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A Fifth Order Compact Difference Method for Singularly Perturbed Singular Boundary Value Problems

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Abstract In this paper, we have developed a fifth order compact difference method for a class of singularly perturbed singular two-point boundary value problems. To avoid the singularity at zero a terminal boundary condition in the implicit form is derived. Using this condition as one of the boundary condition we solve the singularly perturbed singular two-point boundary value problem by the fifth order compact difference scheme. Numerical results are presented to illustrate the proposed method and compared with exact solution.

Keywords: *singular boundary value problem, singularly perturbations, singular point, boundary layer, finite differences*

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1. Introduction

Singularly perturbed singular boundary value Problems arise in many areas of science and engineering such as heat transfer problem with large Peclet numbers, Navier-Stokes flows with large Reynolds numbers, chemical reactor theory, aerodynamics, Reaction-diffusion process, quantum mechanics, optimal control etc. The numerical treatment of singular singularly perturbed boundary value problems present some major computational difficulties due to the boundary layer behavior of the solution and the presence of singularity. It is well known fact that the solution of these problems exhibits a multi scale character, that is, there are thin transition layer(s) where the solution varies rapidly, and while away from the layers (s) the solution behaves regularly and varies slowly.

In general, the classical numerical methods fail to give reliable results for these problems because of the layer behavior and also because of singularity. Detailed theory and numerical treatment of these problems is available in the Ref. [1-13]. Rasidinia, Mohammadi and Ghasemij [5] presented a numerical technique for a class of singularly perturbed two point singular boundary value problems on uniform mess using Polynomial cubic splines. Li [6] described a computational method for solving singularly perturbed two-point singular boundary value problem in which exact solution is represented in the form of series in reproducing kernel space. Kadalbajoo and Aggarwal [17] presented a Fitted mesh B-spline method for the solution of a class of singular singularly perturbed boundary value problems. Mohanty and Jha [10] presented a class of variable mesh spline in compression methods for singularly perturbed two point singular boundary value problems. Mohanty and Arora [11] proposed a family of non-uniform mesh tension spline methods for the solution of singularly perturbed two-point singular boundary value problems with significant first derivatives. Mohanty et. al. [12] suggested a Convergent spline in tension methods for the solution of singularly perturbed two-point singular boundary value problems. Mohanty, Jha, and Evans [13] presented a Spline in compression method for the numerical solution of singularly perturbed two point singular boundary value problems. For a detailed analytical and numerical discussion on singularly perturbed problems one may refer to the books and high level monographs by: Bender and Orszag [1], Miller et. al. [3], Kevorkian and Cole [4], Hemkar et. al. [8] and O'Malley [9].

In this paper, we have presented a fifth order compact difference method for a class of singularly perturbed singular two-point boundary value problems. To avoid the singularity at zero a terminal boundary condition in the implicit form is derived. Using this condition as one of the boundary condition we have solved the singularly perturbed singular two-point boundary value problem by the fifth order compact difference scheme. Numerical results are presented to demonstrate the applicability of the proposed method and compared with exact solution. We have also presented the least square and maximum errors for the problems considered. It is observed from the tables that the present method approximates the exact solution very well.

This paper is organized as follows: Section 2 presents the way of finding terminal boundary condition in the implicit form and the description of the fifth order compact difference scheme. Numerical experiments are performed by considering four standard example problems and presented the computational results in the section 3, show the accuracy and efficiency of the method. In the section 4, based on the numerical experiments performed, and conclusions are presented.

2. Description of the Method

Consider singularly perturbed singular boundary value problems of the form:

$$Ly \equiv \varepsilon y''(x) + \frac{k}{x} y'(x) + q(x)y(x) = r(x), 0 \le x \le 1, \quad (1)$$

with boundary conditions

$$y(0) = \alpha \quad (2a)$$

and

$$y(1) = \beta \quad (2b)$$

where $0 < \varepsilon << 1$, *q(x)*, *r(x)* are bounded continuous functions in (0, 1), *q(x)* > 0 and α, β are finite constants. We know that, if a function is analytic at a point $x = x_0$, then the point x_0 is said to be an ordinary point. The point $x = x_0$ is a singular point if the functions fail to be analytic at x_0. Such problems are called singularly perturbed singular boundary value problems.

To avoid the singular point '0', we introduce δ , a small positive deviating argument, where $0 < \delta << 1$.

