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Generalised Common Fixed Point Theorem for *S* - Compatible Mapping in Metric Space

R K Gujetiya[1], Dheeraj Kumari Mali[2,*], Mala Hakwadiya[2]

[1]Department of Mathematics, Govt. P. G. College, Neemuch, India
[2]Pacific Academy of Higher Education and Research University Udaipur, Rajasthan, India
*Corresponding author: dheerajmali46@gmail.com

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Abstract In this paper we prove a common fixed point theorem of eight self mappings satisfying a generalized inequality using the concept of compatibility.

Keywords: *common fixed point, compatible mapping, complete metric space*

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1. Introduction

A metric space is just a set X equipped with a function d of two variables which measures the distance between points: d(x, y) is the distance between two points x and y in X. It turns out that if we put mild and natural conditions on the function d, we can develop a general notion of distance that covers distances between number, vectors, sequences, functions, sets and much more. The first important result in the theory of fixed point of compatible mappings was obtained by Gerald Jungck in 1986 [4] as a generalization of commuting mappings. In 1993 Jungck, Murthy and Cho [5] introduced the concept of compatible mappings of type (A) by generalizing the definition of weakly uniformly contraction maps. Pathak and Khan [2] introduced the concept of A-compatible and S-compatible by splitting the definition of compatible mappings of type (A). Fixed point results of compatible mappings are found in [1-8]. Shahidur Rahman [9] proved a common fixed point theorem for A-Compatible and S-Compatible mapping. We will generalized the result of Shahidur Rahman [9].

2. Preliminaries

Definition 2.1: A metric space is given by a set X and a distance function $d : X \times X \to \mathbb{R}$ such that

(i) (Positivity) For all $x, y \in X$, $0 \le d(x, y)$.

(ii) (Non-degenerated) For all $x, y \in X, 0 = d(x, y)$ implais that x = y.

(iii) (Symmetry) For all $x, y \in X, d(x, y) = d(y, x)$.

(iv) (Triangle inequality) For all $x, y, z \in X$,

$$d(x, y) \le d(x, z) + d(z, y).$$

Definition 2.2: [4] Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 2.3: [5] Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be compatible of type (A) if $\lim_{n\to\infty} d(ASx_n, SSx_n) = 0$ and $\lim_{n\to\infty} d(SAx_n, AAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that for $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 2.4: [7] Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be S-compatible if $\lim_{n\to\infty} d(SAx_n, AAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that for $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Proposition 2.5: [2] Let A and S be mappings from a complete metric space (X, d) into itself. If a pair (A, S) is S-compatible on X and $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for $t \in X$, then $AAx_n \to St$ if S is continuous at t.

3. Main Result

Lemma: Let A, B, S, T, D and V be self maps of a complete metric space (X, d) satisfying the following conditions:

(1) $AB(X) \subseteq T(X)$ and $BV(X) \subseteq S(X)$

(2) $\left[d(ADx, BVx)\right]^2 \le a_1 d(ADx, Sx) d(BVy, Ty)$
$+a_2 d(BVy, Sx) d(ADx, Ty) + a_3 d(ADx, Sx) d(ADx, Ty)$
$+a_4 d(BVy, Ty) d(BVy, Sx) + a_5 d^2(Sx, Ty)$

Where, $a_1 + a_2 + 2a_3 + a_4 + a_5 < 1$ and $a_1, a_2, a_3, a_4, a_5 \ge 0$.

(3) Let $x_0 \in X$ then by (1) there exists $x_1 \in X$ such that $Tx_1 = ADx_0$ and for x_1 there exists $x_2 \in X$ such that $Sx_2 = BVx_1$ and so on. Continuing this process we define a sequence $\{y_n\}$ in X such that

$y_{2n+1} = Tx_{2n+1} = ADx_{2n}$ and $y_{2n} = Sx_{2n} = BVx_{2n-1}$. Then the sequence $\{y_n\}$ is Cauchy sequence in X.

