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Single-Step Block Method of P-Stable for Solving Third-Order Differential Equations (IVPs): Ninth Order of Accuracy

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Abstract The solution of Differential Equations is an important topic for deliberation among scientists. However, until today, nothing is known on a single-step block method of p-stable for solving third-order Differential Equations (IVPs) whose accuracy is ninth order. This paper focuses on the derivation, analysis, and implementation of the one-step implicit hybrid block method with seven off-step points for direct solution of general third-order ordinary differential equations' initial value problems (IVPs). For the solution of IVPs, the power series functions were utilized as the basis function. To determine the unknown parameters, an approximate solution from the basis function was interpolated at chosen off-grid points. The third derivative of the estimated solution was collocated at all grid and off-grid points to produce a system of linear equations. Consistency, zero stability, convergence, and absolute stability were all evaluated on the method. The numerical results achieved through implementation are quite close to the theoretical solutions and compare well to other novel methods in the literature.

Keywords: *hybrid block method, grid points, off-grid points, 3rd Order ODE, implicit*

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1. Introduction

This research considers a third-order Ordinary Differential Equations (ODEs) of the form

$$y''' = f\left(x, y, y', y''\right)\ y\left(x_0\right) = \gamma_0,\quad y'\left(x_0\right) = \gamma_1, y''\left(x_0\right) = \gamma_2 \tag{1}$$

where f is a given real-valued function that is continuous within the integration interval. The study of thin-film flow, fluid dynamics and mechanics, entry-flow phenomena, hydrodynamics, the constant flow of water in a long rectangular tank, and other problems of the kind Eq. (1) arises. The conventional way of obtaining a numerical solution of Eq. (1) is by reduction to an equivalent system of first-order ODEs of the form

$$y' = f\left(x, y\right)\ y\left(a\right) = \gamma_0\ \ f \in C\left[a, b\right],\ x, y \in R \tag{2}$$

This method is extensively discussed in the works of Refs. [1,2,3], and many others. Despite its enormous success, this approach is not without drawbacks. Computer programs associated with method implementation are frequently complicated, particularly subroutines to supply the starting values for the methods, resulting in longer computer time and requiring more computational work Refs. [4,5,6]. Direct techniques were devised to overcome the disadvantages. The works in this category are implemented in predictor-corrector Refs. [7,8,9] or block mode (Refs. [10-14]), and their stability domain was thoroughly investigated. This work adopted an approach based on collocation and interpolation of power series approximate solution to derive a one-step hybrid scheme with seven off-step points for the direct solution of general third-order ODEs.

2. Derivation of the Method

The series solution techniques appraised by Refs. [15,16,17,18] for obtaining the unknown function of differential equations was adopted as the research methodology. The basis function is considered as an approximate solution of Eq. (1) which is a power series representation of the form

$$y\left(x\right) = \sum_{j=0}^{(r+s)-1} a_j x^j. \tag{3}$$

The third derivative of Eq. (3) gives

$$y'''(x) = \sum_{j=0}^{(r+s)-1} j\left(j-1\right)\left(j-2\right) a_j x^{j-3} \tag{4}$$

Equating Eq. (4) to Eq. (1) yields the differential system

$$\sum_{j=0}^{(r+s)-1} j(j-1)(j-2)a_j x^{j-3} = f(x, y(x), y'(x), y''(x)), \quad (5)$$

Where the's are the parameters to be determined r and s denote the number of collocation and interpolation points respectively. Collocating Eq. (5) at the mesh points $x = x_{n+j},\ j = 0\left(\frac{1}{8}\right)1$ and interpolating Eq. (3) at $x = x_{n+j},\ j = \frac{5}{8}, \frac{6}{8}, \frac{7}{8}$ yields a system of equations

$$\sum_{j=0}^{(r+s)-1} a_j x^j = y_{n+j}, \quad j = \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, \quad (6)$$

$$\sum_{j=0}^{(r+s)-1} j(j-1)(j-2)a_j x^{j-3} = f_{n+j}, \quad j = 0\left(\frac{1}{8}\right)1 \quad (7)$$

By putting these systems of equations in matrix form and then solved to obtain the values of parameters, $j = 0, \frac{1}{8}, \ldots, 1$ which when substituted in Eq. (4), yields, after some simplification, a hybrid linear method with continuous coefficients of the form

$$y(t) = \sum_{j=5}^{7} \alpha_{\frac{j}{8}}(t) y_{n+\frac{j}{8}}(t) + h^3 \sum_{j=0}^{8} \beta_{\frac{j}{8}}(t) f_{n+\frac{j}{8}}(t) \quad (8)$$

Where the coefficients $\alpha_{\frac{j}{8}}(t)$ and $\beta_{\frac{j}{8}}(t)$ are given as

$$\alpha_{\frac{5}{8}} = (4ht-3)(8ht-7)$$

$$\alpha_{\frac{3}{4}} = -(8ht-5)(8ht-7)$$

$$\alpha_{\frac{7}{8}} = (8ht-5)(4ht-3)$$

$$\beta_0 = \frac{1}{20437401600} h^3 (8ht-7)(4ht-3)(8ht-5) \left(33554432h^8t^8 - 132120576h^7t^7 + 208928768h^6t^6 - 171147264h^5t^5 + 78393344h^4t^4 - 20199936h^3t^3 + 2767040h^2t^2 - 174312ht + 2123\right)$$

$$\beta_{\frac{1}{8}} = \frac{1}{5109350400} h^3 (8ht-5)(4ht-3)(8ht-7) \left(67108864h^8t^8 - 252706816h^7t^7 + 371720192h^6t^6 - 288042240h^5t^5 + 94834688h^4t^4 - 11564032h^3t^3 - 1938880h^2t^2 + 645776ht - 48499\right)$$

$$\beta_{\frac{1}{4}} = \frac{1}{2554675200} h^3 (4ht-3)(8ht-5)(8ht-7) \left(11744052h^8t^8 - 420051840h^7t^7 + 576978944h^6t^6 - 3675517696h^5t^5 + 101924864h^4t^4 - 4896896h^3t^3 - 1652224h^2t^2 - 294986ht + 90101\right)$$

$$\beta_{\frac{3}{8}} = -\frac{1}{5109350400} h^3 (8ht-5)(4ht-3)(8ht-7) \left(469762048h^8t^8 - 1607467008h^7t^7 + 2042626048h^6t^6 - 1161166848h^5t^5 + 268251136h^4t^4 - 10773504h^3t^3 - 1387328h^2t^2 + 729744ht - 457457\right)$$