Using Taylor series expansion in the neighbourhood of the point x , we have

$$\begin{aligned} y(x-\delta) &= y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x) \\ y''(x) &= \frac{2y(x-\delta) - 2y(x) + 2\delta y'(x)}{\delta^2} \end{aligned} \quad (3)$$

Substituting $y''(x)$ in (1), we get

$$p(x)y' + q(x)y = r(x) \quad (4)$$

where

$$p(x) = 2\varepsilon\delta + a(x)\delta^2, q(x) = -2\varepsilon + b(x)\delta^2,$$
$$r(x) = \delta^2 f(x) - 2\varepsilon y(x-\delta)$$

At $x = \delta$, Eq. (4) becomes

$$p(\delta)y' + q(\delta)y = r(\delta)$$

We use this equation as the terminal boundary condition.

Then the considered boundary value problem (BVP) (1) with (2a) and (2b) over [δ , 1] is given by

$$\varepsilon y''(x) + \frac{k}{x} y'(x) + q(x)y(x) = r(x) \quad (5)$$

with boundary conditions

$$p(\delta)y' + q(\delta)y = r(\delta) \quad (6)$$

and

$$y(1) = \beta \quad (7)$$

Now we solve this boundary value problem by the **fifth order compact difference scheme** described below.: For this we consider the first order linear system corresponding to the above BVP as:

$$Y' = A(x)Y + R(x), x \in [a,b] \quad (8)$$

with the boundary conditions

$$B_1 Y(a) + B_2 Y(b) = D\,,$$

where A, B_1 and B_2 are 2 x 2 matrices and Y, R, D are two dimensional vectors.

Now we divide the interval $[\delta,1] \equiv [a,b]$ into N equal parts with constant mesh length *h.* Let $a = x_0, x_1, x_2, \ldots\ldots\ldots, x_N = b$ be the mesh points. Again we divide each subinterval $[x_i,\ x_{i+1}]$ into four equal smaller sub intervals. Let $t_1, t_2, \ldots.., t_5$ are the grids in the subinterval $[x_i,\ x_{i+1}]$ and corresponding values of the variables and its derivatives are Y_1, Y_2, Y_3, Y_4, Y_5 and $Y_1', Y_2', Y_3', Y_4', Y_5'$.

By considering Taylor's expansions of Y_1, Y_2, Y_3, Y_4, Y_5 at the fractional grid t_3 (Peng [2]), we have

$$\frac{h^{n+1}}{(n+1)!} Y_3^{(n+1)} = \sum_{j=1}^{5} a_j^n Y_j + a_6^n Y_3' + O(h^6 Y_3^{(6)}), n = 1,2,3,4 \quad (9)$$

where $h = \dfrac{x_{i+1} - x_i}{4}$ and the coefficients a_j^n are given by:

$$a_1^1 = \frac{-1}{24}, a_2^1 = \frac{2}{3}, a_3^1 = \frac{-5}{4}, a_4^1 = \frac{2}{3}, a_5^1 = \frac{-1}{24}, a_6^1 = 0$$
$$a_1^2 = \frac{1}{48}, a_2^2 = \frac{-2}{3}, a_3^2 = 0, a_4^2 = \frac{2}{3}, a_5^2 = \frac{-1}{48}, a_6^2 = \frac{-5}{4}$$
$$a_1^3 = \frac{1}{24}, a_2^3 = \frac{-1}{6}, a_3^3 = \frac{1}{4}, a_4^3 = \frac{-1}{6}, a_5^3 = \frac{1}{24}, a_6^3 = 0$$
$$a_1^4 = \frac{-1}{48}, a_2^4 = \frac{1}{6}, a_3^4 = 0, a_4^4 = \frac{-1}{6}, a_5^4 = \frac{1}{48}, a_6^4 = \frac{1}{4}$$

By taking the Taylor's series expansions of $Y_1', Y_2', Y_3', Y_4', Y_5'$ at the grid point t_3 and substituting (9), we get

$$Y_k' = \frac{1}{h} \sum_{j=1}^{5} b_j^k Y_j + b_6^k Y_3' + O(h^5 Y_3^{(6)}) \text{ for } k = 1,2,4,5 \quad (10)$$

where

$$b_j^1 = -4a_j^1 + 12a_j^2 - 32a_j^3 + 80a_j^4 + Sgn(j-6)$$
$$b_j^2 = -2a_j^1 + 3a_j^2 - 4a_j^3 + 5a_j^4 + Sgn(j-6)$$
$$b_j^4 = 2a_j^1 + 3a_j^2 + 4a_j^3 + 5a_j^4 + Sgn(j-6)$$
$$b_j^5 = 4a_j^1 + 12a_j^2 + 32a_j^3 + 80a_j^4 + Sgn(j-6)$$
$$Sgn(x) = \begin{cases} 1\,, & x \ge 0 \\ 0\,, & x < 0 \end{cases}$$

The variable Y and its derivative Y' at grids $t_1, t_2, ...t_5$ subject to equations

$$Y_j' = A_j Y_j + R_j,\ j = 1,2,3,4,5 \tag{11}$$

where A_j and R_j are values of A and R at grids t_j.

