American Journal of Applied Mathematics and Statistics, 2024, Vol. 12, No. 2, 28-34
Available online at http://pubs.sciepub.com/ajams/12/2/2
Published by Science and Education Publishing
DOI:10.12691/ajams-12-2-2

Solutions of Linear Stochastic Differential Equations for Economic Investments

Azor, P. A.[1,*], Nwobi, F. N.[2], Amadi, I. U.[3]

[1]Department of Mathematics & Statistics, Federal University, Otuoke, Nigeria
[2]Department of Statistics, Imo State University, Owerri, Nigeria
[3]Department of Mathematics and Statistics, Captain Elechi Amadi Polytechnics, Port Harcourt, Nigeria
*Corresponding author: azorpa@fuotuoke.edu.ng

Received March 15, 2024; Revised April 17, 2024; Accepted April 24, 2024

Abstract The advantage of financial assets and their return rates lie in economic investments which accumulate wealth daily, monthly, yearly and probably periodically, etc. For that reason, this paper considered the problem of a system of Stochastic Differential Equations (SDEs) for economic investments whose rate of returns and asset valuation follow series price index, periodic additive effects and periodic multiplicative effects; which are used as major parameters in the model. The problems were accurately solved independently by adopting Ito's theorem which presented closed-form diverse investment results for proper investment decisions. Finally, we presented graphical results which represented the behaviour of the economic investments and discussed the effect of the relevant parameters.

Keywords: *stock price, rate of returns, stochastic analysis, series price effect, periodic additive effect, periodic multiplicative effect, ito's lemma*

Cite This Article: Azor, P. A., Nwobi, F. N., and Amadi, I. U., "Solutions of Linear Stochastic Differential Equations for Economic Investments." *American Journal of Applied Mathematics and Statistics*, vol. 12, no. 2 (2024): 28-34. doi: 10.12691/ajams-12-2-2.

1. Introduction

Basically, one important factor in the financial market which an investor does not downplay is an asset, because of its advantages to the investor. Consequently, the estimation of an asset is a process of determining the market value of asset prices. Asset valuation has become so instrumental for impelling economic variations, distributing economic resources to subdivisions with time and controlling the assets of the whole system that produces different returns. Return on investment for the capital market is an estimate which correctly examines the profit of an investment. Return on investment is useful to the investors in the regulation of gains made out of the expenditure.

On the other hand, looking at this kind of problem, an analytic method that gives an exact solution for appropriate mathematical prediction is therefore required. Problems that are related to asset valuation, proper formulation and exact solutions are necessary requirements for the estimation of accurate valuation of results, thus, following the specific feature of the problem, the analytic solution is utilized.

However, stock market prices have been studied in a wide manner by many researchers. For instance,[1] studied the stochastic analysis of the stock market price model, and considered the stochastic analysis of the behaviour of stock prices using a proposed log-normal distribution model. Their results showed that the proposed model is efficient for the prediction of stock prices. [2] studied the stochastic analysis of stock market expected returns for investors. Their results of comparing the variances of four different stocks indicated that stock 1 is the best among the stocks of different companies, which is consistent with the work of [3]. Also, [4] investigated a stochastic analysis of stock market expected returns and growth rates. In trying to analyse the unstable nature of stock market forces stochastic, [5] considered the stability analysis of a stochastic model for stock market prices and did an analysis of the unstable nature of stock market forces by applying a new differential equation model that can impact the expected returns of investors in stock exchange market with a stochastic volatility in the equation. While [6] suggested in their study, some analytical solutions of stochastic differential equations with respect to Martingale processes and discovered that the solutions of some SDEs are related to other stochastic equations with diffusion part. The second technique is to change SDE to ODE that is tried to omit the diffusion part of the stochastic equation by using Martingale processes.