$$\beta_{\frac{1}{2}} = -\frac{1}{2043740160} h^3 (8ht-5)(4ht-3)(8ht-7) \left(234881024h^8t^8 - 763363328h^7t^7 + 903086080h^6t^6 - 466681856h^5t^5 + 98824192h^4t^4 - 5583872h^3t^3 - 1431104h^2t^2 + 79312ht - 280819\right)$$

$$\beta_{\frac{5}{8}} = -\frac{1}{5109350400} h^3 (8ht-7)(4ht-3)(8ht-5) \left(469762048h^8t^8 - 1445986304h^7t^7 + 1598554112h^6t^6 - 770441216h^5t^5 + 161017856h^4t^4 - 7844864h^3t^3 - 250580h^2t^2 - 990128ht - 1256893\right)$$

$$\beta_{\frac{3}{4}} = \frac{1}{1277337600} h^3 (8ht-5)(4ht-3)(8ht-7) \left(58720256h^8t^8 - 170655744h^7t^7 + 177471488h^6t^6 - 82022400h^5t^5 + 17057792h^4t^4 - 512448h^3t^3 + 50720h^2t^2 + 205629ht + 167684\right)$$

$$\beta_{\frac{7}{8}} = \frac{1}{51093350400} h^3 (4ht-3)(8ht-7)(8ht-5) \left(67108864h^8t^8 - 183500800h^7t^7 + 181403648h^6t^6 - 82051072h^5t^5 + 16437248h^4t^4 - 750592h^3t^3 - 173248h^2t^2 + 3728ht - 9823\right)$$

$$\beta_1 = \frac{1}{20437401600} h^3 (8ht-5)(4ht-3)(8ht-7) \left(33554432h^8t^8 - 85983232h^7t^7 + 82051072h^6t^6 - 36339712h^5t^5 + 7204864h^4t^4 - 318976h^3t^3 - 68672h^2t^2 + 7496ht - 847\right) \quad (9)$$

where $t=\dfrac{x-x_{n+v_i}}{h}$. Evaluating Eq. (9) at $t=0,\frac{1}{8},\frac{2}{8},\frac{3}{8},\frac{4}{8},1$ yields the discrete one-step formulas

$$y_n-21y_{n+\frac{5}{8}}+35y_{n+\frac{3}{4}}-15y_{n+\frac{7}{8}}=\frac{-h^3}{17694720}\begin{pmatrix}193f_n+17636f_{n+\frac{1}{8}}+65528f_{n+\frac{1}{8}}\\+166348f_{n+\frac{3}{8}}+255290f_{n+\frac{1}{2}}\\+457052f_{n+\frac{5}{8}}+243904f_{n+\frac{3}{4}}\\+3572f_{n+\frac{7}{8}}+77f_{n+1}\end{pmatrix}\quad(10a)$$

$$y_{n+\frac{1}{8}}-15y_{n+\frac{5}{8}}+24y_{n+\frac{3}{4}}-10y_{n+\frac{7}{8}}=\frac{h^3}{15482880}\begin{pmatrix}56f_n-797f_{n+\frac{1}{8}}-12190f_{n+\frac{1}{4}}\\-67609f_{n+\frac{3}{8}}-122930f_{n+\frac{1}{2}}\\-257071f_{n+\frac{5}{8}}-142138f_{n+\frac{3}{4}}\\-2075f_{n+\frac{7}{8}}-46f_{n+1}\end{pmatrix}\quad(10b)$$

$$y_{n+\frac{1}{4}}-10y_{n+\frac{5}{8}}+15y_{n+\frac{3}{4}}-6y_{n+\frac{7}{8}}=\frac{h^3}{61931520}\begin{pmatrix}145f_n-1412f_{n+\frac{1}{8}}+5300f_{n+\frac{1}{4}}\\-75788f_{n+\frac{3}{8}}-210650f_{n+\frac{1}{2}}\\-581020f_{n+\frac{5}{8}}-341228f_{n+\frac{3}{4}}\\-4820f_{n+\frac{7}{8}}-127f_{n+1}\end{pmatrix}\quad(10c)$$

$$y_{n+\frac{3}{8}}-6y_{n+\frac{5}{8}}+8y_{n+\frac{3}{4}}-3y_{n+\frac{7}{8}}=\frac{h^3}{77414400}\begin{pmatrix}88f_n-859f_{n+\frac{1}{8}}+3854f_{n+\frac{1}{4}}\\-11903f_{n+\frac{3}{8}}-52510f_{n+\frac{1}{2}}\\-327673f_{n+\frac{5}{8}}-212726f_{n+\frac{3}{4}}\\-2989f_{n+\frac{7}{8}}-82f_{n+1}\end{pmatrix}\quad(10d)$$

$$y_{n+\frac{1}{2}}-3y_{n+\frac{5}{8}}+3y_{n+\frac{3}{4}}-y_{n+\frac{7}{8}}=\frac{h^3}{619315200}\begin{pmatrix}331f_n-3268f_{n+\frac{1}{8}}+14848f_{n+\frac{1}{4}}\\-41516f_{n+\frac{3}{8}}-71030f_{n+\frac{1}{2}}\\-674876f_{n+\frac{5}{8}}-568952f_{n+\frac{3}{4}}\\-6868f_{n+\frac{7}{8}}-329f_{n+1}\end{pmatrix}\quad(10e)$$

$$y_{n+1}-3y_{n+\frac{7}{8}}+3y_{n+\frac{3}{4}}-y_{n+\frac{5}{8}}=\frac{h^3}{619315200}\begin{pmatrix}329f_n-3292f_{n+\frac{1}{8}}+15112f_{n+\frac{1}{4}}\\-42484f_{n+\frac{3}{8}}+82970f_{n+\frac{1}{2}}\\-112484f_{n+\frac{5}{8}}+702512f_{n+\frac{3}{4}}\\+557108f_{n+\frac{7}{8}}+9829f_{n+1}\end{pmatrix}\quad(10f)$$