Substituting (11) in (10), we get six linear algebraic equations with respect to five unknown variables Y_1, Y_2, Y_3, Y_4, Y_5.

By eliminating Y_2, Y_3, Y_4 from the above equations a relation between Y_1 and Y_5 can be obtained as follows:

$$\frac{1}{h} S_i Y_i + \frac{1}{h} T_i Y_{i+1} = F_i \ \ for \text{ i} = 0,1,2,.....\text{N-1} \tag{12}$$

where S_i and T_i are 2 x 2 matrices and F_i is a two dimensional vector. The relation (12) is a fifth order compact difference scheme of Eq. (8) in the $i-$ th subinterval. By assuming

$$c_1 = b_2^1 b_4^5 - b_2^5 b_4^1$$
$$W_1 = (b_5^5 b_4^1 - b_5^1 b_4^5)I - hb_4^1 A_5)/c_1$$
$$W_2 = ((b_3^5 b_4^1 - b_3^1 b_4^5)I + h(b_6^5 b_4^1 - b_6^1 b_4^5)A_3)/c_1$$
$$W_3 = ((b_1^5 b_4^1 - b_4^5 b_1^1)I + hb_4^5 A_1)/c_1$$
$$G_1 = (b_4^5 R_1 - b_4^1 R_5 + (b_6^5 b_4^1 - b_6^1 b_4^5)R_3)/c_1$$
$$W_4 = ((b_1^5 b_2^1 - b_1^1 b_2^5)I + hb_2^5 A_1)/c_2$$
$$W_5 = ((b_2^1 b_3^5 - b_2^5 b_3^1)I + h(b_6^5 b_2^1 - b_6^1 b_2^5)A_3)/c_2$$
$$W_6 = ((b_2^1 b_5^5 - b_2^5 b_5^1)I - hb_2^1 A_5)/c_2$$
$$G_2 = (b_2^5 R_1 - b_2^1 R_5 + (b_6^5 b_2^1 - b_6^1 b_2^5)R_3)/c_2$$
$$W_7 = b_1^2 I + (b_2^2 - hA_2)W_3 + b_4^2 W_4,$$
$$W_8 = b_3^2 I + b_4^2 W_5 + hb_6^2 A_3 + (b_2^2 I - hA_2)W_2,$$
$$W_9 = b_5^2 I + b_4^2 W_6 + (b_2^2 I - hA_2)W_1,$$
$$G_3 = R_2 - b_6^2 R_3 - (b_2^2 I - hA_2)G_1 - b_4^2 G_2$$
$$W_{10} = b_2^4 W_3 + (b_4^4 - hA_4)W_4 + b_1^4 I,$$
$$W_{11} = b_3^4 I + b_2^4 W_2 + hb_6^4 A_3 + (b_4^4 - hA_4)W_5$$
$$W_{12} = b_5^4 I + b_2^4 W_1 + (b_4^4 - hA_4)W_6,$$
$$G_4 = R_4 - b_6^4 R_3 - b_2^4 G_1 - (b_4^4 - hA_4)G_2$$

We get

$$S_i = W_{11} W_7 - W_8 W_{10},$$
$$T_i = W_{11} W_9 - W_8 W_{12},$$
$$F_i = G_3 W_{11} - W_8 G_4$$

Now the system (12) can be written in matrix form as:

$$\begin{bmatrix} S_0 & T_0 & .... & & & F_0 \\ & S_1 & T_1 & .... & & F_1 \\ & & S_2 & T_2 & .... & F_2 \\ \\ & .... & & S_{N-1} & T_{N-1} & F_{N-1} \end{bmatrix}$$

Solving the above system together with the given boundary conditions (6) and (7), we will get the solution.