On the other hand, [7] looked at the numerical techniques of solving stochastic differential equations like the Euler- Maruyama and Milstein methods based on the truncated Ito-Taylor expansion by solving a non-linear stochastic differential equation and approximated numerical solution using Monte Carlo simulation for each scheme. Their results showed that if the discretization value N is increasing, the Euler-Maruyama and Milstein techniques were close to exact solution. [8] worked on the stability of both analytical and numerical solutions for non-linear stochastic delay differential equations with jumps and they observed that the compensated stochastic methods inherit the stability property of the correct solution. [9] studied the solution of differential equations and stochastic differential equations of time-varying investment returns and obtained precise conditions governing asset price returns rate via multiplicative and multiplicative inverse trend series. The proposed model showed an efficient and reliable multiplicative inverse trend series than the multiplicative trend in both deterministic and stochastic systems. [10] studied a stochastic model of the fluctuations of stock market price and obtained precise conditions for determining the equilibrium price. The model constrains the drift parameters of the price process in a manner that is adequately characterized by the volatility. [11] examined the stability behaviours of stochastic differential equations (SDEs) driven by time-changed Brownian motions. A connection between the stability of the solution to the time-changed stochastic differential equations and their corresponding non-time-changed stochastic differential equations was shown using the duality theorem. [12] studied stochastic methods in the practical delineation of financial models and suggested the Euler-Maruyama method as the stochastic differential equation expression as potentially useful for delineation of asset stock price and volatility.

This paper examines a system of stochastic differential equations for economic investments whose rate of returns during periodic events are assumed to follow: series price index, additive effects and multiplicative effects; all having quadratic functions over time. These investment equations were solved independently by adopting Ito's theorem which presented a closed-form analytical solution.

2. Purpose of This Paper

The aim of this paper is to adopt an analytical solution to investment equations. This type of analysis is important because it presents an opportunity for the regular assessment of asset value. It helps investors to make informed decisions about buying, selling or holding unto their investments. More so, periodic valuations help investors identify potential risks and steps to be taken to mitigate them. This paper extends the works of [13] and [14] by incorporating series price index parameter in SDE domain and investments during periodic events which was not considered by the previous efforts.

This paper is prescribed as follows: Section 2 presents the mathematical preliminaries, the method of solution is seen in Subsection 2.2, Results and discussion are seen in Section 3 and the paper is concluded in Section 4.

3. Mathematical Preliminaries

At this juncture, we present a few basic definitions as touching this dynamic area of study, hence we have as follows:

Definition 1: Probability space: This is a triple $(\Omega, \mathbb{F}, \wp)$ where Ω represents a set of sample space, $\mathbb{F}$ represents a collection of subsets of Ω, while $\wp$ is the probability measure defined on each event $A \in \mathbb{F}$. The collection $\mathbb{F}$ is a σ-algebra or σ-field such as $\Omega \in \mathbb{F}$ and $\mathbb{F}$ is closed under the arbitrary unions and finite intersections. Hence it is called probability measure when the following condition holds.

(i) $$P(A) \geq 0 \;\; for\ all\ \ A \subset \Omega \tag{1}$$

(ii) $$P(\Omega) = 1 \tag{2}$$

(iii) $$A, B \subset \Omega,\ A \cap B = \phi\ then\ P(A \cup B) = P(A) + P(B) \tag{3}$$

Definition 2: A σ-algebra is a set $\mathbb{F}$ of subsets of Ω with the following axioms:

(i) $$\phi, \Omega \in \mathbb{F} \tag{4}$$

(ii) $$\text{If } \ A \in \mathbb{F}, then\ A^c \in \mathbb{F} \tag{5}$$

(iii) If $$A_1, A_2, \ldots, \in \mathbb{F}, then \bigcup_{k=1}^{\infty} A_k, \bigcap_{k=1}^{\infty} A_k \in \mathbb{F} \tag{6}$$

Clearly $A^c := \Omega - A$ is the complement of A.

Definition 3: A random variable X is said to be a function if $X: \Omega \to \mathfrak{A}$ that is $\mathbb{F}$-measurable if for all $B \in B(\mathfrak{A})$,

$$f^{-1}(B) \in F \tag{7}$$

Theorem1. $(\sigma - \text{algebra})$:there exists a $\sigma -$ algebra such that $\sigma(X) := \{X^{-1}(B) : B \in B(\mathbb{R})\}$.

Proof: We want to show that the $\sigma -$ algebra generated by X.