By combining the schemes Eq. (10), the first, second derivatives of the schemes and write in block form, using the definition of implicit block method in Eq. (9) to obtain the block formula describe as follows:

$$h^p\sum_{j=0}^{q}a_{i,j}y_{n+j}^{\lambda}=h^{\lambda}\sum_{j=0}^{q}e_{i,j}y_n^{\lambda}+h^{p-\lambda}\begin{pmatrix}\sum_{j=1}^{q}d_{i,j}f_n\\+\sum_{j=1}^{q}b_{i,j}f_{n+j}\end{pmatrix},\quad(11)$$

$$i=0,1,\dots,q$$

λ is the power of the derivative of the continuous method and p is the order of the problem to solve: $q=r+s$. This equation is solved, and values for y_{n+v_i}, y_{n+1}, y'_{n+v_i}, y'_{n+1}, y''_{n+v_i} and y''_{n+1} $i=0\left(\frac{1}{8}\right)1$ are obtained as follows:

$$y_{n+\frac{1}{8}}=y_n+\frac{1}{8}hy'_n+\frac{1}{128}h^2y''_n+\frac{h^3}{20437401600}\begin{pmatrix}3619903f_n+6779886f_{n+\frac{1}{8}}\\-9359135f_{n+\frac{1}{4}}+11774146f_{n+\frac{3}{8}}\\-276129795f_{n+\frac{1}{2}}+6771082f_{n+\frac{5}{8}}\\-5920898f_{n+\frac{3}{4}}+679110f_{n+\frac{7}{8}}\\-73886f_{n+1}\end{pmatrix}$$

$$y_{n+\frac{1}{4}}=y_n+\frac{1}{4}hy'_n+\frac{1}{32}h^2y''_n+\frac{h^3}{10218700800}\begin{pmatrix}9182944f_n+29158528f_{n+\frac{1}{8}}\\-29652672f_{n+\frac{1}{4}}+37540480f_{n+\frac{3}{8}}\\-166866575f_{n+\frac{1}{2}}+21492096f_{n+\frac{5}{8}}\\-15108362f_{n+\frac{3}{4}}+2150272f_{n+\frac{7}{8}}\\-233706f_{n+1}\end{pmatrix}$$

$$y_{n+\frac{3}{8}}=y_n+\frac{3}{8}hy'_n+\frac{9}{128}h^2y''_n+\frac{h^3}{756940800}\begin{pmatrix}1650456f_n+6407262f_{n+\frac{1}{8}}\\-4690953f_{n+\frac{1}{4}}+6896610f_{n+\frac{3}{8}}\\-16128015f_{n+\frac{1}{2}}+3971754f_{n+\frac{5}{8}}\\-2681066f_{n+\frac{3}{4}}+398358f_{n+\frac{7}{8}}\\-43335f_{n+1}\end{pmatrix}$$

$$y_{n+\frac{1}{2}}=y_n+\frac{1}{2}hy'_n+\frac{1}{8}h^2y''_n+\frac{h^3}{10218700800}\begin{pmatrix}41110016f_n+175521792f_{n+\frac{1}{8}}\\-96317440f_{n+\frac{1}{4}}+183713792f_{n+\frac{3}{8}}\\-294165765f_{n+\frac{1}{2}}+100450304f_{n+\frac{5}{8}}\\-66465328f_{n+\frac{3}{4}}+10076160f_{n+\frac{7}{8}}\\-1096192f_{n+1}\end{pmatrix}$$

$$y_{n+\frac{5}{8}}=y_n+\frac{5}{8}hy'_n+\frac{25}{128}h^2y''_n+\frac{h^3}{817496064}\begin{pmatrix}5253125f_n+23702750f_{n+\frac{1}{8}}\\-10296375f_{n+\frac{1}{4}}+25537250f_{n+\frac{3}{8}}\\-30295999f_{n+\frac{1}{2}}+12920250f_{n+\frac{5}{8}}\\-8455162f_{n+\frac{3}{4}}+1295750f_{n+\frac{7}{8}}-141000f_{n+1}\end{pmatrix}$$

$$y_{n+\frac{3}{4}}=y_n+\frac{3}{4}hy'_n+\frac{9}{32}h^2y''_n+\frac{h^3}{378470400}\begin{pmatrix}3553632f_n+16619904f_{n+\frac{1}{8}}\\-5935680f_{n+\frac{1}{4}}+18486144f_{n+\frac{3}{8}}\\-17330205f_{n+\frac{1}{2}}+12920250f_{n+\frac{5}{8}}\\-8455162f_{n+\frac{3}{4}}+1295750f_{n+\frac{7}{8}}\\-141000f_{n+1}\end{pmatrix}$$

$$y_{n+\frac{7}{8}}=y_n+\frac{7}{8}hy'_n+\frac{49}{128}h^2y''_n+\frac{h^3}{2919628800}\begin{pmatrix}37701874f_n+180838518f_{n+\frac{1}{8}}\\-54639557f_{n+\frac{1}{4}}+206894170f_{n+\frac{3}{8}}\\-160182975f_{n+\frac{1}{2}}+104842066f_{n+\frac{5}{8}}\\-57678362f_{n+\frac{3}{4}}+9423582f_{n+\frac{7}{8}}\\-1020425f_{n+1}\end{pmatrix}$$

$$y_{n+1}=y_n+hy'_n+\frac{1}{2}h^2y''_n+\frac{h^3}{10218700800}\begin{pmatrix}173694976f_n+848822272f_{n+\frac{1}{8}}\\-221380608f_{n+\frac{1}{4}}+993525760f_{n+\frac{3}{8}}\\-655505615f_{n+\frac{1}{2}}+520617984f_{n+\frac{5}{8}}\\-237853856f_{n+\frac{3}{4}}+52953088f_{n+\frac{7}{8}}\\-4546560f_{n+1}\end{pmatrix} \quad (12)$$

$$y'_{n+\frac{1}{8}}=hy'_n+\frac{1}{8}h^2y''_n+\frac{h^2}{464486400}\begin{pmatrix}1624505f_n+4124232f_{n+\frac{1}{8}}\\-5225624f_{n+\frac{1}{4}}+6488192f_{n+\frac{3}{8}}\\-5888310f_{n+\frac{1}{2}}+3698920f_{n+\frac{5}{8}}\\-2660140f_{n+\frac{3}{4}}\\+369744f_{n+\frac{7}{8}}-40187f_{n+1}\end{pmatrix}$$

$$y'_{n+\frac{1}{4}}=hy'_n+\frac{1}{4}h^2y''_n+\frac{h^2}{58060800}\begin{pmatrix}465544f_n+1880576f_{n+\frac{1}{8}}\\-1469664f_{n+\frac{1}{4}}+1978624f_{n+\frac{3}{8}}\\-1816240f_{n+\frac{1}{2}}+1145856f_{n+\frac{5}{8}}\\-757103f_{n+\frac{3}{4}}+114944f_{n+\frac{7}{8}}\\-12504f_{n+1}\end{pmatrix}$$