In the boundary condition (6), we replace the $y'(\delta)$ by the following fifth order approximation which is obtained by the expansion

$$y(x-\delta) = y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x) - \frac{\delta^3}{3!} y'''(x) + \frac{\delta^4}{4!} y^{iv}(x) - \frac{\delta^5}{5!} y^{v}(x) \tag{13}$$

We calculate the required derivatives from the differential equation and at $x = \delta$ we write Eq. (13), so that we have $y'(\delta)$ in terms of $y(\delta)$. Substitute this $y'(\delta)$ in Eq. (6) so that we have the boundary condition for $y(\delta)$.

3. Numerical Experiments

To demonstrate the applicability of fifth order compact difference method computationally, we consider four singularly perturbed two-point singular boundary value problems. These problems have been chosen because they have been widely discussed in the literature and because exact solutions are available for comparison.

Example 1. Consider the singularly perturbed singular boundary value problem

$$-\varepsilon y'' + (1/x)y' + (1+x^2)y = f(x), \quad 0 < \text{x} < 1$$

The exact solution of this problem is $y(x) = \exp(x^2)$ The numerical results are shown in Table 1 and Table 2 for $\varepsilon = 0.01$ and $\varepsilon = 0.001$ respectively.

Example 2. Consider the following singularly perturbed singular boundary value problem:

$$\varepsilon y'' - \frac{1}{x} y' - y = 0$$

With boundary conditions $y(0) = 1, y(1) = 1$ The uniform solution of this problem is

$$y(x) = e^{-x^2/2}\left[1 + \frac{\varepsilon\left(x^2-1\right)^2}{4}\right] + \left(1 - e^{-1/2}\right)\left[1 - \frac{\varepsilon\left(X^2 - 4X\right)}{2}\right]e^{-X}$$

where $X = \dfrac{1-x}{\varepsilon}$

The numerical results are shown in Table 3 and Table 4 for $\varepsilon = 0.01$ and $\varepsilon = 0.001$ respectively.

Example 3. Consider the following singularly perturbed singular boundary value problem where *q(x)* is also not continuous at x = 0

$$\varepsilon y'' + \frac{1}{x} y' + \frac{1}{x^2} y = \frac{2}{x} - 2\varepsilon - 3$$

subject to boundary conditions $y(0) = 0, y(1) = 0$. The exact solution of this problem is $y(x) = x - x^2$.

The numerical results are shown in Table 5 and Table 6 for $\varepsilon = 0.01$ and $\varepsilon = 0.001$ respectively.

Example 4. Consider the following singularly perturbed singular boundary value problem

$$\varepsilon y'' + \frac{1}{x} y' + y = 0, \quad 0 < x < 1,$$

with boundary conditions $y(0) = 0, \ y(1) = \exp\left(\frac{-1}{2}\right)$

whose exact solution is not known. This problem has regular singularity at x = 0 and boundary layer also at x = 0. However, the condition on y (0) is so weak that the solution does not exhibit a boundary layer at x = 0 as $\varepsilon \to 0_+$, even though $\frac{1}{x} > 0$ for x > 1. The numerical results are shown in Table 7 for $\varepsilon = 0.01$ and $\varepsilon = 0.001$ respectively.

Table 1. Numerical solution of example 1 with $\varepsilon = 0.01$

x	Exact solution	Numerical solution:
0.01	1.00010000500017	0.99990265931703
0.02	1.00040008001067	1.00019218079790
0.03	1.00090040512153	1.00068362713605
0.04	1.00160128068294	1.00137715986688
0.05	1.00250312760580	1.00227304746931
0.10	1.01005016708417	1.00980559020511
0.20	1.04081077419239	1.04055445710034
0.30	1.09417428370521	1.09381428789038
0.40	1.17351087099181	1.17282555210895
0.50	1.28402541668774	1.28266791788160
0.60	1.43332941456034	1.43083019282642
0.70	1.63231621995538	1.62808725848883
0.80	1.89648087930495	1.88982497716907
0.90	2.24790798667647	2.23803599894365
0.95	2.46575981160379	2.45403278154133
0.96	2.51330846816559	2.50130795422577
0.97	2.56228643870935	2.55026370000716
0.98	2.61274136097607	2.60142598862026
0.99	2.66472270087634	2.65618134866874
1.00	2.71828182845905	2.71828182845905

Least square error = 4.777048157769524e-002
Maximum error = 1.202273870219051e-002