By condition (2) and (3),we have

$$\begin{aligned}[d(y_{2n+1}, y_{2n})]^2 &= [d(ADx_{2n}, BVx_{2n-1})]^2 \\ &\le a_1 d\left(ADx_{2n}, Sx_{2n}\right) d\left(BVx_{2n-1}, Tx_{2n-1}\right) \\ &+a_2 d\left(BVx_{2n-1}, Sx_{2n}\right) d\left(ADx_{2n}, Tx_{2n-1}\right) \\ &+a_3 d\left(Ax_{2n}, Sx_{2n}\right) d\left(ADx_{2n}, Tx_{2n-1}\right) \\ &+a_4 d(BVx_{2n-1}, Tx_{2n-1}) d(BVx_{2n-1}, Sx_{2n}) \\ &+a_5 d^2\left(Sx_{2n}, Tx_{2n-1}\right)\end{aligned}$$

$$\begin{aligned}&[d(y_{2n+1}, y_{2n})]^2 \\ &\le a_1 d\left(y_{2n+1}, y_{2n}\right) d\left(y_{2n}, y_{2n-1}\right) \\ &+a_2 d\left(y_{2n}, y_{2n}\right) d\left(y_{2n+1}, y_{2n-1}\right) \\ &+a_3 d\left(y_{2n+1}, y_{2n}\right) d\left(y_{2n+1}, y_{2n-1}\right) \\ &+a_4 d\left(y_{2n}, y_{2n-1}\right) d + a_5 d^2\left(y_{2n}, y_{2n-1}\right)\end{aligned}$$

$$\begin{aligned}&[d(y_{2n+1}, y_{2n})]^2 \\ &\le a_1 d\left(y_{2n+1}, y_{2n}\right) d\left(y_{2n}, y_{2n-1}\right) \\ &+a_3 d^2\left(y_{2n+1}, y_{2n}\right) + a_3 d\left(y_{2n+1}, y_{2n}\right) \\ &+d\left(y_{2n}, y_{2n-1}\right) + a_5 d^2(y_{2n}, y_{2n-1})\end{aligned}$$

$$\begin{aligned}&(1-a_3)\left[\frac{d(y_{2n+1}, y_{2n})}{d(y_{2n}, y_{2n-1})}\right]^2 \\ &\le \left(a_1 + a_3\right)\left[\frac{d(y_{2n+1}, y_{2n})}{d(y_{2n}, y_{2n-1})}\right] + a_5 \\ &\Rightarrow \lambda^2 - \lambda B - c \le 0.\end{aligned}$$

Where $\lambda = \dfrac{d(y_{2n+1}, y_{2n})}{d(y_{2n}, y_{2n-1})}$

$B = \dfrac{a_1 + a_3}{1 - a_3}$

$C = \dfrac{a_5}{1 - a_3}$.

Since $a_1 + a_2 + 2a_3 + a_4 + a_5 < 1$ and $a_1, a_2, a_3, a_4, a_5 \ge 0$.

In order to satisfy the inequation, one value of λ will be positive and the other will be negative. We also note that the sum and product of the two values of λ is less than 1 and -1 respectively. Neglecting the negative value, we have $\dfrac{d(y_{2n+1}, y_{2n})}{d(y_{2n}, y_{2n-1})}$ < p where 0 < p < 1.

$$d\left(y_{2n+1}, y_{2n}\right) < pd\left(y_{2n}, y_{2n-1}\right).$$

Hence $\{y_n\}$ is Cauchy sequence.

Now we prove the following theorem.