$$y'_{n+\frac{3}{8}}=hy'_n+\frac{3}{8}h^2y''_n+\frac{h^2}{51609600}\begin{pmatrix}644949f_n+2957472f_{n+\frac{1}{8}}\\-1355616f_{n+\frac{1}{4}}+2835000f_{n+\frac{3}{8}}\\-2532870f_{n+\frac{1}{2}}+1595376f_{n+\frac{5}{8}}\\-1037308f_{n+\frac{3}{4}}\\+160056f_{n+\frac{7}{8}}-17415f_{n+1}\end{pmatrix}$$

$$y'_{n+\frac{1}{2}}=hy'_n+\frac{1}{2}h^2y''_n+\frac{h^2}{29030400}\begin{pmatrix}492992f_n+2383872f_{n+\frac{1}{8}}\\-742400f_{n+\frac{1}{4}}+2564096f_{n+\frac{3}{8}}\\-1895040f_{n+\frac{1}{2}}+1220608f_{n+\frac{5}{8}}\\-789199f_{n+\frac{3}{4}}\\+122880f_{n+\frac{7}{8}}-13376f_{n+1}\end{pmatrix}$$

$$y'_{n+\frac{5}{8}} = hy'_n + \tfrac{5}{8}h^2 y''_n + \frac{h^2}{92897280}\left(\begin{array}{l} 1994125\, f_n + 9935000\, f_{n+\frac{1}{8}} \\ -2325000\, f_{n+\frac{1}{4}} + 11470000\, f_{n+\frac{3}{8}} \\ -6418750\, f_{n+\frac{1}{2}} + 5103000\, f_{n+\frac{5}{8}} \\ -3197468\, f_{n+\frac{3}{4}} \\ +500000\, f_{n+\frac{7}{8}} - 54375\, f_{n+1} \end{array}\right)$$

$$y'_{n+\frac{3}{4}} = hy'_n + \tfrac{3}{4}h^2 y''_n + \frac{h^2}{6451200}\left(\begin{array}{l} 167400\, f_n + 850176\, f_{n+\frac{1}{8}} \\ -158112\, f_{n+\frac{1}{4}} + 1022976\, f_{n+\frac{3}{8}} \\ -460080\, f_{n+\frac{1}{2}} + 518400\, f_{n+\frac{5}{8}} \\ -258085\, f_{n+\frac{3}{4}} \\ +41472\, f_{n+\frac{7}{8}} - 4536\, f_{n+1} \end{array}\right)$$

$$y'_{n+\frac{7}{8}} = hy'_n + \tfrac{7}{8}h^2 y''_n + \frac{h^2}{66355200}\left(\begin{array}{l} 2019731\, f_n + 10388784\, f_{n+\frac{1}{8}} \\ -1575056\, f_{n+\frac{1}{4}} + 12811736\, f_{n+\frac{3}{8}} \\ -4826010\, f_{n+\frac{1}{2}} + 7068544\, f_{n+\frac{5}{8}} \\ -2155492\, f_{n+\frac{3}{4}} \\ +589176\, f_{n+\frac{7}{8}} - 57281\, f_{n+1} \end{array}\right)$$

$$y'_{n+1} = hy'_n + h^2 y''_n + \frac{h^2}{1277337600}\left(\begin{array}{l} 44560384\, f_n + 232128512\, f_{n+\frac{1}{8}} \\ -3135897\, f_{n+1/4} + 295567360\, f_{n+\frac{3}{8}} \\ -97700625\, f_{n+\frac{1}{2}} + 177340416\, f_{n+\frac{5}{8}} \\ -35477288\, f_{n+\frac{3}{4}} + 33161216\, f_{n+\frac{7}{8}} \end{array}\right) \quad (13)$$

$$y''_{n+\frac{1}{8}} = h^2 y''_n + \frac{h}{29030400}\left(\begin{array}{l} 1070017\, f_n + 4467094\, f_{n+\frac{1}{8}} \\ -4604594\, f_{n+\frac{1}{4}} + 5595358\, f_{n+\frac{3}{8}} \\ -5033120\, f_{n+\frac{1}{2}} + 3146338\, f_{n+\frac{5}{8}} \\ -1859948\, f_{n+\frac{3}{4}} + 312874\, f_{n+\frac{7}{8}} \\ -33953\, f_{n+1} \end{array}\right)$$

$$y''_{n+\frac{1}{4}} = h^2 y''_n + \frac{h}{29030400}\left(\begin{array}{l} 1036064\, f_n + 5842688\, f_{n+\frac{1}{8}} \\ -1359808\, f_{n+\frac{1}{4}} + 3842816\, f_{n+\frac{3}{8}} \\ -3715840\, f_{n+\frac{1}{2}} + 2391296\, f_{n+\frac{5}{8}} \\ -1565662\, f_{n+\frac{3}{4}} + 284627\, f_{n+\frac{7}{8}} - \\ 26656\, f_{n+1} \end{array}\right)$$

$$y''_{n+\frac{3}{8}} = h^2 y''_n + \frac{h}{9676800}\left(\begin{array}{l} 347787\, f_n + 1914354\, f_{n+\frac{1}{8}} \\ -92826\, f_{n+\frac{1}{4}} + 2158218\, f_{n+\frac{3}{8}} \\ -1516320\, f_{n+\frac{1}{2}} + 929718\, f_{n+\frac{5}{8}} \\ +611380\, f_{n+\frac{3}{4}} + 91854\, f_{n+\frac{7}{8}} \\ -9963\, f_{n+1} \end{array}\right)$$

$$y''_{n+\frac{1}{2}} = h^2 y''_n + \frac{h}{14515200}\left(\begin{array}{l} 520064\, f_n + 2889728\, f_{n+\frac{1}{8}} \\ +31232\, f_{n+\frac{1}{4}} + 4192256\, f_{n+\frac{3}{8}} \\ -1162240\, f_{n+\frac{1}{2}} + 1181696\, f_{n+\frac{5}{8}} \\ +853927\, f_{n+\frac{3}{4}} + 124928\, f_{n+\frac{7}{8}} \\ -13696\, f_{n+1} \end{array}\right)$$