Now, we have that $\varphi = X^{-1}(\varphi)$, So there exists $B_K \in B(\mathbb{R})$ such that $X^{-1}(B_K) = K$. We note that:

$$K^c = \left(X^{-1}(B_K)\right)^c = X^{-1}\left(B_K^c\right) \in \sigma(\mathrm{X}). \tag{8}$$

Let us assume $K_1, K_2, \ldots, \in \sigma(X)$. Now, there exists:

$$B_1, B_2, \ldots, \in B(\mathbb{R}) \text{ such that } \mathrm{X}^{-1}(B_i) = K_i \tag{9}$$

We find that

$$\cup_{i=1}^{\infty} K_i = \cup_{i=1}^{\infty}\left[X^{-1}(B_i)\right] = X^{-1}\left(\cup_{i=1}^{\infty} B_i\right) \in \sigma(\mathrm{X}). \tag{10}$$

Therefore $\sigma(\mathrm{X})$ is $\sigma -$ algebra generated by X having satisfied the axioms.

Definition 4: A stochastic process $X(t)$ is a relation of random variables $\{X_t(\gamma), t \in T, \gamma \in \Omega\}$, i.e, for each t in the index set $T, X(t)$ is a random variable. Now we understand t as time and call $X(t)$ the state of the procedure at time t. In view of the fact that a stochastic process is a relation of random variables, its requirement is similar to that for random vectors.

Definition 5. Random Walk: There are different methods to which we can state a stochastic process. Then relating the process in terms of movement of a particle which moves in discrete steps with probabilities from a point $x = a$ to a point $x = b$. A random walk is a stochastic sequence $\{S_n\}$ with $S_0 = 0$, defined by

$$S_n = \sum_{k-1}^{n} X_k \tag{11}$$

where X_k are independent and identically distributed random variables

Definition 6: A standard Brownian motion is simply a stochastic process $\{B_t\}_{t\in\tau}$ with the following properties:

i). With probability 1, $B_0 = 0$.

ii). For all $0 \le t_1 \le t_2 \le \cdots \le t_n$, the increments $B_{t2-}B_{t1}, B_{t3-}B_{t2}, B_{t2}, \cdots, B_{tn} - B_{tn-1}$ are independent.

iii). For $t \ge s \ge 0$, $B_t - B_s \sim N(0, t-s)$.

iv). With probability 1, the function $\to B_i$ is continuous.

Theorem 1.2: (Ito's formula) Let $(\Omega, \beta, \mu, F(\beta))$ be a filtered probability space $X = \{X, t \ge 0\}$ be an adaptive stochastic process on $(\Omega, \beta, \mu, F(\beta))$ possessing a quadratic variation (X) with SDE defined as:

$$dX(t) = g(t, X(t))dt + f(t, X(t))dW(t)$$

$t \in \mathbb{R}$ and for $u = u(t, X(t) \in C^{1\times 2}(\Pi \times \mathbb{R}))$

Details of the above can be seen in [15,16,17].

3.1. Mathematical Formulation of the Problem

Though, the price evolution of a risky assets are usually modelled as the trajectory of a risky assets that are usually of a diffusion process defined on some underlying probability space, with the geometric Brownian motion the paramount tool used as the established reference model, [10].Therefore, we assume economic investment whose rate of return grows in the form as follows: series index price function, periodic additive effect series and multiplicative effect series respectively at time t. Hence the rate of return is defined following the method of [9] and [14]:

$$R_t := Cos(\lambda_1 + \lambda_2)^2 \tag{12}$$

Where $t = 1, 2, ..., \mathrm{N}$

$$R_t := Sin(\lambda_1 \lambda_2)^2 \tag{13}$$

Where $t = 1, 2, ..., \mathrm{N}$

Thus, the stochastic process describing the process is of the form:

$$dS(t) = \lambda dt + \beta dZ^{(1)}(\mathrm{t}) \tag{14}$$

where λ is an expected rate of returns on stock, β is the volatility of the stock, dt is the relative change in the price during the period of time and $Z^{(1)}$ is a Wiener process. Using (12)-(14) gives the following mathematical structure:

$$dS_\omega(t) = \alpha_\omega \mathrm{S}_\omega(t)dt + \sigma S_\omega(t)dZ(\mathrm{t}) \tag{15}$$

$$dS_\theta(t) = Cos(\alpha_1 + \alpha_2)^2 \mathrm{S}_\theta(t)dt + \sigma S_\theta(t)dZ(\mathrm{t}) \tag{16}$$

$$dS_\phi(t) = Sin(\alpha_1 \alpha_2)^2 \mathrm{S}_\phi(t)dt + \sigma S_\phi(t)dZ(\mathrm{t}) \tag{17}$$

where $S_\omega(\mathrm{t}), S_\theta(\mathrm{t})\, and\, S_\phi(\mathrm{t})$ are underlying stocks with the following initial conditions;

$$S_\omega(0) = S_0(\omega), t > 0 \tag{18}$$

$$S_\theta(0) = S_0(\theta), t > 0 \tag{19}$$

$$S_\phi(0) = S_0(\phi), t > 0 \tag{20}$$

Where, $\alpha_\omega = \left(\frac{ks}{rt}\right)^2 + \left(\frac{ks}{rt}\right)^4 + \left(\frac{ks}{rt}\right)^6 + \left(\frac{ks}{rt}\right)^8 +, ...,$ [13], is the drift which is the expected rate of returns and σ the volatility of the stock, $dz(t)$ is a Brownian motion or Wiener's process, $S(t)$ be the price of some risky asset at time t and dt as a relative change during the trading days

3.2. Method of Solution

The models (15)-(17) are made up of a system of variable coefficient stochastic differential equations whose solutions are not trivial. We implement the methods of Ito's lemma in solving for $\mathrm{S}_\omega(t), \mathrm{S}_\theta(t)\, and\, S_\phi(t)$. To grab this problem, we note that we can forecast the future worth of the asset with sureness. The expression dZ, which contains the randomness that is certainly a characteristic of asset prices is called a Wiener process or Brownian motion.

$$dZ^2 \to dt\ as\ dt \to 0 \tag{21}$$

So, the smaller dt becomes, the more certainly dZ^2 is equal to dt. Suppose that $f(S_\omega)$ is a smooth function of S_ω and forget for the moment that S_ω is stochastic. If we vary S_ω by a small amount, dS_ω then clearly f also is varied by a small amount provided we are not close to singularities of f. Applying Taylor series expansion, we can write

$$df = \frac{df}{dS}dS + \frac{1}{2}\frac{d^2 f}{dS^2}dS^2 + \ldots \tag{22}$$

3.2.1. Series Price Index

From (15) and squaring both sides

$$dS^2 = \left(\sigma S dx + \mu S dt\right)^2$$

$$= \sigma^2 S^2 dx^2 + 2\sigma\mu S^2 dt dx + \mu^2 S^2 dt^2$$

But

$$dx = o\left(\sqrt{dt}\right)$$

$$dS^2 = \sigma^2 S^2 dx^2 + \ldots$$

Since

$$dx^2 \to dt,\ dS^2 \to \sigma^2 S^2 dt$$

Substituting (22) and retain only those terms which are at least as large as $o(dt)$ using the definition of dS from (15), we find that

$$df = \frac{df}{dS}\left(\sigma S dx + \mu S dt\right) + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2}dt$$

$$= \sigma S\frac{df}{dS}dx + \left(\mu S\frac{df}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2}\right)dt \tag{23}$$

Thus, generalizing from (15) and considering a function $f\left(S_\omega(t),t\right)$, of the random variable S_ω and of time, t. There are two independent variables S_ω and, t, hence it has to do with partial derivatives. Expansion of $f\left(S_\omega(t), \mathrm{d}S_\omega(\mathrm{t}), t+dt\right)$ in a Taylor series about $\left(S_\omega(t),t\right)$ gives

$$df = \frac{\partial f}{\partial S_\omega(t)}dS_\omega(t) + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial S_\omega^{\ 2}(t)}dS_\omega^2(t) + \ldots \tag{24}$$