Table 2. Numerical solution of example 3 with $\varepsilon = 0.001$

x	Exact solution:	Approximate solution:
0.01	1.00010000500017	0.99990205927314
0.02	1.00040008001067	1.00018905639825
0.03	1.00090040512153	1.00067646871064
0.04	1.00160128068294	1.00136457001547
0.05	1.00250312760580	1.00225375870636
0.10	1.01005016708417	1.00973849889269
0.20	1.04081077419239	1.04037087963570
0.30	1.09417428370521	1.09358326435083
0.40	1.17351087099181	1.17272329597310
0.50	1.28402541668774	1.28297095588620
0.60	1.43332941456034	1.43190869695686
0.70	1.63231621995538	1.63039456558919
0.80	1.89648087930495	1.89387916501748
0.90	2.24790798667647	2.24438923316360
0.95	2.46575981160379	2.46167033033493
0.96	2.51330846816559	2.50909440029684
0.97	2.56228643870935	2.55794405903290
0.98	2.61274136097607	2.60826699026021
0.99	2.66472270087634	2.66013923700702
1.00	2.71828182845905	2.71828182845905

Least square error = 1.904631456944122e-002
Maximum error = 4.583463869312965e-003

Table 3. Numerical solution of example 2 with $\varepsilon = 0.01$

x	Exact solution:	Approximate solution:
0.01	0.997450626196856	1.000049022457190
0.02	0.997302519148790	0.999904225086626
0.03	0.997055721933094	0.999658487774658
0.04	0.996710306322574	0.999311873898195
0.05	0.996266372751550	0.998864486169478
0.10	0.992574449865540	0.995121852839487
0.20	0.977940295563457	0.980227935514073
0.30	0.954018328046335	0.955865521911172
0.40	0.921487969151610	0.922767296625126
0.50	0.881255891315336	0.881906900045628
0.60	0.834414894714787	0.834450471423083
0.70	0.782195584615876	0.781700511217907
0.80	0.725913764785679	0.725035887695443
0.90	0.666916616201294	0.665860411655690
0.95	0.636816479670408	0.637851270188140
0.96	0.630769127747760	0.635834496080972
0.97	0.624715625453869	0.641302565624547
0.98	0.618657222109340	0.667839162157120
0.99	0.612595157145356	0.753050309220415
1.00	1	1

Least square error = 1.505656477919014e-001
Maximum error = 1.404551520750587e-001

Table 4. Numerical solution of example 2 with $\varepsilon = 0.001$

x	Exact solution:	Approximate solution:
0.01	0.999700063744667	1.000048997651690
0.02	0.999550269913679	0.999905539210212
0.03	0.999300663304642	0.999662033112511
0.04	0.998951318555473	0.999318554174597
0.05	0.998502340107278	0.998875205378224
0.10	0.994768676259968	0.995166101518143
0.20	0.979972835532425	0.980401474291418
0.30	0.955799566454423	0.956237102247845
0.40	0.922953508663133	0.923381407662209
0.50	0.88237280145767	0.882777947819268
0.60	0.835184679741624	0.835560336606245
0.70	0.782653642879269	0.782999677707841
0.80	0.72612550984489	0.726447900949766
0.90	0.666970791392756	0.667280511703749
0.95	0.63683010090161	0.637138738359658
0.96	0.630777851267461	0.631086732569697
0.97	0.624720535025953	0.625029898567321
0.98	0.618659404963294	0.618982539886557
0.99	0.612595702981965	0.615146041057813
1.00	1	1

Least square error = 4.563325208909310e-003
Maximum error = 2.550338075847702e-003

Table 5. Numerical solution of example 3 with $\varepsilon = 0.01$

x	Exact solution:	Approximate solution:
0.01	0.0099	0.0100960003041015
0.02	0.0196	0.0360366653987273
0.03	0.0291	0.0303670657072325
0.04	0.0384	0.0260149475694329
0.05	0.0475	0.0332649887416231
0.10	0.0900	0.0823839464178892
0.20	0.1600	0.1561204591585380
0.30	0.2100	0.2074355496974660
0.40	0.2400	0.2381548425145110
0.50	0.2500	0.2486379172432640
0.60	0.2400	0.2390036020695580
0.70	0.2100	0.2093025182617390
0.80	0.1600	0.1595598605309490
0.90	0.0900	0.0897895740811211
1.00	0.0000	0.0000000000000000

Least square error = 3.982185182162525e-002
Maximum error = 1.643666539872733e-002