$$y''_{n+\frac{5}{8}} = h^2 y''_n + \frac{h}{29030400}\left(\begin{array}{l} 1042625\, f_n + 5753750\, f_{n+\frac{1}{8}} \\ +932548\, f_{n+\frac{1}{4}} + 7958750\, f_{n+\frac{3}{8}} \\ -100000\, f_{n+\frac{1}{2}} + 4273250\, f_{n+\frac{5}{8}} \\ -1797484\, f_{n+\frac{3}{4}} + 286250\, f_{n+\frac{7}{8}} \\ -30625\, f_{n+1} \end{array}\right)$$

$$y''_{n+\frac{3}{4}} = h^2 y''_n + \frac{h}{4838400}\left(\begin{array}{l} 173232\, f_n + 964224\, f_{n+\frac{1}{8}} \\ +7776\, f_{n+\frac{1}{4}} + 1392768\, f_{n+\frac{3}{8}} \\ -155520\, f_{n+\frac{1}{2}} + 1150848\, f_{n+\frac{5}{8}} \\ -26533\, f_{n+\frac{3}{4}} + 31104\, f_{n+\frac{7}{8}} \\ -3888\, f_{n+1} \end{array}\right)$$

$$y''_{n+\frac{7}{8}} = h^2 y''_n + \frac{h}{29030400}\begin{pmatrix} 1046689 f_n + 5716438 f_{n+\frac{1}{8}} \\ +340942 f_{n+\frac{1}{4}} + 7601566 f_{n+\frac{3}{8}} \\ +384160 f_{n+\frac{1}{2}} + 5152564 f_{n+\frac{5}{8}} \\ +3085588 f_{n+\frac{3}{4}} + 1562218 f_{n+\frac{7}{8}} \\ -57281 f_{n+1} \end{pmatrix}$$

$$y''_{n+1} = h^2 y''_n + \frac{h}{14515200}\begin{pmatrix} 506368 f_n + 3014656 f_{n+\frac{1}{8}} \\ -475136 f_{n+\frac{1}{4}} + 5373952 f_{n+\frac{3}{8}} \\ -2324480 f_{n+\frac{1}{2}} + 5373952 f_{n+\frac{5}{8}} \\ -759503 f_{n+\frac{3}{4}} + 3014656 f_{n+\frac{7}{8}} \\ +506368 f_{n+1} \end{pmatrix} \quad (14)$$

3. Analysis of the Method

In this section, the analysis of the basic properties of the method was carried out as follows.

3.1. Order and Error Constant of the Method

The formula in Eq. (10f) in a conventional linear multistep method can be expressed as:

$$\sum_{j=5}^{7} \alpha_{\frac{j}{5}} y_{n+\frac{5}{8}} = h^3 \sum_{j=0}^{8} \beta_{\frac{j}{8}} y'''_{n+\frac{j}{8}} \quad (15)$$

Following Ref. [1], the local truncation error associated with Eq. (15) was defined by the difference operator

$$L_{\frac{j}{8}}\{y(x):h\} = \sum_{j=o}^{k}\left\{\alpha_{\frac{j}{8}} y(x_n + \frac{j}{8}h) - h^3 \beta_{\frac{j}{8}} y'''(x_n + \frac{j}{8}h)\right\} \quad (16)$$

where $y(x)$ is assumed to have continuous derivatives of a sufficiently high order. Therefore expanding (10f) in Taylor series about the point x to obtain the expression

$$\begin{aligned} L_{\frac{j}{8}}\{y(x):h\} &= C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \ldots \\ &+ C_{p+2} h^{p+2} y^{(p+2)}(x) \\ &+ C_{p+3} h^{p+3} y^{(p+3)}(x) \end{aligned} \quad (17)$$

The term C_{p+3} is called the error constant and implies that the local truncation error is given by:

$$t_{n+k} = C_{p+4} h^{(p+3)} y^{(p+3)}(x_n) + 0h^{(p+4)} \quad (18)$$

since $C_0 = C_1 = \ldots = C_{p+2} = 0, C_{p+3} \neq 0$. see Ref. [19]; then the method has ordered $p = 9$ with error constant $c_{p+3} = \frac{19}{11083077207982080}$.

3.2. Definition: Zero Stability of the Method

According to Ref. [2] a block method is zero stable provided the roots $z_j, j = 1(1)k$ of the first characteristic polynomial $\rho(r)$ specified as

$$\rho(z) = \det\left[\sum_{j=0}^{k} A^{(j)} z^{k-j}\right] = 0,\ A^{(0)} = -1 \quad (19)$$

satisfies $|z_j| \leq 1$, and for those roots with $|z_j| = 1$, the multiplicity must not exceed 2. By definition (3.2) block Eq. (11) is zero stable since the roots of the characteristic polynomial satisfy $|z| \leq 1$ and the root $|z| = 1$ has multiplicity not exceeding the order of the differential equation. Moreover, as $h^\mu \to 0, \rho(z) = z^{r-\mu}(\lambda - 1)^\mu$, where μ is the order of the differential equation, for the block method, $r = 24$, and $\mu = 3$

$$\rho(z) = \lambda^{21}(\lambda - 1)^3 = 0$$

Implies that

$\lambda = 0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1$

Hence, the method is Zero stable.

3.3. Consistency of the Method

From Eq. (10f), the first and second characteristics polynomials of the method are given by

$$\rho(r) = r - 3r^{7/8} + 3r^{3/4} - r^{5/8}$$

$$\begin{aligned} \sigma(r) = &\frac{329}{619315200} - \frac{3292}{619315200} r^{1/8} + \frac{15112}{619315200} r^{1/4} \\ &- \frac{42484}{619315200} r^{3/8} + \frac{82970}{619315200} r^{1/2} \\ &- \frac{112484}{619315200} r^{5/8} + \frac{702512}{619315200} r^{3/4} \\ &+ \frac{557108}{619315200} r^{7/8} + \frac{9829}{619315200} r \end{aligned}$$

This implies that the method presented in this report is consistent since it satisfies the following conditions:

i) The order of the method is $p = 9 \geq 1$ which is obvious.
ii) For the method $\alpha_1 = 1$, $\alpha_{\frac{7}{8}} = -3$, $\alpha_{\frac{3}{4}} = -3$, and $\alpha_{\frac{5}{8}} = -1$ thus

iii) If $\rho(r) = r - 3r^{\frac{7}{8}} + 3r^{\frac{3}{4}} - r^{\frac{5}{8}}$ and

$$\rho'(r) = 1 - \frac{21}{8}r^{-\frac{1}{8}} + \frac{9}{4}r^{-\frac{1}{4}} - \frac{5}{8}r^{-\frac{3}{8}}$$

it follows from here that $\rho(1) = 0 = \rho'(1)$ shows that the condition (iii) is satisfied as well

iv) Note that

$$\rho'''(r) = -\frac{189}{512}r^{-\frac{17}{8}} + \frac{45}{64}r^{-\frac{9}{4}} - \frac{165}{512}r^{-\frac{19}{8}}$$

$$r = 1 \Rightarrow \rho'''(1) = \frac{1}{512} = 3!\sigma(1).$$

For the principal root is observed that the last condition above is satisfied. Hence the method is consistent.