Substituting (15) in (21) gives;

$$df = \frac{\partial f}{\partial S_\omega(t)}\left\{\sigma S_\omega(t)dZ(\mathrm{t}) + \alpha_\omega S_\omega(t)dt\right\}$$
$$+\frac{1}{2}\frac{\partial^2 f}{\partial S_\omega^{\ 2}(t)}dS_\omega^2(t)\,\mathrm{d}S_\omega^2(\mathrm{t})$$

$$df = \sigma S_\omega(t)\frac{\partial f}{\partial S_\omega(t)}dZ(t)$$
$$+\left(\alpha_\omega S_\omega(t)\frac{\partial f}{\partial S_\omega(t)} + \frac{1}{2}\sigma^2 S_\omega^2(t)\frac{\partial^2 f}{\partial S_\omega^{\ 2}(t)}dS_\omega^2(t)\right)dt$$

Now considering the SDE in (15) and letting $f\left(S_\omega(\mathrm{t})\right) = \ln S_\omega(t)$, the partial derivatives becomes

$$\frac{\partial f}{\partial S_\omega(t)} = \frac{1}{S_\omega(t)},\ \frac{\partial^2 f}{\partial S^2{}_\omega(t)} = -\frac{1}{S_\omega^2(t)},\ \frac{\partial f}{\partial t} = 0 \tag{25}$$

Following Theorem 1.2 (Ito's), substituting in (25) and simplifying gives;

$$df = \left\{\alpha_\omega - \frac{1}{2}\sigma^2\right\}dt + \sigma dZ \tag{26}$$

Since the RHS of (26) is independent of $f\left(S_\omega(t)\right)$, the stochastic is computed as follows:

$$f\left(S_\omega(t)\right) = f_0 + \int_0^t\left\{\alpha_\omega - \frac{1}{2}\sigma^2\right\}dt + \int_0^t \sigma dZ(t)$$

$$= f_0 + \left\{\alpha_\omega - \frac{1}{2}\sigma^2\right\}t + \sigma dZ(t),$$

Since $f\left(S_\omega(t)\right) = \ln S_\omega(t),$ a found solution for $S_\omega(t)$ becomes

$$\ln S_\omega(t) = \ln S_0(t) + \left\{\alpha_\omega - \frac{1}{2}\sigma^2\right\}t + \sigma dZ(t),$$

$$\ln S_\omega(t) - \ln S_0(t) = \left\{\alpha_\omega - \frac{1}{2}\sigma^2\right\}t + \sigma dZ(t)$$

$$\ln\left(\frac{S_\omega(t)}{S_0(t)}\right) = \left\{\alpha_\omega - \frac{1}{2}\sigma^2\right\}t + \sigma dZ(t)$$

$$S_\omega(t) = S_0(t)\exp\left\{\left(\alpha_\omega - \frac{1}{2}\sigma^2\right)t + \sigma dZ(t)\right\} \tag{27}$$

$$\text{where, } \alpha_\omega = \left(\frac{ks}{rt}\right)^2 + \left(\frac{ks}{rt}\right)^4 + \left(\frac{ks}{rt}\right)^6 + \left(\frac{ks}{rt}\right)^8 +, \ldots,$$

This is the complete solution of investment equation whose rate of returns and asset price valuation follows series price index.

3.2.2. Periodic Additive Effects

From (16), Thus, generalizing and considering a function $f\left(S_\theta(t),t\right)$, of the random variable S_θ and of time, t. There are two independent variables S_θ and, t hence it has to do with partial derivatives. Expansion of $f\left(S_\theta(t), \mathrm{d}S_\theta(\mathrm{t}), t+dt\right)$ in a Taylor series about $\left(S_\theta(t),t\right)$ gives

$$df = \frac{\partial f}{\partial S_\theta(t)}dS_\theta(t) + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial S_\theta^{\ 2}(t)}dS_\theta^2(t) + \ldots \tag{28}$$