Table 6. Numerical solution of example 3 with $\varepsilon = 0.001$

x	Exact solution:	Approximate solution:
0.01	0.0099	0.0099196036011825
0.02	0.0196	0.0151466085205695
0.03	0.0291	0.0262760006709565
0.04	0.0384	0.0381164248027364
0.05	0.0475	0.0491380867375322
0.10	0.0900	0.0877019458347869
0.20	0.1600	0.1562179650662710
0.30	0.2100	0.2074818316449780
0.40	0.2400	0.2381810050094690
0.50	0.2500	0.2486536820928910
0.60	0.2400	0.2390132758174820
0.70	0.2100	0.2093083167324720
0.80	0.1600	0.1595630447654590
0.90	0.0900	0.0897909162179272
1.00	0.0000	0.0000000000000000

Least square error = 2.037236974106488e-002
Maximum error = 4.760843428055148e-003

Table 7. Numerical solution of example 4 with $\varepsilon = 0.01$ and $\varepsilon = 0.001$

x	Approximate solution with $\varepsilon = 0.01$	Approximate solution with $\varepsilon = 0.001$
0.01	0.999999050001247	0.99999905000124
0.02	1.00138293563935	1.00058198682535
0.03	0.998925873414962	1.00122226664290
0.04	0.996674926803085	1.00163440638376
0.05	0.995679052387157	1.00161046638509
0.10	0.991937113766411	0.99564009909598
0.20	0.977375503365064	0.97944747230656
0.30	0.953528615082815	0.95535034155804
0.40	0.921074458355628	0.92258155140267
0.50	0.880918490815012	0.88207633075946
0.60	0.834152379939586	0.83496006907256
0.70	0.782005569199992	0.78249568824034
0.80	0.725792600064296	0.72602775110387
0.90	0.666859273971753	0.66692586481171
1.00	0.606530659712633	0.60653065971263

4. Discussions and Conclusions

We have described and demonstrated the applicability of the fifth order compact difference scheme for a class of singularly perturbed singular two-point boundary value problems. To avoid the singularity at zero a terminal boundary condition in the implicit form is derived. Using this condition as one of the boundary condition we solve the singularly perturbed singular two-point boundary value problem by the fifth order compact difference scheme. We have implemented this method on four examples and tabulated the computational results obtained by present method as well as the exact solutions. We have also presented the least square and maximum errors for the problems considered. It can be observed from the tables that the present method approximates the exact solution very well.

References

[1] C.M. Bender, S.A. Orszag, *Advanced mathematical methods for scientists and engineers,* Mc. Graw-Hill, New York, 1978.

[2] D. Peng, *High-Order numerical method for two-point boundary value problems*, Journal of Computational Physics, 120, (1995), 253-259.

[3] J. J. H. Miller, E. O. Riordan and G. I. Shishkin, *Fitted numerical Methods for Singular Perturbation Problems,* World Scientific, Singapore, 1996.

[4] J. Kevorkian, J.D. Cole, *Multiple Scale and Singular Perturbation Methods,* Springer-Verlag, New York, 1996.

[5] J. Rashidinia, R. Mohammadi, M. Ghasemi, *Cubic spline solution of singularly perturbed boundary value problems with significant first derivatives*, Appl. Math. Comput. 190 (2007) 1762-1766.

[6] Jian Li, *A computational method for solving singularly perturbed two-point singular boundary value problem,* Int. Journal of Math. Analysis, 2 (2008) 1089-1096.

[7] M.K. Kadalbajoo, V.K. Aggarwal, *Fitted mesh B-spline method for solving a class of singular singularly perturbed boundary value problems*, Int. J. Comput. Math. 82 (2005) 67-76.

[8] P.W. Hemker, J.J.H. Miller, *Numerical Analysis of Singular Perturbation Problems,* Academic Press, New York, 1979.

[9] R.E. O'Malley, *Introduction to Singular Perturbations,* Academic Press, New York, 1974.

[10] R.K. Mohanty, N. Jha, *A class of variable mesh spline in compression method for singularly perturbed two point singular boundary value problems,* Appl. Math. Comput. 168 (2005) 704-716.

[11] R.K. Mohanty, Urvashi Arora, *A family of non-uniform mesh tension spline methods for singularly perturbed two-point singular boundary value problems with significant first derivatives*, Appl. Math. Comput. 172 (2006) 531-544.

[12] R.K. Mohanty, D.J. Evans, U. Arora, *Convergent spline in tension methods for singularly perturbed two point singular boundary value problems*, Int. J. Comput. Math. 82 (2005) 55-66.

[13] R.K. Mohanty, N. Jha, D.J. Evans, *Spline in compression method for the numerical solutionof singularly perturbed two point singular boundary value problems,* Int. J. Comput. Math. 81(2004) 615-627.