Theorem 3.1: Let A, B, S, T, D and V be self maps of a complete metric space (X, d) satisfying the following conditions:

(1) $AB(X) \subseteq T(X)$ and $BV(X) \subseteq S(X)$

(2) $\left[d\left(ADx, BVx\right)\right]^2 \le a_1 d\left(ADx, Sx\right) d\left(BVy, Ty\right)$
$+a_2 d\left(BVy, Sx\right) d\left(ADx, Ty\right) + a_3 d\left(ADx, Sx\right) d\left(ADx, Ty\right)$
$a_4 d\left(BVy, Ty\right) d\left(BVy, Sx\right) + a_5 d^2(Sx, Ty)$

Where, $a_1 + a_2 + 2a_3 + a_4 + a_5 < 1$ and $a_1, a_2, a_3, a_4, a_5 \ge 0$.

(3) one of AD, BV, S or T is continuous.

(4) [AD,S] and [BV,T] are S-compatible mapping on X.

(5) Let $x_0 \in X$ then by (1) there exists $x_1 \in X$ such that $Tx_1 = ADx_0$ and for x_1 there exists $x_2 \in X$ such that $Sx_2 = BVx_1$ and so on. Continuing this process we define a sequence $\{y_n\}$ in X such that

$$\begin{aligned}&y_{2n+1} = Tx_{2n+1} = ADx_{2n} \\ &\textit{and} \quad y_{2n} = Sx_{2n} = BVx_{2n-1}.\end{aligned}$$

Then the sequence $\{y_n\}$ is Cauchy sequence in X.

Then A, D, B, V, S and T have a unique common fixed point.

Proof: By lemma, we have

$\{y_n\}$ is Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $\lim y_n = z$ as $n \to \infty$. Consequently subsequence ADx_{2n}, Sx_{2n}, BVx_{2n-1} and Tx_{2n+1} converges to z.

Let S be a continuous mapping, since AD and S are S-compatible mapping on X. Then by proposition 2.5 we have $(AD)Ax_{2n} \to Sz$ and $SAx_{2n} \to Sz$ as $n \to \infty$.

Now by condition (2) of lemma, we have

$$\begin{aligned}&d^2\left[ADx_{2n}, BVx_{2n-1}\right] \\ &\le a_1 d\left(ADAx_{2n}, SAx_{2n}\right) d\left(BVx_{2n-1}, Tx_{2n-1}\right) \\ &+a_2 d\left(BVx_{2n-1}, SAx_{2n}\right) d\left(ADAx_{2n-1}, Tx_{2n-1}\right) \\ &+a_3 d\left(ADAx_{2n}, SAx_{2n}\right) d\left(ADAx_{2n}, Tx_{2n-1}\right) \\ &+a_4 d\left(BVx_{2n-1}, Tx_{2n-1}\right) d\left(BVx_{2n-1}, SAx_{2n}\right) \\ &+a_5 d^2\left(SAx_{2n}, Tx_{2n-1}\right)\end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned}\left[d\left(Sz, z\right)\right]^2 &\le a_1 d\left(Sz, Sz\right) d\left(z, z\right) + a_2 d\left(z, Sz\right) d\left(Sz, z\right) \\ &+ a_3 d\left(Sz, Sz\right) d\left(Sz, z\right) \\ &+ a_4 d\left(z, z\right) d\left(z, Sz\right) + a_5 d^2(Sz, z) \qquad ,\end{aligned}$$

$$d^2\left(Sz, z\right) \le a_2 d\left(z, Sz\right) d\left(Sz, z\right) + a_5 d^2(Sz, z)$$

$$d^2\left(Sz, z\right) \le (a_2 + a_5)[d(Sz, z)]^2$$

Which is a contradiction.

