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Related Fixed Point Theorem for Mappings on Three Metric Spaces

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Abstract In this note we obtained a new related fixed point theorem on three metric spaces of which one is compact. Here we consider three mappings, not all of which are necessarily continuous. Our result generalizes some earlier results.

Keywords: *compact metric space, related fixed point*

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1. Introduction

Related fixed point theorems on three metric spaces have been studied by several authors [1-8]. Fisher and Rao [8] proved a related fixed point theorem for three mappings on three metric spaces of which one is a compact metric space. The aim of this paper is to improve the result of Rao *et. al.* [6] and Fisher and Rao [8].

2. Main Results

We prove the following theorem.

2.1. Theorem. Let (X, d), (Y, ρ) and (Z, σ) be three metric spaces and $T: X\to Y$, $S:Y\to Z$ and $R:Z\to X$ be mappings satisfying the inequalities

$$d(RSTx, RSTx') < \max\left\{\begin{matrix} d(x,x'), d(x,RSTx), \\ d(x',RSTx'), \rho(Tx,Tx'), \\ \sigma(STx,STx') \end{matrix}\right\} \quad (1)$$

$$\rho(TRSy, TRSy') < \max\left\{\begin{matrix} \rho(y,y'), \rho(y,TRSy), \\ \rho(y',TRSy'), \sigma(Sy,Sy'), \\ d(RSy,RSy') \end{matrix}\right\} \quad (2)$$

$$\sigma(STRz, STRz') < \max\left\{\begin{matrix} \sigma(z,z'), \sigma(z,STRz), \\ \sigma(z',STRz'), d(Rz,Rz'), \\ \rho(TRz,TRz') \end{matrix}\right\} \quad (3)$$

for all x, x' in X, y, y' in Y and z, z' in Z. Further assume one of the following conditions:

(i) (X, d) is compact and RST is continuous.
(ii) (Y, ρ) is compact and TRS is continuous.
(iii) (Z, σ) is compact and STR is continuous.

Then RST has a unique fixed point w in X, TRS has a unique fixed point u in Y and STR has a unique fixed point v in Z. Further $Su = v$, $Rv = w$ and $Tw = u$.

Proof: Suppose (i) holds. Define $\Phi(x) = d(x, RSTx)$ for $x \in X$. Then there exists p in X such that

$$\Phi(p) = d(p, RSTp) = \inf\{\Phi(x) : xX\}.$$

Suppose that $RSTRSTRSTp \neq RSTRSTp$.

Then $STRSTRSTp\neq STRSTp$, $TRSTRSTp\neq TRSTp$, $RSTRSTp\neq RSTp$, $STRSTp\neq STp$, $TRSTp\neq Tp$, $RSTp\neq p$.

Using (1) with $x=RSTp$ and $x'=RSTRSTp$

$$d(RSTRSTp, RSTRSTRSTp) < \max\{d(RSTp, RSTRSTp),$$

$$d(RSTRSTp, RSTRSTRSTp)$$

$$< \max\left\{\begin{matrix} d(RSTp, RSTRSTp), \\ d(RSTp, RSTRSTp), \\ d(RSTRSTp, RSTRSTRSTp), \\ \rho(TRSTp, TRSTRSTp), \\ \sigma(STRSTp, STRSTRSTp) \end{matrix}\right\}$$

so that

$$\Phi(RSTRSTp) < \max\left\{\begin{matrix} \rho(TRSTp, TRSTRSTp), \\ \sigma(STRSTp, STRSTRSTp) \end{matrix}\right\} \quad (4)$$