3.4. Convergence of the Method

According to Ref. [20], the necessary and sufficient condition for a numerical method to be convergent is to be consistent and Zero stable. Thus since it has been successfully shown from the above condition, it could be seen that method is convergent.

3.5. Region of Absolute Stability of the Method.

The boundary locus method was adopted by considering the stability polynomial written in general form:

$$\pi\left(r, \bar{h}\right) = \rho(r) - \bar{h}\sigma(r) = 0 \tag{20}$$

$\bar{h} = h^2\lambda$ and $\lambda = \dfrac{df}{dy}$ is assumed constant. The stability polynomial of the formula (10f) becomes:

$$\left(r - 3r^{\frac{7}{8}} + 3r^{\frac{3}{4}} - r^{\frac{5}{8}}\right) - \bar{h}\left(\begin{array}{l} \frac{329}{619315200} - \frac{3292}{619315200}r^{\frac{1}{8}} \\ + \frac{15112}{619315200}r^{\frac{1}{4}} - \frac{42484}{619315200}r^{\frac{3}{8}} \\ + \frac{82970}{619315200}r^{\frac{1}{2}} - \frac{112484}{619315200}r^{\frac{5}{8}} \\ + \frac{702512}{619315200}r^{\frac{3}{4}} + \frac{557108}{619315200}r^{\frac{7}{8}} \\ + \frac{9829}{619315200}r \end{array}\right) = 0 \tag{21}$$

where,

$$\rho(r) = r - 3r^{\frac{7}{8}} + 3r^{\frac{3}{4}} - r^{\frac{5}{8}},$$

and

$$\sigma(r) = \frac{329}{619315200} - \frac{3292}{619315200}r^{\frac{1}{8}} + \frac{15112}{619315200}r^{\frac{1}{4}} - \frac{42484}{619315200}r^{\frac{3}{8}} + \frac{82970}{619315200}r^{\frac{1}{2}} - \frac{112484}{619315200}r^{\frac{5}{8}} + \frac{702512}{619315200}r^{\frac{3}{4}} + \frac{557108}{619315200}r^{\frac{7}{8}} + \frac{9829}{619315200}r$$

From Eq. (20),

$$\bar{h} = \frac{\rho(r)}{\sigma(r)}. \tag{22}$$

Substituting $\rho(r)$ and $\sigma(r)$ into Eq. (21), evaluate, and equate the imaginary part to zero leads to

$$\bar{h}(\theta) = \frac{619315200\left(\begin{array}{l} 329\cos\theta - 1031525\cos\frac{1}{8}\theta \\ +803737\cos\frac{1}{4}\theta - 513787\cos\frac{3}{8}\theta \\ +259050\cos\frac{1}{2}\theta - 98025\cos\frac{5}{8}\theta \\ +25975\cos\frac{3}{4}\theta - 4279\cos\frac{7}{8}\theta \\ +558525 \end{array}\right)}{\left(\begin{array}{l} 6467482\cos\theta + 608557930160\cos\frac{1}{8}\theta \\ +17408233776\cos\frac{1}{4}\theta + 26570306064\cos\frac{3}{8}\theta \\ -19677416680\cos\frac{1}{2}\theta + 11303528240\cos\frac{5}{8}\theta \\ -2908674480\cos\frac{3}{4}\theta + 301862928\frac{7}{8}\theta \\ +825569922510 \end{array}\right)} \tag{23}$$

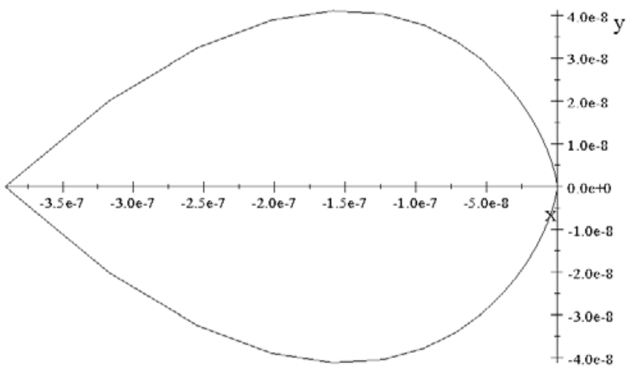

Figure 1. Region of absolute stability of the proposed method

4. Numerical Experiments

The method was utilized to solve specific initial value problems of third-order ordinary differential equations to verify its accuracy, workability, and applicability. The following notations are used to represent the current findings:

XVAL: Value of the independent variable where a numerical value is taken.

ERC: Exact result at XVAL
NRC: Numerical result at XVAL
ERR: Error in proposed method at XVAL

4.1. Problem 1

Consider a non linear third order ODE problem:

$$y''' = \frac{1+2\sin^2(y)}{\cos^2(y)} \qquad 0 \le x \le \frac{\pi}{4}$$

$$y(0)=0,\ y'(0)=1,\ y''(0)=0$$

whose exact solution is given by $y(x)=\arcsin(x)$. The method was used to solve the problem, and the results were compared with Ref. [21] as shown in Table 1.

Table 1. Comparison of results obtained with the proposed method and that of Ref. [21] on problem 1

XVAL	ERC	NRC	ERR P = 9, K = 1 (Single Step)	ERR in [21] P = 9 K= 3 (Three Steps)
0.1	0.10016742	0.10016742	5.5511E-17	0.0000+00
0.2	0.20135792	0.20135792	8.3266E-17	5.5511E−17
0.3	0.30469265	0.30469265	5.5511E-17	1.1102E−16
0.4	0.41151684	0.41151684	2.7755E-16	3.3306E−16
0.5	0.52359877	0.52359877	2.2204E-16	4.4408E−16
0.6	0.64350110	0.64350110	2.2204E-16	4.4408E−16
0.7	0.77539749	0.77539749	6.6613E-16	5.5511E−16
0.8	0.92729521	0.92729521	1.6653E-15	8.8817E−16