Substituting (16) in (28) gives;

$$df = \frac{\partial f}{\partial S_\theta(t)}\{\sigma S_\theta(t)dZ(\mathrm{t}) + \alpha_\theta S_\theta(t)dt \ \} + \frac{1}{2}\frac{\partial^2 f}{\partial {S_\theta}^2(t)} dS_\theta^2(t)\,\mathrm{d}S_\theta^2(\mathrm{t})$$

$$df = \sigma S_\theta(t)\frac{\partial f}{\partial S_\theta(t)}dZ(t) + \left(\alpha_\theta S_\theta(t)\frac{\partial f}{\partial S_\theta(t)} + \frac{1}{2}\sigma^2 S_\theta^2(t)\frac{\partial^2 f}{\partial {S_\theta}^2(t)}dS_\theta^2(t)\right)dt$$

Now considering the SDE in (16);

Let $f\left(S_\theta(\mathrm{t})\right) = \ln S_\theta(t)$, the partial derivatives becomes

$$\frac{\partial f}{\partial S_\theta(t)} = \frac{1}{S_\theta(t)}, \ \frac{\partial^2 f}{\partial S_\theta^2(t)} = -\frac{1}{S_\theta^2(t)}, \ \frac{\partial f}{\partial t} = 0 \tag{29}$$

following Theorem 1.1(Ito's), substituting in (29) and simplifying gives

$$df = \left\{Cos\left(\alpha_1 + \alpha_2\right)^2 - \frac{1}{2}\sigma^2\right\}dt + \sigma dZ \tag{30}$$

Since the RHS of (29) is independent of $f\left(S_\theta(t)\right)$,the stochastic is computed as follows:

$$f\left(S_\theta(t)\right) = f_0 + \int_0^t \left\{Cos\left(\alpha_1 + \alpha_2\right)^2 - \frac{1}{2}\sigma^2\right\}dt + \int_0^t \sigma dZ(t)$$

$$= f_0 + \left\{Cos\left(\alpha_1 + \alpha_2\right)^2 - \frac{1}{2}\sigma^2\right\}t + \sigma dZ(t),$$

Since $f\left(S_\theta(t)\right) = \ln S_\theta(t)$ a found solution for $S_\theta(t)$ becomes;

$$\ln S_\theta(t) = \ln S_0(t) + \left\{Cos\left(\alpha_1 + \alpha_2\right)^2 - \frac{1}{2}\sigma^2\right\}t + \sigma dZ(t),$$

$$\ln S_\theta(t) - \ln S_0(t) = \left\{Cos\left(\alpha_1 + \alpha_2\right)^2 - \frac{1}{2}\sigma^2\right\}t + \sigma dZ(t)$$

$$\ln\left(\frac{S_\theta(t)}{S_0(t)}\right) = \left\{Cos\left(\alpha_1 + \alpha_2\right)^2 - \frac{1}{2}\sigma^2\right\}t + \sigma dZ(t)$$

$$S_\theta(t) = S_0(t)\exp\left\{\left(Cos(\alpha_1 + \alpha_2)^2 - \frac{1}{2}\sigma^2\right)t + \sigma dZ(t)\right\} \tag{31}$$

This is the complete solution of investment equation whose rate of returns and asset price valuation follows periodic additive series with quadratic function.

3.2.3. Periodic Multiplicative Effects

From (17), Thus, generalizing and considering a function $f\left(S_\phi(t), t\right)$, of the random variable S_ϕ and of time, t . There are two independent variables S_ϕ and t hence it has to do with partial derivatives. Expansion of $f\left(S_\phi(t), \mathrm{d}S_\phi(\mathrm{t}), t + dt\right)$ in a Taylor series about $\left(S_\phi(t), t\right)$ gives;

$$df = \frac{\partial f}{\partial S_\phi(t)}dS_\phi(t) + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial {S_\phi}^2(t)}dS_\phi^2(t) + \ldots \tag{32}$$

Substituting (17) in (32) gives;

$$df = \frac{\partial f}{\partial S_\phi(t)}\{\sigma S_\phi(t)dZ(\mathrm{t}) + \alpha_\phi S_\phi(t)dt \ \} + \frac{1}{2}\frac{\partial^2 f}{\partial {S_\phi}^2(t)} dS_\phi^2(t)\,\mathrm{d}S_\phi^2(\mathrm{t})$$

$$df = \sigma S_\phi(t)\frac{\partial f}{\partial S_\phi(t)}dZ(t) + \left(\alpha_\phi S_\phi(t)\frac{\partial f}{\partial S_\phi(t)} + \frac{1}{2}\sigma^2 S_\phi^2(t)\frac{\partial^2 f}{\partial {S_\phi}^2(t)}dS_\phi^2(t)\right)dt$$