Using (2) with $y = Tp$ and $y' = TRSTp$

$$\rho(TRSTp, TRSTRSTp) < \max\left\{\begin{matrix} \rho(Tp, TRSTp), \\ \rho(Tp, TRSTp), \\ \rho(TRSTp, TRSTRSTp), \\ \sigma(STp, STRSTp), \\ d(RSTp, RSTRSTp) \end{matrix}\right\}$$

so that

$$\rho(TRSTp,TRSTRSTp) < \max\{\sigma(STp,STRSTp),\Phi(RSTp)\} \quad (5)$$

Using (3) with $z = STp$ and $z' = STRSTp$

$$\sigma(STRSTp,STRSTRSTp) < \max\begin{Bmatrix}\sigma(STp,STRSTp),\sigma(STp,STRSTp),\\ \sigma(STRSTp,STRSTRSTp),d(RSTp,RSTRSTp),\\ \rho(TRSTp,TRSTRSTp)\end{Bmatrix}$$

so that

$$\sigma(STRSTp,STRSTRSTp) < \max\{\Phi(RSTp),\rho(TRSTp,TRSTRSTp)\} \quad (6)$$

From (4), (5) and (6) it follows that Φ(*RSTRSTp*) < Φ(*RSTp*), contradicting the existence of *p*.

Hence RSTRSTRSTp = RSTRSTp

Putting *RSTRSTp* = *w* in *X*, we have

$$RSTw = w.$$

Now let *Tw* = *u* in *Y* and *Su* = *v* in *Z*. Then *Rv* = *RSu* = *RSTw* = *w* and it follows that

$$STRv = STw = Su = v$$

and

$$TRSu = TRv = Tw = u.$$

To prove uniqueness, suppose *RST* has a second distinct fixed point *w′* in *X*.

Then

$$RSTw \neq RSTw', STw \neq STw', Tw \neq Tw'.$$

Using (1) with $x = w$ and $x' = w'$

$$d(RSTw,RSTw') < \max\begin{Bmatrix}d(w,w'),d(w,RSTw),\\ d(w',RSTw'),\rho(Tw,Tw'),\\ \sigma(STw,STw')\end{Bmatrix}$$

so that

$$d(w,w') < \max\{\rho(Tw,Tw'),\sigma(STw,STw')\} \quad (7)$$

Using (2) with $y = Tw$ and $y' = Tw'$

$$\rho(TRSTw,TRSTw') < \max\begin{Bmatrix}\rho(Tw,Tw'),\rho(Tw,TRSTw),\\ \rho(Tw',TRSTw'),\sigma(STw,STw'),\\ d(RSTw,RSTw')\end{Bmatrix}$$

so that

$$\rho(Tw,Tw') < \max\{\sigma(STw,STw'),d(w,w')\} \quad (8)$$

Using (3) with $z = STw$ and $z' = STw'$

$$\sigma(STRSTw,STRSTw') < \max\begin{Bmatrix}\sigma(STw,STw'),\sigma(STw,STRSTw),\\ \sigma(STw,STRSTw'),d(RSTw,RSTw'),\\ \rho(TRSTw,TRSTw')\end{Bmatrix}$$

so that

$$\sigma(STw,STw') < \max\{d(w,w'),\rho(Tw,Tw')\} \quad (9)$$

From (7), (8) and (9), it follows that

$$d(w,w') < d(w,w')$$

so that *w* = *w′*, proving the uniqueness of *w*.

Similarly we can show that *v* is the unique fixed point of *STR* and *u* is the unique fixed point of *TRS*.

It follows similarly that the theorem holds if (ii) or (iii) holds instead of (i).

Now we give the following example to illustrate our theorem.

Example. Let *X* = [0, 1], *Y* = (1, 2], *Z* = (2, 3] and let *d* = *ρ* = *σ* be the usual metric for the real numbers. Define *T*: *X*→*Y*, S: *Y*→*Z* and *R*: *Z*→*X* by

$$Tx = \begin{cases} 1 & \text{if } x \in [0,4/5) \\ 3/2 & \text{if } x \in [4/5, 1] \end{cases}$$

$$Sy = 3 \text{ for all } y \text{ in } Y$$

$$Rz = \begin{cases} 3/4 & \text{if } z \in (2,7/3] \\ 1 & \text{if } z \in (7/3, 3] \end{cases}$$

Here *Y* and *Z* are not compact spaces and *T* and *R* not continuous. However all the conditions of theorem 2.1 are satisfied. Clearly,

$$RST(1) = 1, TRS(3/2) = 3/2,$$
$$STR(3) = 3, S(3/2) = 3,$$
$$R\text{-}(3) = 1 \text{ and } T(1) = 3/2.$$

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