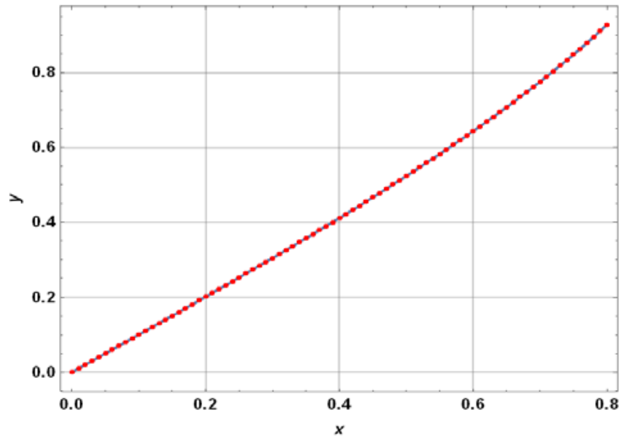

Figure 2. curve of problem 1 as compared with the exact solution

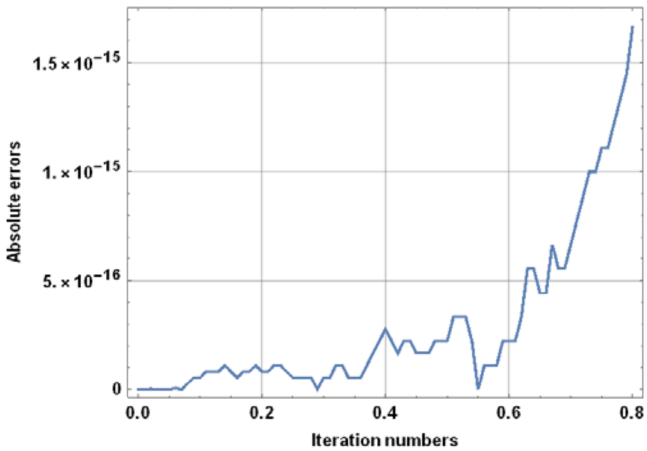

Figure 3. Behaviours of absolute errors obtained by the proposed method on problem

4.2. Problem 2

Consider the linear problem:

$$y''' = 2y'' + 3y' - 10y + 34xe^{-2x}$$
$$-16e^{-2x} - 10x^2 + 6x + 34,$$
$$0 \le x \le 1 \quad y(0)=3, y'(0)=0, y''(0)=0$$

Exact solution: $y(x) = x^2 e^{-2x} - 2x - x^2 + 3.$

The proposed method was applied to this example and the results obtained are compared with that of Ref. [21] in Table 2. The result is as shown in Table 2.

Table 2. Comparison of results obtained with the proposed method and that of Ref. [21] on problem 2

XVAL	ERC	NRC	ERR P = 9, K = 1	ERR in [21]: P =9 K= 3
0.1	2.99818730	2.99818730	3.8646E-19	2.5934E−13
0.2	2.98681280	2.98681280	1.6071E-18	4.3611E−11
0.3	2.95939304	2.95939304	3.7550E-18	2.9672E−11
0.4	2.91189263	2.91189263	6.9813E-18	9.9812E−11
0.5	2.84196986	2.84196986	1.1497E-17	2.3423E−10
0.6	2.74842991	2.74842991	1.7577E-17	4.5508E−10
0.7	2.63083251	2.63083251	2.5561E-17	7.9121E−10
0.8	2.48921377	2.48921377	3.5862E-17	1.2750E−09
0.9	2.32389209	2.32389209	4.8968E-17	1.9452E−09
1.0	2.13533528	2.13533528	6.5444E-17	2.8494E−08

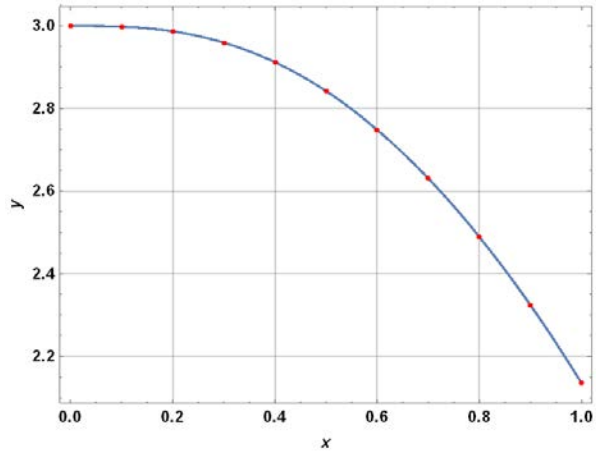

Figure 4. Solution curve of problem as compared with the exact solution

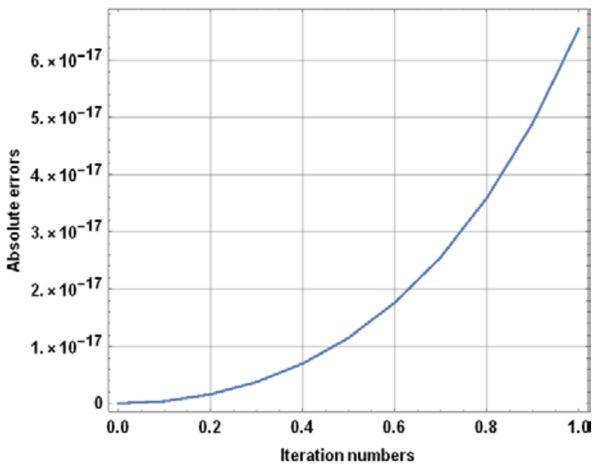

Figure 5. Nature of absolute errors obtained by the proposed method on problem 2

4.3. Problem 3

Consider the problem:

$$y''' = -y, \quad 0 \le x \le 1, \quad h = 0.1$$
$$y(0) = 1, y'(0) = -1, y''(0) = 1$$

Exact solution: $y(x) = e^{-x}$. The proposed method was applied to this example and the results obtained are compared with that of Ref. [6] in Table 3.