Now considering the SDE in (16)

Let $f\left(S_\phi(\mathrm{t})\right) = \ln S_\phi(t)$, the partial derivatives becomes

$$\frac{\partial f}{\partial S_\phi(t)} = \frac{1}{S_\phi(t)}, \ \frac{\partial^2 f}{\partial S_\phi^2(t)} = -\frac{1}{S_\phi^2(t)}, \ \frac{\partial f}{\partial t} = 0 \tag{33}$$

following theorem 1.2 (Ito's), substituting (33) and simplifying gives;

$$df = \left\{Sin\left(\alpha_1\alpha_2\right)^2 - \frac{1}{2}\sigma^2\right\}dt + \sigma dZ \tag{34}$$

Since the RHS of (33) is independent of $f\left(S_\phi(t)\right)$, the stochastic is computed as follows:

$$f\left(S_\theta(t)\right) = f_0 + \int_0^t \left\{Sin\left(\alpha_1\alpha_2\right)^2 - \frac{1}{2}\sigma^2\right\}dt + \int_0^t \sigma dZ(t)$$

$$= f_0 + \left\{Sin\left(\alpha_1\alpha_2\right)^2 - \frac{1}{2}\sigma^2\right\}t + \sigma dZ(t),$$

Since $f\left(S_\theta(t)\right) = \ln S_\theta(t)$ a found solution for $S_\phi(t)$ becomes;

$$\ln S_\theta(t) = \ln S_0(t) + \left\{Sin\left(\alpha_1\alpha_2\right)^2 - \frac{1}{2}\sigma^2\right\}t + \sigma dZ(t),$$

$$\ln S_\phi(t) - \ln S_0(t) = \left\{Sin\left(\alpha_1\alpha_2\right)^2 - \frac{1}{2}\sigma^2\right\}t + \sigma dZ(t),$$

$$\ln\left(\frac{S_\phi(t)}{S_0(t)}\right) = \left\{Sin\left(\alpha_1\alpha_2\right)^2 - \frac{1}{2}\sigma^2\right\}t + \sigma dZ(t)$$

$$S_\phi(t) = S_0(t)\exp\left\{\left(Sin\left(\alpha_1\alpha_2\right)^2 - \frac{1}{2}\sigma^2\right)t + \sigma dZ(t)\right\} \tag{35}$$

This is the complete solution of investment equation whose rate of returns and asset price valuation is periodic multiplicative effects series with a quadratic function.

4. Results

This section presents the graphical results for the problems in (15) - (17) whose solutions are in (27), (31) and (35). Hence, the following parameter values were used in the simulation study:

$S_{\omega}0 = 128.91, \sigma = 0.03, dZ = 1.00, t = 1, 2, ...,$
$10, r = 2, 4, 6, 8, 10, k = 2, S_{\phi}0 = 128.91, S_{\theta}0 = 128.91,$

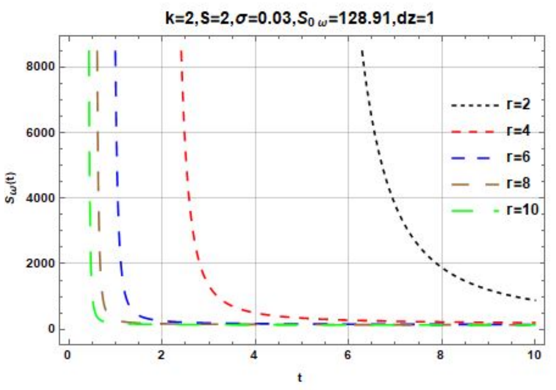

Figure 1. The index price function on the value of asset price over time, t with variations of interest rates