Hence $Sz = z$. Now by (2), we have

$$\begin{aligned}&\left[d\left(ADz, BVx_{2n-1}\right)\right]^2 \\ &\le a_1 d\left(ADz, Sz\right) d\left(BVx_{2n-1}, Tx_{2n-1}\right) \\ &+a_2 d\left(BVx_{2n-1}, Sz\right) d\left(ADz, Tx_{2n-1}\right) \\ &+a_3 d\left(ADz, Sz\right) d\left(ADz, Tx_{2n-1}\right) \\ &+a_4 d\left(BVx_{2n-1}, Tx_{2n-1}\right) d\left(BVx_{2n-1}, Sz\right) \\ &+a_5 d^2(Sz, Tx_{2n-1})\end{aligned}$$

Letting $n \to \infty$, we have

$$\left[d\left(ADz,z\right)\right]^2 \le a_1 d\left(ADz,z\right) d\left(z,z\right) + a_2 d\left(z,z\right) d\left(ADz,z\right) + a_3 d\left(ADz,z\right) d\left(ADz,z\right) + a_4 d\left(z,z\right) d\left(z,z\right) + a_5 d^2(z,z)$$

Hence $ADz = z$.

Now since by condition (1), $z \in T(X)$. Also T is self map of X so there exists a point $u \in X$ such that $ADz = z = Tu$. Moreover by condition (2) we obtain,

$$[d(z,BVu)]^2 = \left[d\left(ADz,BVu\right)\right]^2 \le a_1 d\left(ADz,Sz\right) d\left(BVu,Tu\right) + a_2 d\left(BVu,Sz\right) d\left(ADz,Tu\right) + a_3 d\left(ADz,Sz\right) d\left(ADz,Tu\right) + a_4 d\left(BVu,Tu\right) d\left(BVu,Sz\right) + a_5 d^2(Sz,Tu)$$

$$\left[d\left(z,BVu\right)\right]^2 \le a_1 d\left(z,z\right) d\left(BVu,z\right) + a_2 d\left(BVu,z\right) d\left(z,z\right) + a_3 d\left(z,z\right) d\left(z,z\right) + a_4 d\left(BVu,z\right) d\left(BVu,z\right) + a_5 d^2(z,z)$$

$$\left[d\left(z,BVu\right)\right]^2 \le a_4 \left[d\left(z,BVu\right)\right]^2.$$

Hence $BVu = z$ i.e., $z = BVu = Tu$.

By condition (4), we have $d(TBVu, BVTu) = 0$.

Hence $d(Tz, BVz) = 0$ i.e., $Tz = BVz$.

Now by condition (2), we have

$$[d(z,Tz)]^2 = \left[d\left(ADz,BVz\right)\right]^2$$

$$[d(z,Tz)]^2 \le a_1 d\left(ADz,Sz\right) d\left(BVz,Tz\right) + a_2 d\left(BVz,Sz\right) d\left(ADz,Tz\right) + a_3 d\left(ADz,Sz\right) d\left(ADz,Tz\right) a_4 d\left(BVz,Tz\right) d\left(BVz,Sz\right) + a_5 d^2(Sz,Tz)$$

$$\left[d\left(z,Tz\right)\right]^2 \le a_1 d\left(z,z\right) d\left(Tz,Tz\right) + a_2 d\left(Tz,z\right) d\left(z,Tz\right) + a_3 d\left(z,z\right) d\left(z,Tz\right) + a_4 d\left(Tz,Tz\right) d\left(Tz,z\right) + a_5 d^2(z,Tz)$$

$$[d(z,Tz)]^2 \le a_2 d(Tz,z) d(z,Tz) + a_5 d^2(z,Tz)$$

$$[d(z,Tz)]^2 \le (a_2 + a_5)[d(z,Tz)]^2.$$

Which is a contradiction.

Hence $z = Tz$ i.e., $z = Tz = BVz$.