Table 3. Comparison of results obtained with the proposed method and that of Ref. [6] on problem 3

XVAL	ERC	NRC	ERR P = 9, K = 1	ERR in [6]: P =9 K= 5
0.1	0.90483741	0.90483741	2.8160E-24	0.0000+00
0.2	0.81873075	0.81873075	1.1025E-23	2.7756E−14
0.3	0.74081822	0.74081822	2.4162E-23	1.5838E−12
0.4	0.67032004	0.67032004	4.1797E-23	2.7879E−11
0.5	0.60653065	0.60653065	6.3522E-23	2.9477E−11
0.6	0.54881163	0.54881163	8.8946E-23	8.5048E−11
0.7	0.49658530	0.49658530	1.1768E-22	8.0357E−11
0.8	0.44932896	0.44932896	1.4936E-22	1.6601E−10
0.9	0.40656965	0.40656965	1.8358E-22	1.1176E−10
1.0	0.36787944	0.36787944	2.1997E-22	1.4871E−10

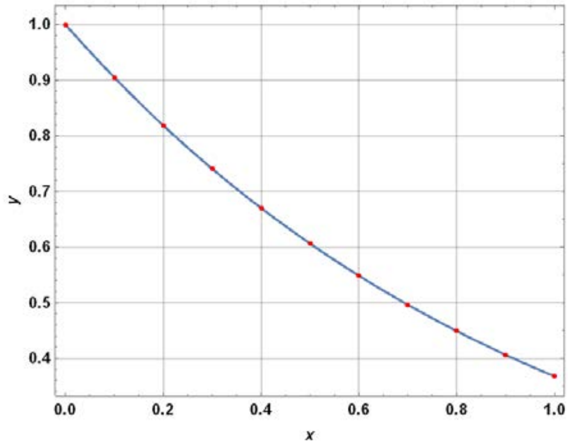

Figure 6. Solution curve of problem as compared with the exact solution

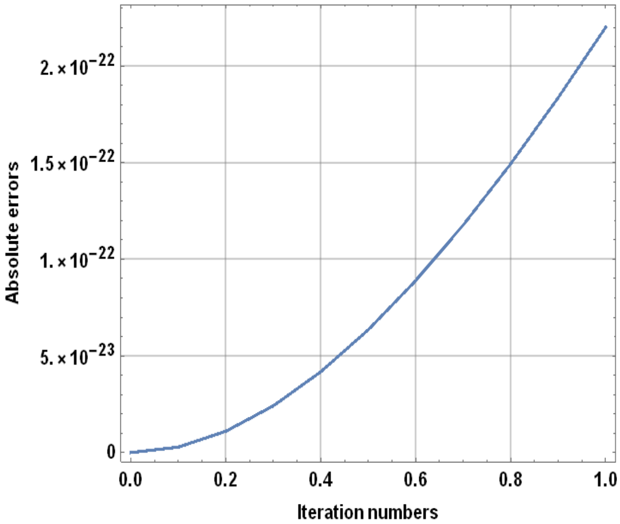

Figure 7. Nature of absolute errors obtained by the proposed method on problem 3

4.4. Problem 4.

Consider the problem:

$$y''' = e^{x}, \quad 0 \le x \le 1, \quad h = 0.1$$
$$y(0) = 3, y'(0) = 1, y''(0) = 5$$

Exact solution: $y(x) = 2 + 2x^2 + e^{x}$. The proposed method was applied to this example and the results obtained are compared with that of Ref. [6] in Table 4.

Table 4. Comparison of results obtained with the proposed method and that of [6] on problem 4

XVAL	ERC	NRC	ERR in P = 9, K = 1	ERR in [6]: P =9 K= 5
0.1	3.12517091	3.12517091	3.0834E-24	0.0000E+00
0.2	3.30140275	3.30140275	1.2625E-23	2.8422E−13
0.3	3.52985880	3.52985880	2.9242E23	1.6729E−12
0.4	3.81182469	3.81182469	5.3616E-23	2.9983E−11
0.5	4.14872127	4.14872127	8.6502E-23	3.1673E−11
0.6	4.54211880	4.54211880	1.2873E-22	9.1899E−11
0.7	4.99375270	4.99375270	1.8122E-22	8.9531E−11
0.8	5.50554092	5.50554092	2.4500E-22	1.9168E−10
0.9	6.07960311	6.07960311	3.2119E-22	2.1110E−10
1.0	6.71828182	6.71828182	4.1103E-22	4.9398E−10

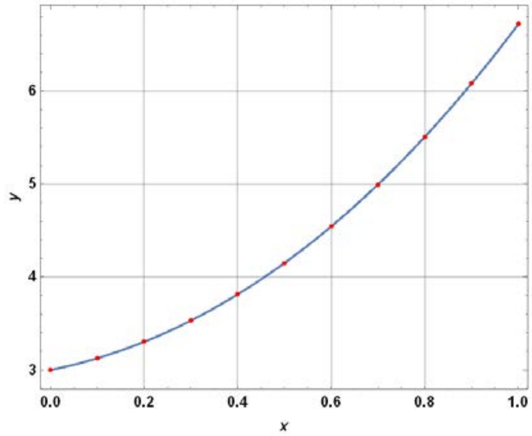

Figure 8. Solution curve of problem as compared with the exact solution

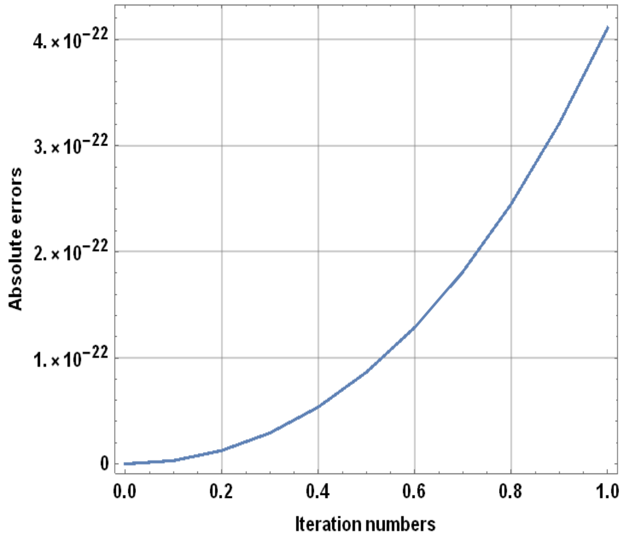

Figure 9. Nature of absolute errors obtained by the proposed method on problem 4

5. Conclusion

This work developed a one-step collocation approach with seven off-steps to directly solve initial value problems of general third-order ODEs. A step size with seven off-step locations is chosen for improved technique performance within the stability interval. In fact, when the new approach's results were compared to the block method proposed by Allogmany and Ismail [20], the new method was more accurate.

Competing Interests

The authors declare no competing interests.

Author Contributions

Conceptualization, Methodology, Validation, Formal analysis, Writing-original, Draft Preparation, Writing-review and Editing, and Revision, Dr. M. K. Duromola

Data Availability

The author would make the associated data available upon a reasonable request.

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