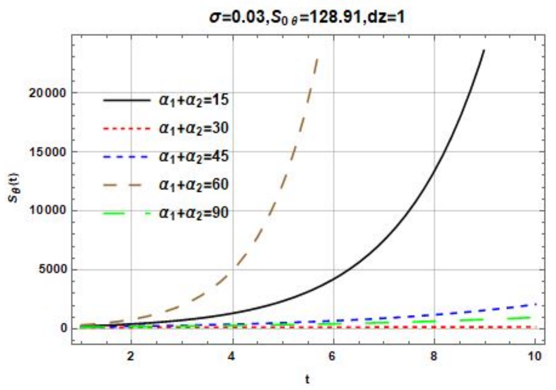

Figure 2. The value of asset price profiles against time (t) with variations of additive periodic effect parameter $(\alpha_1 + \alpha_2)$

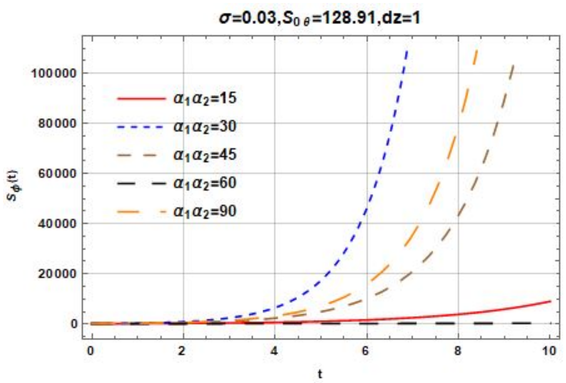

Figure 3. The value of asset price profiles with variations of multiplicative periodic parameter $(\alpha_1\alpha_2)$

As a guide in describing the plots, investment in this work is represented by $S_{\omega}(t), S_{\theta}(t), S_{\phi}(t)$ for series price index, periodic additive series, and periodic multiplicative series respectively. Time, t, ranges from zero to 10. So, investment is plotted against time.

The asset value $S_{\omega}(t)$ is influenced by interest rate, γ, as shown in Figure 1. It shows that $\lim_{\gamma \to \infty} S_{\omega}(t) = 0$ that is, the higher the interest rate, the faster the convergence of $S_{\omega}(t)$ to zero at the shortest time limit. In other words, an investor stands to benefit if the interest is low.

The plot of the additive periodic effect model is shown in Figure 2. It shows that asset price $S_{\theta}(t)$ obeys the exponential growth law when $\alpha_1 + \alpha_2 = 60$ and 15. The investment reaches optimum fastest when $\alpha_1 + \alpha_2 = 60$ at time approximately 4.75, while the optimum is achieved at $t \approx 8.5$ when $\alpha_1 + \alpha_2 = 15$. It however has a linear function with zero growth rate when $\alpha_1 + \alpha_2 = 30$.

In Figure 3, the investment $S_{\phi}(t)$ has a linear growth function with zero rate of growth when $\alpha_1\alpha_2 = 60$ and $\alpha_1\alpha_2 = 15$, the growth rate becomes slightly above zero. The model displays the characteristic of an exponential growth function when $\alpha_1\alpha_2 = 30, 90$ and 45 with $\alpha_1\alpha_2 = 30$ being the fastest. The investment reaches its optimal value in the shortest time when $\alpha_1\alpha_2 = 30$, followed by $\alpha_1\alpha_2 = 90$ and then $\alpha_1\alpha_2 = 45$. In general, $S_{\phi}(t)$ depends on the values of $\alpha_1\alpha_2$.

5. Conclusion

The analysis of asset value and its return rates for economic investments have been readily established by means of series price index, periodic additive effects and periodic multiplicative effects, which was accurately solved for proper investment for future plans. From the closed form analytical solution of the investment equations, we presented diverse results for proper investments decision making. More so, graphical results which represent the behaviour of the economic investments and discussions of the relevant parameters were achieved. Nevertheless, the implications of this paper is to aid the tracking of asset values over time, such that investors could now see their investment performances and make adjustment as required. Also, regular asset valuation can investors to plan properly for taxes and optimize their returns.

Finally, in the next study we shall be looking at the controllability studies of these investment equations its further implications in financial markets.

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