Now putting x = Dz and $y = x_{2n-1}$ in (2),then we have

$$\left[d\left(AD(Dz),BVx_{2n-1}\right)\right]^2 \le a_1 d\left(AD(Dz),S(Dz)\right) d\left(BVx_{2n-1},Tx_{2n-1}\right) + a_2 d\left(BVx_{2n-1},S(Dz)\right) d\left(AD\left(Dz\right),Tx_{2n-1}\right) + a_3 d\left(AD(Dz),S(Dz)\right) d\left(AD(Dz),Tx_{2n-1}\right) + a_4 d\left(BVx_{2n-1},Tx_{2n-1}\right) d\left(BVx_{2n-1},S(Dz)\right) + a_5 d^2(S(Dz),Tx_{2n-1})$$

Letting n→∞,we have

$$\left[d\left(Dz,z\right)\right]^2 \le a_1 d\left(Dz,Dz\right) d\left(z,z\right) + a_2 d\left(z,Dz\right) d\left(Dz,z\right) + a_3 d\left(Dz,Dz\right) d\left(Dz,z\right) + a_4 d\left(z,z\right) d\left(z,Dz\right) + a_5 d^2(z,Dz)$$

$$\left[d\left(Dz,z\right)\right]^2 \le \left(a_2 + a_5\right)\left[d\left(Dz,z\right)\right]^2$$

Which is a contradiction.

Hence $z = Dz$. Since $ADz = z$ which implies that $Az = z$.

Now putting $x = x_{2n}$ and $y = Vz$ in (2), then we have

$$[d(ADx_{2n}, BV(Vz)]^2 \le a_1 d\left(ADx_{2n},Sx_{2n}\right) d\left(BV(Vz),T(Vz)\right) + a_2 d\left(BV(Vz),Sx_{2n}\right) d\left(ADx_{2n},T(Vz)\right) + a_3 d\left(Ax_{2n},Sx_{2n}\right) d\left(ADx_{2n},T(Vz)\right) + a_4 d(BV(Vz),T(Vz)) d(BV(Vz),Sx_{2n}) + a_5 d^2\left(Sx_{2n},T(Vz)\right)$$

Letting n→∞,we have

$$\left[d\left(z,Vz\right)\right]^2 \le a_1 d\left(z,z\right) d\left(Vz,Vz\right) + a_2 d\left(Vz,z\right) d\left(z,Vz\right) + a_3 d\left(z,z\right) d\left(z,Vz\right) + a_4 d\left(Vz,Vz\right) d\left(Vz,z\right) + a_5 d^2(z,Vz)$$

$$[d(z,Vz)]^2 \le a_2 d\left(Vz,z\right) d\left(z,Vz\right) + a_5 d^2(z,Vz)$$

$$[d(z,Vz)]^2 \le (a_2 + a_5)[d(z,Vz)]^2.$$

Which is a contradiction.

Hence $z = Vz$. Since BVz = z which implies that Bz = z.

Therefore z is a common fixed point of A, D, B, V, S and T.

Uniqueness: Finally, to prove the uniqueness of z, suppose w be another common fixed point of A, D, B, V, S and T. Then we have,

$$[d(z,w)]^2 = \left[d\left(ADz,BVw\right)\right]^2 \le a_1 d\left(ADz,Sz\right) d\left(BVw,Tw\right) + a_2 d\left(BVw,Sz\right) d\left(ADz,Tw\right) + a_3 d\left(ADz,Sz\right) d\left(ADz,Tw\right) + a_4 d\left(BVw,Tw\right) d\left(BVw,Sz\right) + a_5 d^2(Sz,Tw)$$

$$\left[d\left(z,w\right)\right]^2 \le a_1 d\left(z,z\right) d\left(w,w\right) + a_2 d\left(w,z\right) d\left(z,w\right) + a_3 d\left(z,z\right) d\left(z,w\right) + a_4 d\left(w,w\right) d\left(w,z\right) + a_5 d^2\left(z,w\right)$$

$$[d(z,w)]^2 \le \left(a_2 + a_5\right)[d(z,w)]^2,$$

Which is a contradiction. Hence $z = w$.

Thus z is a unique common fixed point of A, D, B, V, S and T.

4. Conclusion

In this paper, we have presented common fixed point theorem for six self mappings in metric spaces through concept of compatibility.

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