniformly distributed random variable in the interval [0,1].

From (6) and (19), one can get the median of X as follows,

$$Me = \begin{cases} \sqrt{2ab/\theta} - b/\theta & if\ \ \ \theta < 2b/a \\ a/2 + b/\theta & if\ \ \ \theta \ge 2b/a \end{cases} \quad (20)$$

7. Summary and Conclusions

In spite of the great importance of the uniform distribution uses, but unfortunately the form of the distribution and its properties reduced the distribution applications, especially in real life. This issue has made us think to construct other distributions based on the uniform distribution, So that the new distribution have flexible form and properties to represent a lot of other applications.

In this paper, we construct a new skewed distribution with pdf of the form $2f(x)G(\theta x)$, where θ is a real number, $f(.)$ is taken to be uniform $(-a,a)$ while $G(.)$ comes from uniform $(-b,b)$. We derive some properties of the new skewed distribution, the $r\ th$ moment, mean, variance, skewness, kurtosis, moment generating function, characteristic function, hazard rate function, median, Rènyi entropy and Shannon entropy. We also consider the generating issues.

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